Mathematica Slovaca

Ján Jakubík

On direct and subdirect product decompositions of partially ordered sets

Mathematica Slovaca, Vol. 52 (2002), No. 4, 377--395

Persistent URL: http://dml.cz/dmlcz/129466

Math. Slovaca, **52** (2002), No. 4, 377–395

Mathematica Slovaca

ON DIRECT AND SUBDIRECT PRODUCT DECOMPOSITIONS OF PARTIALLY ORDERED SETS

JÁN JAKUBÍK

(*Communicated by Pavol Zlatoš*)

ABSTRACT. This paper concerns direct and subdirect product decompositions of some types of partially ordered sets; in particular, we deal with certain forms of the cancellation rule for such decompositions.

1. Introduction

In the first part of the present paper (Sections 2–6), we characterize two-factor internal direct product decompositions of a lattice by means of the properties of pairs of convex sublattices.

For the following results (A) and (B) cf. Grätzer [4; pp. 152, 157, Chapt. III].

(A) The direct decompositions of a bounded lattice L into two factors are (up to isomorphism) in a one-to-one correspondence with the complemented neutral ideals of L.

(B) Representations of a lattice L with 0 as a direct product of two lattices are (up to isomorphism) in one-to-one correspondence with pairs of ideals $\langle I, J\rangle$ satisfying $I \cap J = \{0\}$ and every element of L has exactly one representation of the form $a = i \vee j$, $i \in I$, $j \in J$.

Let S be a directed set having the least element. Direct product decompositions of S into two factors were investigated by Halaš [5]. Similarly as in (A), Halaš applied the notion of complemented neutral ideals of S. The main results of [5] are Theorem 1 and Theorem 2. Pringerová [12] generalized [5;

2000 Mathematics Subject Classification: Primary 06A06, 06A12, 06B05.
Keywords: directed set, semilattice, lattice, direct product, internal direct product, regular subdirect product.

Supported by VEGA Grant 2/1131/21.

Theorem 1] to the case when S is a directed set which need not have the least element.

We deal with the question in which way we have to modify (B) in the case when the existence of the least element of L is not assumed and when instead of a pair of ideals of L we have a pair of convex sublattices of L.

In the second part of the article (Sections 7, 8) we investigate a cancellation rule for direct product decompositions of a directed set of finite length.

A partially ordered set L will be said to satisfy the *strong cancellation rule for direct product decompositions* if, whenever

$$L \simeq A \times B, \qquad L \simeq C \times D \qquad \text{and} \qquad A \simeq C,$$

then $B \simeq D$.

In the present paper we prove:

(∗) Each directed set of finite length satisfies the strong cancellation rule for direct product decompositions.

Related results concerning the cancellation for internal direct product decompositions of certain types of partially ordered sets have been proved by Csontóová and the author [9], [10].

The third part of the paper contains Sections 9–11. Here we define the notion of a regular subdirect decomposition of a semilattice S.

The corresponding condition in this definition concerns the intervals of S; it is related to a condition dealt with by Kolibiar [11] for prime intervals of a semilattice.

The well-known relation between subdirect decompositions of S (cf. Birkhoff [2; Chapter VI, §5] yields that to each subdirect decomposition φ of S there corresponds a subdirect decomposition $\overline{\varphi}$ of S such that the underlying sets of the subdirect factors from $\overline{\varphi}$ are certain partitions of the set S; we call $\overline{\varphi}$ a *p-subdirect decomposition.* (For details, cf. Section 10 below.)

We prove a cancellation rule for regular p-subdirect product decompositions of a semilattice. (In fact, we deal with slightly more general structures including semilattices.)

2. Preliminaries

Suppose that L is a lattice and $c \in L$. For the notion of an internal direct product decomposition of L with the central element c, cf. [9]; the definition (for two factor decompositions) is recalled in Section 3 below.

We remark that if I and J are as in (B), then, in fact, L is an internal direct product of I and J with the central element $c = 0$.

For $A \subseteq L$ we put

$$A^+ = \{a \in A : \ a \geqq c\}, \qquad A^- = \{a \in A : \ a \leqq c\}.$$

Let A and B be convex sublattices of L with $A \cap B = \{c\}$. Consider the following conditions for A and B:

(α_1) Each element $x \in L^+$ has exactly one representation of the form $x = x_1 \vee x_2$, $x_1 \in A^+$, $x_2 \in B^+$.

(α_2) Each element $y \in L^-$ has exactly one representation of the form $y = y_1 \wedge y_2$, $y_1 \in A^-$, $y_2 \in B^-$.

