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A NOTE ON IMPERFECT MONOMIAL CURVES IN $\mathbf{P}^3$

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One of the most interesting problems in algebraic geometry started with Kronecker's result in 1882 is the following: What is the smallest number of (homogeneous) equations defining an algebraic set in an affine (or projective) n-space. Lately several authors have obtained strong results in the affine case and particular ones also in the projective case. For more detail see, e.g., [12].

There are papers dealing with curves in a 3-dimensional projective space $\mathbf{P}^3_k$ over a field k. In 1979 R. Hartshorne (see [6]) published a short but very nice proof of the fact that every curve $\mathbf{C}_d$ given parametrically by $(s^d, s^{d-1}t, st^{d-1}, t^d)$ in $\mathbf{P}^3_k$ is a set-theoretic complete intersection for $d \geqslant 4$ and the characteristics char(k) $= p > 0$. Bresinsky, Stückrad and Renschuch proved in [4] the same for the curves $\mathbf{C}(d, b, a)$ given parametrically by $(s^d, s^b t^{d-b}, s^a t^{d-a}, t^d)$ in $\mathbf{P}^3_k$ with g.c.d. $(d, b, a) = 1$ (also in the case of finite characteristics of k). More complicated is the situation in the case of char(k) $= 0$. Stückrad and Vogel showed in [12] that the above mentioned curve $\mathbf{C}(d, b, a)$ is a set-theoretic complete intersection for any characteristics, if $\mathbf{C}(d, b, a)$ is arithmetically Cohen-Macaulay. Note that a curve $\mathbf{C}$ is arithmetically Cohen-Macaulay iff the local ring of the vertex of the affine cone over $\mathbf{C}$ is Cohen-Macaulay.

During his stay in Bratislava W. Vogel posed the question: Is there an irreducible arithmetically non-Cohen-Macaulay (equivalently: imperfect) curve in $\mathbf{P}^3_k$, char(k) $= 0$, which is a set-theoretic complete intersection?

Using a proposition with an algebraic formulation of the problem we are investigating some classes of curves in $\mathbf{P}^3_k$ with char(k) $= 0$. We get sufficient conditions for these curves to be a set-theoretic complete intersection.

The notation in this paper is the standard one, for the basic facts and definitions (systems of parameters, multiplicity e_0, regular and Cohen-Macaulay local rings, ...) see, e.g., [14]. We denote by $L_A(M)$ the length of an A-module $\mathbf{M}$ and by ht($\mathfrak{a}$) the height of the ideal $\mathfrak{a}$, see, e.g., [7]. Dim(A) means the Krull-dimension of the ring A. The notion of a "set-theoretic complete intersection" is explained in Proposition 1.

With respect to the above mentioned results we will assume in the following that char(k) $= 0$.

First of all we formulate two conditions to abbreviate our explanation.

1. Let $(\mathbf{A}, \mathfrak{m})$ be a local ring with the maximal ideal $\mathfrak{m}$. We say that the condition (E) in $\mathbf{A}$ holds if for every ideal $\mathfrak{a}$ in $\mathbf{A}$ there is

$$\dim(\mathbf{A}/\mathfrak{a}) + \mathrm{ht}(\mathfrak{a}) = \dim(\mathbf{A}). \tag{E}$$

2. Let $(\mathbf{A}, \mathfrak{m})$ be a local ring and $\mathfrak{p}$ a prime ideal of $\mathbf{A}$ with $\dim(\mathbf{A}/\mathfrak{p}) = r$. We say that the multiplicity condition (M) for $\mathfrak{p}$ holds when there exist r elements $x_1, \dots, x_r$ of $\mathfrak{m}$ such that $\boldsymbol{x} = \{x_1, \dots, x_r\}$ is a system of parameters for $\mathbf{A}/\mathfrak{p}$ and the following condition is true

$$e_0((\mathfrak{p}, \boldsymbol{x}), \mathbf{A}) = e_0((\mathfrak{p}, \boldsymbol{x})/\mathfrak{p}, \mathbf{A}/\mathfrak{p}) \,.\, e_0(\mathfrak{p} \,.\, \mathbf{A}_\mathfrak{p}, \mathbf{A}_\mathfrak{p}). \tag{M}$$

Proposition 1. Let $(\mathbf{A}, \mathfrak{m})$ be a local ring with an infinite residue field $\mathbf{A}/\mathfrak{m}$ in which the condition (E) holds. Let $\mathfrak{p}$ be a prime ideal of $\mathbf{A}$. When (M) for $\mathfrak{p}$ is true, then $\mathfrak{p}$ is the set-theoretic complete intersection, i.e. there are $s = \mathrm{ht}(\mathfrak{p})$ elements $a_1, \dots, a_s$ of $\mathfrak{p}$ such that $\mathrm{rad}((a_1, \dots, a_s)) = \mathfrak{p}$.