If (α_1) and (α_2) are valid, then for each $z \in L$ we denote

$$z^A = (x_1 \wedge z) \vee y_1, \qquad z^B = (x_2 \wedge z) \vee y_2,$$

where $x = z \vee c$, $y = z \wedge c$ and x_i, y_i $(i = 1,2)$ are as in (α_1) and (α_2). Then we can consider the condition:

(α_3) If $z, z' \in L$ and $z^A \leqq (z')^A$, $z^B \leqq (z')^B$, then $z \leqq z'$.

Further, we shall deal with the condition:

(α_4) Let $p \in A$, $q \in B$. Then there is a sublattice L_1 of L such that
- (i) L_1 is a Boolean algebra;
- (ii) the Boolean algebra L_1 is generated by its subset $\{p, q, c\}$;
- (iii) if x is the complement of c in L_1, then $x^A = p$ and $x^B = q$.

We prove:

(C) Let A and B be convex sublattices of a lattice L such that $A \cap B = \{c\}$. Then L is an internal direct product of A and B with the central element c if and only if the conditions (α_1)–(α_4) are satisfied.

3. Internal direct products

Assume that L is a lattice and $c \in L$. Let A and B be lattices and let us consider an isomorphism

$$\varphi \colon L \to A \times B \tag{1a}$$

of L onto the direct product $A \times B$. For $x \in L$ we denote

$$\varphi(x) = \big(x(A), x(B)\big).$$

Put

$$A(c) = \big\{x \in L : \ x(B) = c(B)\big\}, \qquad B(c) = \big\{x \in L : \ x(A) = c(A)\big\}.$$

Further, for each $x \in L$ we set

$$\varphi^c(x) = (x^0, y^0),$$

where $x^0 \in A(c)$, $y^0 \in B(c)$ and

$$x^0(A) = x(A), \qquad y^0(B) = y(B).$$

Then φ^c is an isomorphism of L onto $A(c) \times B(c)$; moreover, $A(c)$ and $B(c)$ are convex sublattices of L with $A(c) \cap B(c) = \{c\}$. We express this situation by writing

$$\varphi^c : L = (\mathrm{int})A(c) \times B(c) \tag{1}$$

and we say that φ^c is an *internal direct product decomposition of L with the central c*. (Cf. [3].)

It is obvious that the lattice $A(c)$ is isomorphic to A and that $B(c)$ is isomorphic to B.

From this definition we immediately obtain:

3.1. Lemma. *Let φ be as in* (1a). *Suppose that A and B are convex sublattices of L with $A \cap B = \{c\}$. Then φ is an internal direct product decomposition of L with the central element c if and only if the following conditions are satisfied:*

$$z \in A \iff z(A) = z \iff z(B) = c;$$
$$z \in B \iff z(B) = z \iff z(A) = c.$$

In the remaining part of the present section we assume that (1) is valid. We write A and B instead of $A(c)$ or $B(c)$, respectively.

Suppose that A_1 and B_1 is a sublattice of A or of B, respectively. Denote

$$L_1 = \{x \in L : \ x(A) \in A_1 \text{ and } x(B) \in B_1\}. \tag{2}$$

Consider the partial mapping $\varphi^1 = \varphi^c|_{L_1}$. Then (1) yields:

3.2. Lemma. *φ^1 is an isomorphism of L_1 onto $A_1 \times B_1$. If, moreover, $c \in A_1 \cap B_1$, then we have*

$$\varphi^1 : L_1 = (\mathrm{int})A_1 \times B_1. \tag{3}$$

Let $p \in A$, $q \in B$. Put

$$u_1 = p \wedge c, \quad v_1 = p \vee c, \quad u_2 = q \wedge c, \quad v_2 = q \vee c,$$
$$A_1 = \{p, c, u_1, v_1\}, \quad B_1 = \{q, c, u_2, v_2\}.$$

Further, let L_1 be as in (2).

3.3. Lemma. *The relation* (3) *is valid and the conditions* (i), (ii), (iii) *from* (α_4) *are satisfied.*

P r o o f. The validity of (3) is a consequence of 3.2. Since A_1 and B_1 are Boolean algebras, (3) yields that L_1 is a Boolean algebra as well.

Put $u = u_1 \wedge u_2$, $v = v_1 \vee v_2$. Hence $u, v \in L_1$. It is clear that u is least element of L_1 and v is the greatest element of L_1.

Let L_1^0 be the subalgebra of the Boolean algebra L_1 which is generated by the set $\{p, q, c\}$. Then we have $u_1, u_2, v_1, v_2 \in L_1^0$, hence $u, v \in L_1^0$.

Let x be the complement of the element c in L_1. Thus $x \in L_1^0$; moreover, both the elements $x \wedge v_1$ and $x \wedge v_2$ belong to L_1^0.

Now, each element $t \in L_1$ can be written in the form $t = t_1 \vee t_2$, where

$$t_1 \in \{u, u_2, v_2, c\}, \qquad t_2 \in \{u, x \wedge v_1, x \wedge v_2, x\}.$$

Hence $t \in L_1^0$ and thus $L_1 = L_1^0$. Therefore (ii) from (α_4) is valid.

Finally, in view of (ii), each element of L_1 has a unique complement in L_1. A simple calculation shows that the element $(\varphi_1)^{-1}\big((p,q)\big)$ is a complement of c in L_1. □

4. Necessary condition

In this section we assume that relation (1) from Section 3 is satisfied. Similarly as in the previous section we write A and B instead of $A(c)$ or $B(c)$, respectively.

4.1. Lemma. *The condition (α_1) is satisfied.*

Proof. Let $x \in L^+$. Put $x_1 = x(A)$, $x_2 = x(B)$. Then $x_1 \in A^+$, $x_2 \in B^+$. Further,

$$\varphi^c(x_1) = (x_1, c), \qquad \varphi^c(x_2) = (c, x_2),$$
$$\varphi^c(c) = (c, c), \qquad \varphi^c(x) = (x_1, x_2),$$

whence

$$\varphi^c(x) = \varphi^c(x_1) \vee \varphi^c(x_2).$$

Thus $x = x_1 \vee x_2$. Let $x_1' \in A^+$, $x_2' \in B^+$, $x = x_1' \vee x_2'$. We obtain

$$\varphi^c(x_1') = (x_1', c), \qquad x_1' \leqq x,$$

hence $(x_1', c) \leqq (x_1, x_2)$ and thus $x_1' \leqq x_1$. Similarly we get $x_1 \leqq x_1'$. Thus $x_1 = x_1'$. Analogously, $x_2 = x_2'$. □

By a dual reasoning, we have

4.2. Lemma. *The condition (α_2) is valid.*

Moreover, when looking at the proof of 4.1 we conclude:

4.3. LEMMA. *Let $x \in L^+$ and let x_1, x_2 be as in (α_1). Then $x_1 = x(A)$, $x_2 = x(B)$.*

Similarly, we have:

4.4. LEMMA. *Let $y \in L^-$ and let y_1, y_2 be as in (α_2). Then $y_1 = y(A)$, $y_2 = y(B)$.*

4.5. LEMMA. *Let $z \in L$ and let z^A, z^B be as in Section 2. Then $z^A = z(A)$, $z^B = z(B)$.*

Proof. We have

$$z^A = (x_1 \wedge z) \vee y_1 ,$$

where

$$\begin{aligned} z \vee c = x = x_1 \vee x_2 , \qquad & x_1 \in A^+ , \ x_2 \in B^+ , \\ z \wedge c = y = y_1 \wedge y_2 , \qquad & y_1 \in A^- , \ y_2 \in B^- . \end{aligned}$$

Thus in view of 4.3 and 4.4,

$$z^A(B) = \big(x_1(B) \wedge z(B)\big) \vee y_1(B) = \big(c \wedge z(B)\big) \vee c = c .$$

Therefore $z^A \in A$ and hence $z^A(A) = z^A$. Further,

$$\begin{aligned} z^A(A) &= \big(x_1(A) \wedge z(A)\big) \vee y_1(A) = \big(x_1 \wedge z(A)\big) \vee y_1 \\ &= \big(x(A) \wedge z(A)\big) \vee y(A) = \big((x \wedge z) \vee y\big)(A) = z(A) . \end{aligned}$$

Summarizing, we get $z^A = z(A)$. Analogously we obtain $z^B = z(B)$. □

4.6. LEMMA. *The condition (α_3) is satisfied.*

Proof. It suffices to apply 4.1, 4.2, 4.6 and 3.3. □

5. Sufficient condition

In this section we assume that A and B are convex sublattices of a lattice L, $c \in L$, $A \cap B = \{c\}$ and that the conditions (α_1)–(α_4) are satisfied.

Let $z \in L$ and let z^A, z^B be as in Section 2. Then $z^A \in A$ and $z^B \in B$. Consider the mapping

$$\varphi^0(z) = \big(z^A, z^B\big)$$

of L into $A \times B$.