For the proof of proposition 1 see [1] Proposition 2 or [10].

The following lemma shows that Proposition 1 is useless for defining primes of curves in $\mathbf{P}^3_k$ which are imperfect, i.e. arithmetically non-Cohen-Macaulay.

Lemma 2. Let $(\mathbf{A}, \mathfrak{m})$ be a regular local ring with $\mathbf{A}/\mathfrak{m}$ infinite and $\mathfrak{p}$ is a prime ideal of $\mathbf{A}$. If (M) for $\mathfrak{p}$ holds, then $\mathbf{A}/\mathfrak{p}$ is Cohen-Macaulay.

Proof. Let (M) be true for $\mathfrak{p}$. Put $\mathfrak{q} = (\mathfrak{p}, \boldsymbol{x})$, where $\boldsymbol{x} = \{x_1, \dots, x_r\}$ is a system of parameters for $\mathbf{A}/\mathfrak{p}$. By virtue of (M) there is then $e_0(\mathfrak{q}, \mathbf{A}) = e_0(\mathfrak{q}/\mathfrak{p}, \mathbf{A}/\mathfrak{p}) \,.\, e_0(\mathfrak{p} \,.\, \mathbf{A}_\mathfrak{p}, \mathbf{A}_\mathfrak{p})$.

We will count $e_0(\mathfrak{q}/\mathfrak{p}, \mathbf{A}/\mathfrak{p})$. Set $\mathbf{A}/\mathfrak{p} = \bar{\mathbf{A}}$ and $\bar{\mathfrak{q}} = \mathfrak{q} \,.\, \bar{\mathbf{A}} = (\bar{x}_1, \dots, \bar{x}_r)$. For the system of parameters $\{\bar{x}_1, \dots, \bar{x}_r\}$ in $\bar{\mathbf{A}}$ we set $\mathfrak{b}_0 = (0) \,.\, \bar{\mathbf{A}}$ and $\mathfrak{b}_k = \mathrm{U}(\mathfrak{b}_{k-1}) + (\bar{x}_k)$ for $0 < k \leqslant r$. The symbol $\mathrm{U}(\mathfrak{a})$ denotes the intersection of all primary ideals $\mathfrak{q}_j$ belonging to $\mathfrak{a}$ such that $\dim(\bar{\mathbf{A}}/\mathfrak{q}_j) = \dim(\bar{\mathbf{A}}/\mathfrak{a})$. Then $e_0(\bar{\mathfrak{q}}, \bar{\mathbf{A}}) = \mathrm{L}(\bar{\mathbf{A}}/\mathfrak{b}_r)$, see [2]. Counting in $\mathbf{A}$ we get $\mathfrak{b}'_0 = \mathrm{U}_{(\mathfrak{p})}$ $\mathfrak{b}'_k = \mathrm{U}(\mathfrak{b}'_{k-1}) + (x_k)$, $0 < k \leqslant r$. Put $\mathfrak{b}'_r = \mathfrak{q}^*$. Because of $\mathfrak{p} \subseteq \mathfrak{q} \subseteq \mathfrak{q}^*$ (see [2]), we have

$$e_0(\bar{\mathfrak{q}}, \bar{\mathbf{A}}) = \mathrm{L}(\bar{\mathbf{A}}/\mathfrak{q}^* . \bar{\mathbf{A}}) = \mathrm{L}(\bar{\mathbf{A}}/\mathfrak{q}^*). \tag{1}$$

The regularity of $\mathbf{A}$ implies $e_0(\mathfrak{p} \,.\, \mathbf{A}_\mathfrak{p}, \mathbf{A}_\mathfrak{p}) = 1$ and together with the condition (M) we get $e_0(\mathfrak{q}, \mathbf{A}) = e_0(\bar{\mathfrak{q}}, \bar{\mathbf{A}})$. With trivial $\mathrm{L}(\mathbf{A}/\mathfrak{q}) \leqslant e_0(\mathfrak{q}, \mathbf{A})$ (see, e.g., [5], p. 255) there then holds $\mathrm{L}(\mathbf{A}/\mathfrak{q}) \leqslant \mathrm{L}(\mathbf{A}/\mathfrak{q}^*)$. On the other hand, we have from $\mathfrak{q} \subseteq \mathfrak{q}^*$ that $\mathrm{L}(\mathbf{A}/\mathfrak{q}) \geqslant \mathrm{L}(\mathbf{A}/\mathfrak{q}^*)$ and $\mathfrak{q} = \mathfrak{q}^*$. Then we get

$$e_0(\bar{\mathfrak{q}}, \bar{\mathbf{A}}) = \mathrm{L}\bar{\mathbf{A}}/\bar{\mathfrak{q}}), \tag{2}$$

i.e. in $\bar{\mathbf{A}}$ there is an ideal $\bar{\mathfrak{q}} = (\bar{x}_1, \dots, \bar{x}_r)$ generated by a system of parameters such that (2) holds. This means that $\bar{\mathbf{A}} = \mathbf{A}/\mathfrak{p}$ is Cohen-Macaulay (see, e.g., [14]) as required.