5.1. Lemma. *Let z and z' be elements of L. Then*

$$z \leqq z' \iff \varphi^0(z) \leqq \varphi^0(z') .$$

Proof. The implication $\Longrightarrow$ is an immediate consequence of the definition of φ^0. The converse implication follows from (α_3). □

5.2. Lemma. *We have $\varphi^0(L) = A \times B$.*

Proof. This is implied by 3.3. □

5.3. Corollary. *φ^0 is an isomorphism of L onto $A \times B$.*

5.4. Lemma. *Let $z \in A$, $z' \in B$. Then*

$$\varphi^0(z) = (z,c) , \qquad \varphi^0(z') = (c,z') .$$

Proof. Let x, y, x_i, y_i $(i = 1,2)$ be as in Section 2. From $z \in A$ we conclude that x and z also belong to A. Hence we must have

$$x_1 = x , \qquad x_2 = c , \qquad y_1 = y , \qquad y_2 = c .$$

This yields

$$z^A = z , \qquad z^B = c ,$$

whence $\varphi^0(z) = (z,c)$. Analogously we obtain the relation $\varphi^0(z') = (c,z')$. □

Summarizing, from 3.1, 5.3 and 5.4 we obtain:

5.5. Lemma. *The assertion "if" from* (C) *is valid.*

In view of 4.7 and 5.5, the assertion (C) holds.

6. Additional remarks

Again, let L be a lattice and $c \in L$. Assume that A and B are convex sublattices of L with $A \cap B = \{c\}$.

6.1. Suppose that the relation

$$\varphi : L = (\mathrm{int})A \times B \tag{6.1}$$

is valid. Consider the partial mappings

$$\varphi^+ = \varphi|_{L^+} , \qquad \varphi^- = \varphi|_{L^-} .$$

Then we have

$$\varphi^+ : L^+ = (\mathrm{int})A^+ \times B^+ , \tag{6.2}$$

$$\varphi^- : L^- = (\mathrm{int})A^- \times B^- . \tag{6.3}$$

6.2. If the relations

$$\varphi_1 : L^+ = (\mathrm{int})A^+ \times B^+ \,, \tag{6.4}$$

$$\varphi_2 : L^- = (\mathrm{int})A^- \times B^- \tag{6.5}$$

are valid, then L need not be an internal direct product of A and B. Example: Let L be the lattice in Fig. 1. Put $A = \{p, c, s\}$, $B = \{q, c, r\}$. Then A and B are convex sublattices of L with $A \cap B = \{c\}$. Moreover, (6.4) and (6.5) are valid (the meanings of φ_1 and φ_2 are obvious). But (6.1) does not hold; it is easy to verify that L is directly indecomposable.

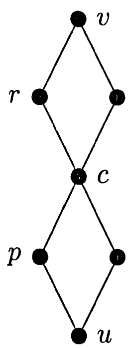


Figure 1.

6.3. Suppose that the relations (6.1), (6.2) and (6.3) are satisfied.

If $t \in L$ and $\varphi(t) = (t_A, t_B)$, then we denote

$$t_A = \varphi_A(t)\,, \quad t_B = \varphi_B(t)\,;$$

thus

$$\varphi(t) = \bigl(\varphi_A(t), \varphi_B(t)\bigr)\,.$$

Similarly, for $x \in L^+$ and $y \in L^-$ we write

$$\varphi^+(x) = \bigl(\varphi_A^+(x), \varphi_B^+(x)\bigr)$$

(for typographical reasons we write here A rather than A^+); analogously we put

$$\varphi^-(y) = \bigl(\varphi_A^-(y), \varphi_B^-(y)\bigr)\,.$$

The results of Section 4 above show that the mapping φ can be explicitly described if the mappings φ^+ and φ^- are given. Namely, according to 4.5, for each $z \in L$ we have

$$\varphi_A(z) = \bigl(\varphi_A^+(z \vee c) \wedge z\bigr) \vee \varphi_A^-(z \wedge c)\,,$$

$$\varphi_B(z) = \bigl(\varphi_B^+(z \vee c) \wedge z\bigr) \vee \varphi_B^-(z \wedge c)\,.$$

7. Connected partially ordered sets

In this section we assume that L is a connected partially ordered set. We apply Hashimoto's theorem ([6]) on direct product decompositions of L.

The below investigation would be trivial in the case $\operatorname{card} L = 1$; thus we suppose that L has more than one element.

A partially ordered set A is called *directly indecomposable* if, whenever $A \simeq B \times C$, then either $\operatorname{card} B = 1$ or $\operatorname{card} C = 1$.

7.1. NOTATION. Suppose that L possesses a direct product decomposition

$$L \simeq \prod_{i \in I} L_i \tag{1}$$

such that all L_i are directly indecomposable and $\operatorname{card} L_i \neq 1$. For $i \in I$ we denote

$$\bar{i} = \{j \in I : \ L_j \simeq L_i\}.$$

7.2. LEMMA. *Assume that there is $i(0) \in I$ such that the set $\overline{i}(0)$ is infinite. Then L does not satisfy the strong cancellation rule for direct decompositions.*

Proof. Let X be a one-element partially ordered set. Then we have

$$L \simeq L \times L_{i(0)}, \qquad L \simeq L \times X$$

and $L_{i(0)}$ fails to be isomorphic to X. □

7.3. LEMMA. *Suppose that L has a direct product decomposition*

$$L \simeq \prod_{j \in J} T_j \tag{2}$$

such that $\operatorname{card} T_j \neq 1$ for each $j \in J$. Then there are subsets $I(T_j)$ of I such that

a) $\bigcup_{j \in J} I(T_j) = I$;

b) $I(T_{j(1)}) \cap I(T_{j(2)}) = \emptyset$ *whenever* $j(1) \neq j(2)$;

c) $T_j \simeq \prod_{i \in I(T_j)} L_i$ *for each* $j \in J$.

Proof. This is a consequence of Hashimoto's theorem on the refinements of direct product decompositions of L; cf. [6]. □

7.4. Corollary. *Let T_j $(j \in J)$ be as in 7.3. Suppose that all T_j are directly indecomposable. Then there is a bijection $\varphi\colon I \to J$ such that*

$$L_i \simeq T_{\varphi(i)} \qquad \textit{for each} \quad i \in I\,.$$

Assume that

$$L \simeq A \times B\,, \qquad L \simeq C \times D \tag{3}$$

such that each of the sets A, B, C and D has more than one element.

Then in view of 7.3 there are subsets $I(A)$ and $I(B)$ with

$$I(A) \cap I(B) = \emptyset\,, \qquad I(A) \cup I(B) = I\,,$$
$$A \simeq \prod_{i\in I(A)} L_i\,, \qquad B \simeq \prod_{i \in I(B)} L_i\,.$$

Let $I(C)$ and $I(D)$ have analogous meanings.

Denote

$$A_1 = \prod_{i\in I(A)\cap I(C)} L_i\,, \qquad A_2 = \prod_{i\in I(A)\cap I(D)} L_i\,,$$
$$B_1 = \prod_{i\in I(B)\cap I(C)} L_i\,, \qquad B_2 = \prod_{i\in I(B)\cap I(D)} L_i\,,$$
$$C_1 = A_1\,, \quad C_2 = B_1\,, \quad D_1 = A_2\,, \quad D_2 = B_2\,.$$

Then we have

$$A \simeq A_1 \times A_2\,, \qquad B \simeq B_1 \times B_2\,, \qquad C \simeq C_1 \times C_2\,, \qquad D \simeq D_1 \times D_2\,. \tag{4}$$

7.5. Lemma. *Suppose that for each $i \in I$, the set $\bar{i}$ is finite. Then L satisfies the strong cancellation rule for direct product decompositions.*

Proof. Assume that (3) holds and that $A \simeq C$. We have to verify that $B \simeq D$. In view of (4) it suffices to show that the relation

$$B_1 \simeq A_2 \tag{+}$$

is valid.

For $i \in I$ we put $\bar{i}(A) = \bar{i} \cap I(A)$, and analogously for B, C and D. Further, we set

$$\bar{i}(A_1) = \bar{i} \cap \big(I(A) \cap I(C)\big)\,,$$

and similarly for A_2, B_j, C_j, D_j $(j = 1, 2)$.

Let $i \in I(A_2)$. In view of the relation $A \simeq C$ we have

$$\operatorname{card} \bar{i}(A) = \operatorname{card} \bar{i}(C)\,.$$

Moreover, (4) yields

$$\operatorname{card}\bar{i}(A) = \operatorname{card}\bar{i}(A_1) + \operatorname{card}\bar{i}(A_2).$$

Similarly,

$$\operatorname{card}\bar{i}(C) = \operatorname{card}\bar{i}(C_1) + \operatorname{card}\bar{i}(C_2).$$