As in our case $\mathbf{R} = k[X_0, X_1, X_2, X_3]_{(X_0, X_1, X_2, X_3)}$ is regular, we formulate an easy modification of Proposition 1.

Proposition 3. Let $\mathbf{R} = k[X_0, X_1, X_2, X_3]_{(X_0, 1, X_2, X_3)}$ and $\mathfrak{p}$ be a prime ideal in $\mathbf{R}$, $\dim(\mathbf{R}/\mathfrak{P}) = 2$. Assume there are elements a_1, a_2 of $\mathbf{R}$ and $F \in \mathfrak{p}$ such that $\boldsymbol{a} = \{a_1, a_2\}$ is a system of parameters for $\mathbf{R}/\mathfrak{p}$ and

$$e_0((F, \boldsymbol{a})/(F), \mathbf{R}') = e_0((\mathfrak{p}, a)/\mathfrak{p}, \mathbf{R}/\mathfrak{p}) \,.\, e_0(\mathfrak{p}' \,.\, \mathbf{R}'_{\mathfrak{p}'}, \mathbf{R}'_{\mathfrak{p}'}),$$

where $\mathbf{R}' = \mathbf{R}/(F)$ and $\mathfrak{p}' = \mathfrak{p} \,.\, \mathbf{R}'$; then there eists an element $G \in \mathfrak{p}$ such that $\mathfrak{p} = \mathrm{rad}((F, G))$, i.e. $\mathfrak{p}$ is a set-theoretic complete intersection.

In order to describe the way how to find such an element F in some special cases we need the following lemma.

Lemma 4. Let $\mathfrak{q} = (X_1^n, X_1X_2, X_2^n) \subset k[X_1, X_2]_{(X_1, X_2)} = \mathbf{A}$, $n \geqslant 2$. Then $e_0(\mathfrak{q}, \mathbf{A}) = 2n$.

Proof. Put $\mathfrak{q}' = (X_1^n + X_2^n, X_1X_2)$. Then $\mathfrak{q}'$ is a reduction of $\mathfrak{q}$ and $e_0(\mathfrak{q}', \mathbf{A}) = E_0(\mathfrak{q}, \mathbf{A})$, see [8]. Since $\mathfrak{q}'$ is an ideal generated by a system of parameters in a regular local ring, the claim follows from the fact that $e_0(\mathfrak{q}', \mathbf{A}) = L(\mathbf{A}/\mathfrak{q}')$ by counting the length. In fact $\mathfrak{q}'' = (X_1^{n+1}, X_2^{n+1}, X_1X_2) \subset \mathfrak{q}' \subset \mathfrak{q}$ and $L(\mathbf{A}/\mathfrak{q}'') = 2n + 1$, $L(\mathbf{A}/\mathfrak{q}) = 2n - 1$, thus $e_0(\mathfrak{q}, \mathbf{A}) = L(\mathbf{A}/\mathfrak{q}') = 2n$.

Note that Gröbner in [5], p. 256 counted $e_0(\mathfrak{q}, \mathbf{A})$ for the above $\mathfrak{q}$ in the case $n = 3$, but his calculations cannot be used for $n > 3$.

Let $\mathbf{R}$ be as in Proposition 3 and $\mathbf{C}_n$ the curve in $\mathbf{P}_k^3$ given parametrically by $(s^n, s^{n-1}t, st^{n-1}, t^n)$ with the defining ideal $\mathfrak{p} = (F_1, \ldots, F_n)$, $F_1 = X_0X_3 - X_1X_2$, $F_2 = X_0^{n-2}X_2 - X_1^{n-1}$, $F_3 = X_0^{n-3}X_2^2 - X_1^{n-2}X_3$, $\ldots$, $F_{n-1} = X_0X_2^{n-2} - X_1^2X_3^{n-3}$, $F_n = X_2^{n-1} - X_1X_3^{n-2}$, see [9], p. 320. It is known that $\mathbf{C}_n$ is nonsingular for every n and it is arithmetically Cohen-Macaulay for $n = 3$, arithmetically non-Cohen-Macaulay Buchsbaum for $n = 4$ and arithmeticaly non-Buchsbaum whenever $n \geqslant 5$, see, e.g., [13]. Put $\mathfrak{q} = (\mathfrak{p}, X_0, X_3) = (X_0, X_3, X_1^{n-1}, X_1X_2, X_2^{n-1})$. From Lemma 4 it follows that $e_0(\mathfrak{q}, \mathbf{R}) = 2 \,.\, (n - 1)$. Let us count $e_0(\mathfrak{q}/\mathfrak{p}, \mathbf{R}/\mathfrak{p})$ as in the proof of Lemma 2. We use the so-called U-process and we get $e_0(\mathfrak{q}/\mathfrak{p}, \mathbf{R}/\mathfrak{p}) = L(\mathbf{R}\mathfrak{q}^*) = 2 \,.\, (n - 2)$.