Since the cardinalities under consideration are finite and $A_1 = C_1$, $B_1 = C_2$, we obtain

$$\operatorname{card}\bar{i}(A_2) = \operatorname{card}\bar{i}(B_1). \tag{5}$$

Analogously, for each $j \in I(B_1) = I(C_2)$ we get

$$\operatorname{card}\bar{j}(B_1) = \operatorname{card}\bar{j}(A_2). \tag{6}$$

The relations (5) and (6) imply that there exists a bijection

$$\varphi\colon I(A_2) \to I(B_1)$$

such that for each $i \in I(A_2)$ we have

$$L_i \simeq L_{\varphi(i)}.$$

Therefore the relation (+) is valid. □

Summarizing, from 7.4 and 7.5 we obtain:

7.6. THEOREM. *Let L be a connected partially ordered set possessing a direct product decomposition* (1) *with directly indecomposable factors L_i. For $i \in I$ let $\bar{i}$ be as above. Then the following conditions are equivalent:*

(i) *L satisfies the strong cancellation rule for direct product decompositions.*
(ii) *For each $i \in I$, the set $\bar{i}$ is finite.*

8. Weak product decompositions

For the sake of completeness we recall the definitions of some relevant notions.

Again, let L be a partially ordered set.

L is said to be *discrete* (or *locally finite*) if each bounded chain in L is finite.

If there is a positive integer n such that $\operatorname{card} C \leqq n$ whenever C is a chain in L, then L is said to be a *poset of finite length.*

Each partially ordered set of finite length is discrete.

Let I be a nonempty set of indices and for each $i \in I$ let L_i be a partially ordered set. Put

$$P = \prod_{i\in I} L_i.$$

If $p \in P$ with $p = (p_i)_{i\in I}$, then we set $p(L_i) = p_i$. For $p, p' \in P$ we denote

$$d(p,p') = \{i \in I : \ p(L_i) \neq p'(L_i)\}.$$

Let Q be a nonempty subset of P such that the following conditions are satisfied:

(i) If $i \in I$ and $p^i \in L_i$, then there is $q \in Q$ with $q(L_i) = p^i$.
(ii) If q and q' are elements of Q, then the set $d(q, q')$ is finite.
(iii) If $q \in Q$ and $p \in P$ such that the set $d(p, q)$ is finite, then p belongs to Q.

Under these assumptions, Q is said to be a *weak product of the partially ordered sets* L_i $(i \in I)$. (Cf., e.g., [3].)

If Q is as above and if the set I is finite, then Q is a direct product of partially ordered sets L_i $(i \in I)$.

Weak product decompositions of discrete partially ordered sets and, in particular, of discrete lattices, were investigated in [7] and [8].

8.1. PROPOSITION. *Assume that the partially ordered set* L *is directed and discrete. Then* L *is isomorphic to a weak product of directly indecomposable partially ordered sets.*

Proof. This is a consequence of [4; Theorem 4.1]. □

8.2. PROPOSITION. *Assume that* L *is a partially ordered set of finite length. Further, suppose that* L *is directed. Then* L *is isomorphic to a direct product of a finite number of directly indecomposable partially ordered sets.*

Proof. It suffices to consider the case when $\operatorname{card} L > 1$. Then, in view of 8.1, we can suppose without loss of generality that L is a weak product of directly indecomposable partially ordered sets L_i $(i \in I)$ such that $\operatorname{card} L_i \neq 1$ for each $i \in I$. It is obvious that all L_i must be directed.

Since L has finite length, it must possess the least element, which will be denoted by x_0. There exists a positive integer $n \geqq 2$ such that, whenever C is a chain of L, then $\operatorname{card} C \leqq n$.

We want to show that $\operatorname{card} I \leqq n-1$. By way of contradiction, suppose that $\operatorname{card} I > n-1$. Thus there exist distinct elements $i(1), i(2), \dots, i(n)$ in the set I.

For each $i \in I$, $x_0(L_i)$ is the least element of L_i. Hence there is $y^i \in L_i$ such that $y^i > x_0(L_i)$.

Let $j \in \{1, 2, \dots, n\}$. In view of the definition of the weak product there exists $z_j \in L$ such that

$$z_j(L_i) = \begin{cases} y^i & \text{if } i \in \{i(1), i(2), \dots, i(j)\}, \\ x_0(L_i) & \text{otherwise.} \end{cases}$$

Put $C = \{x_0, z_1, z_2, \dots, z_n\}$. Then C is a chain of L with $\operatorname{card} C = n + 1$, which is a contradiction.

Therefore the set I is finite and hence L is a direct product of the partially ordered sets L_i $(i \in I)$. □

8.3. Theorem. *Let L be a directed set of finite length. Then L satisfies the strong cancellation rule for direct product decompositions.*

Proof. This is a consequence of 7.6 and 8.2. □

9. Subdirect decompositions

For elements x, y of a partially ordered set A we denote by $(x,y)_\wedge$ the set of all lower bounds of $\{x,y\}$. Further, let $x \wedge_m y$ be the system of all maximal elements of the set $(x,y)_\wedge$.

9.1. Definition. $\mathcal{M}_\wedge$ is defined to be the *class of partially ordered sets* A such that, whenever, x, y are elements of A, then

(i) $(x,y)_\wedge \neq \emptyset$;
(ii) for each $z \in (x,y)_\wedge$ there exists $z_1 \in x \wedge_m y$ such that $z \leqq z_1$.

Recall that if the class $\mathcal{M}_\vee$ is defined in a dual way, then $\mathcal{M}_\wedge \cap \mathcal{M}_\vee$ is the class of all directed multilattices (in the sense defined by Benado [1]).

By a semilattice we always understand a $\wedge$-semilattice. It is obvious that each semilattice belongs to the class $\mathcal{M}_\wedge$.

It is also clear that if A and B are elements of $\mathcal{M}_\wedge$, then their direct product $A \times B$ belongs to $\mathcal{M}_\wedge$ as well.

Let $A, B \in \mathcal{M}_\wedge$ and let $f\colon A \to B$ be a mapping such that

$$f(x \wedge_m y) = f(x) \wedge_m f(y)$$

for each $x, y \in A$. Then f is said to be a *homomorphism of A into B*; if f is injective, then it is an *isomorphism of A into B*.

Assume that $A, B, C \in \mathcal{M}_\wedge$ and that

$$\varphi\colon A \to B \times C \tag{1}$$

is an isomorphism of A into $B \times C$ such that for each $b \in B$ and $c \in C$ there exist $c_1 \in C$ and $b_1 \in B$ with

$$(b, c_1), (b_1, c) \in \varphi(A).$$

Then φ is called a *subdirect product decomposition of A*.

If φ is a fixed subdirect product decomposition of A and $a \in A$, $\varphi(a) = (b,c)$, then we often write

$$a(B) = b, \qquad a(C) = c.$$

9.2. DEFINITION. The subdirect product decomposition (1) is called *regular* if, whenever a and a' are elements of A with $a < a'$, then there are $a_0, a_1, \dots, a_n \in A$ with

$$a = a_0 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_n = a'$$

such that for each $i \in \{0, 1, 2, \dots, n-1\}$ we have either $a_i(B) = a_{i+1}(B)$ or $a_i(C) = a_{i+1}(C)$.

If φ is a direct product decomposition, then it is regular. In fact, let a and a' be elements of A with $a < a'$. There exists $a_1 \in A$ with $\varphi(a_1) = \big(a(B), a'(C)\big)$; then the system $\{a, a_1, a'\}$ satisfies the required conditions.

9.3. EXAMPLE. This example shows that a subdirect decomposition of an element of $\mathcal{M}_\wedge$ need not be regular.

Let X be a Boolean algebra, $\operatorname{card} X = 8$, with the least element 0, the greatest element 1 and with three atoms x_1, x_2, x_3. Each subset of X is partially ordered by the partial order induced from X. Put

$$B = \{0, x_1, x_2, x_1 \vee x_2\}, \qquad C = \{0, x_3\}, \qquad A = \{0, x_1, x_2, 1\}.$$

Let the mapping $\varphi\colon A \to B \times C$ be defined by

$$\varphi(0) = (0,0), \quad \varphi(x_1) = (x_1, 0), \quad \varphi(x_2) = (x_2, 0), \quad \varphi(1) = (x_1 \vee x_2, x_3).$$

Then φ is a subdirect product decomposition of A which fails to be regular.

Let (1) be a subdirect decomposition of A and suppose that

$$\varphi^*\colon A \to B^* \times C^* \tag{1'}$$

is also a subdirect decomposition of A. We consider (1) and (1') as equal if $B = B^*$, $C = C^*$ and $\varphi = \varphi^*$. Hence for each $A \in \mathcal{M}_\wedge$, the collection $\operatorname{Sd}(A)$ of all subdirect product decompositions of A is a proper class.