Now we formulate the main result.

Theorem 5. Let $\mathbf{C}_n$, $\mathfrak{p}$, $\mathfrak{q}$ be as above, $n \geqslant 4$. If there exists a form $F \in \mathfrak{p}^{(n-1)} - \mathfrak{p}^{n-1}$, which is superficial of degree $n - 2$ with respect to $\mathfrak{q}$, then $\mathbf{C}_n$ is a set-theoretic complete intersection.

Remarks.

1. The symbol $\mathfrak{p}^{(i)}$ denotes the ith symbolic power of $\mathfrak{p}$, i.e. $\mathfrak{p}^{(i)} = \mathfrak{p}^i \,.\, \mathbf{R}_{\mathfrak{p}} \cap \mathbf{R}$.
2. We say that an element F of a local ring $(\mathbf{A}, \mathfrak{m})$ is superficial of degree s with respect to the $\mathfrak{m}$-primary ideal $\mathfrak{q}$ if $F \in \mathfrak{q}^s - \mathfrak{q}^{s+1}$ and there exists a positive

integer c such that $(\mathfrak{q}^n : F) \cap \mathfrak{q}^c = \mathfrak{q}^{n-s}$ for all $n >> 0$. For more facts about superficial elements see [14].

Proof of Theorem 5. The assumptions for F imply $e_n(\mathfrak{q} . \bar{\mathbf{R}}, \bar{\mathbf{R}}_\mathfrak{p}) = 2 . (n-1) . (n-2)$ and $e_0(\mathfrak{p} . \bar{\mathbf{R}}_\mathfrak{p}, \bar{\mathbf{R}}_\mathfrak{p}) = n - 1$, $\bar{\mathbf{R}} = \mathbf{R}/(F)$. The assertion now follows from Proposition 3.

We finish this paper by an example which shows that the idea of Theorem 5 is useful also for the arithmetically Buchsbaum curves. Note that the Buchsbaum property is a simple generalization of the Cohen-Macaulay one, see [11].

Example. In [3], Theorem 3, there is a characterization of arithmetically non-Cohen-Macaulay Buchsbaum curves over an algebraically closed field k. Curves are given parametrically by $(s^{4n}, s^{2n+1}t^{2n-1}, s^{2n-1}t^{2n+1}, t^{4n})$ with the defining ideal $\mathfrak{p} = (X_0X_3 - X_1X_2,\ X_0^2X_2^{2n-1} - X_1^{2n+1},\ X_0X_2^{2n} - X_1^{2n}X_3,\ X_2^{2n+1} - X_1^{2n-1}X_3^2)$. As before we put $\mathfrak{q} = (\mathfrak{p}, X_0, X_3) = (X_0, X_3, X_1X_2, X_1^{2n+1}, X_2^{2n+1})$. Then we get $e_0(\mathfrak{q}, \mathbf{R}) = 2 . (2n+1)$ by virtue of Lemma 4. For $\mathfrak{q}/\mathfrak{p}$ we get $e_0(\mathfrak{q}/\mathfrak{p}, \mathbf{R}/\mathfrak{p}) = 4n = 2 . 2n$. Comparing with the curve $\mathbf{C}_n$ from Theorem 5 we see that the only difference is in the degree of the required superficial element F.

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ЗАМЕЧАНИЯ О НЕСОВЕРШЕННЫХ МОНОМИАЛИНЬІХ КРИВЫХ В P_k^3

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Резюме

В работе исследуются некоторые классы неприводимых несовершенных мономиальных кривых пространства $\mathbf{P}_k^3$, char (k) = 0, рассматривая их как теоретико-множественное полное пересечение. Доказывается, что если дла кривой **C** с общим нулем $(s^d, s^{d-1}t, st^{d-1}, t^d)$ существует однородный многочлен $F \in \mathfrak{p}_C^{(d-1)} - \mathfrak{p}_C^{d-1}$, который является поверхностным элементом порядка $d-2$ относительно идеала $(\mathfrak{p}_C, X_0, X_3)$, то **C** — теоретико- множественное полное прересечение.