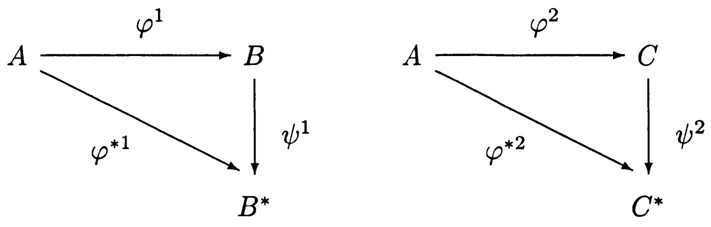
Again, let (1) and (1') be subdirect decompositions of A. Under the notation as above we put

$$\varphi^1(a) = a(B), \qquad \varphi^2(a) = a(C)$$

for each $a \in A$. Similarly, we set

$$\varphi^{*1}(a) = a(B^*), \qquad \varphi^{*2}(a) = a(C^*).$$

9.4. Definition. The subdirect decompositions (1) and (1') are called *equivalent* if there exist an isomorphism ψ^1 of B onto B^* and an isomorphism ψ^2 of C onto C^* such that both the diagrams



are commutative.

10. p-subdirect decompositions

Let X, Y be elements of $\mathcal{M}_\wedge$ and let ψ be a homomorphism of X into Y. By the kernel of ψ we mean the partition $P(\psi)$ on the set X which is defined by

$$x_1\, P(\psi)\, x_2 \iff \psi(x_1) = \psi(x_2)$$

(in fact, we do not distinguish between a partition of X and the corresponding equivalence relation on X).

The class of $P(\psi)$ containing an element $x \in X$ will be denoted by $x\big[P(\psi)\big]$. For x and x' from X we put

$$x\big[P(\psi)\big] \leqq x'\big[P(\psi)\big]$$

if there are $x_1 \in x\big[P(\psi)\big]$ and $x_1' \in x'\big[P(\psi)\big]$ with $x_1 \leqq x_1'$. Then the system

$$X/\psi = \big\{x[P(\psi)] : \ x \in X\big\}$$

turns out to be a poset. The mapping

$$\psi_1 : x/\psi \to Y$$

defined by

$$\psi_1\big(x[P(\psi)]\big) = \psi(x)$$

is an isomorphism of X/φ onto the poset $\psi(X)$. Hence X/ψ is an element of $\mathcal{M}_\wedge$.

Now consider the subdirect decomposition (1). As in the previous section, for each $a \in A$ we put

$$\varphi^1(a) = a(B)\,, \qquad \varphi^2(a) = a(C)\,.$$

Then φ^1 (or φ^2) is a homomorphism of A onto B (or onto C, respectively).

Hence we can define φ_1^1 and φ_1^2 analogously as for ψ_1 above.

Thus A/φ_1^1 is isomorphic to B; analogously, A/φ_1^2 is isomorphic to C.

It is obvious that the mapping

$$\overline{\varphi}\colon A \to (A/\varphi_1^1) \times (A/\varphi_1^2) \tag{1''}$$

defined by

$$\overline{\varphi}(a) = \left((\varphi_1^1)^{-1}(a(B)), (\varphi_1^2)^{-1}(a(C))\right)$$

is a subdirect product decomposition of A.

In another notation, for each $a \in A$ we have

$$\overline{\varphi}(a) = \left(a[P(\varphi^1)], a[P(\varphi^2)]\right).$$

The underlying sets of the posets A/φ_1^i $(i = 1, 2)$ are partitions of A; we say that (1'') is a *p-subdirect decomposition of* A. The collection of all p-subdirect decompositions of A will be denoted by $\mathrm{Sd}_p(A)$.

From the above definitions we obtain:

10.1. Lemma.

1) *The collection* $\mathrm{Sd}_p(A)$ *is a set.*
2) *To each* $\varphi \in \mathrm{Sd}(A)$ *there corresponds an element* $\overline{\varphi} \in \mathrm{Sd}_p(A)$ *such that* φ *and* $\overline{\varphi}$ *are equivalent.*
3) *If* $\chi \in \mathrm{Sd}_p(A)$, *then* $\overline{\chi} = \chi$.
4) *Let* $\varphi, \varphi^* \in \mathrm{Sd}(A)$; *then* φ *and* φ^* *are equivalent if and only if* $\overline{\varphi} = \overline{\varphi^*}$.
5) *Let* $\varphi \in \mathrm{Sd}(A)$; *then* φ *is regular if and only if* $\overline{\varphi}$ *is regular.*

The above results show that by investigating subdirect product decompositions of elements $\mathcal{M}_\wedge$ we can restrict our considerations, without loss of generality, to the case of p-subdirect decompositions.

Under the notation as above put

$$\overline{B} = A/\varphi_1^1, \qquad \overline{C} = A/\varphi_1^2,$$

and let $\overline{B^*}$, $\overline{C^*}$ be defined analogously.

10.2. Theorem. *Let* $A \in \mathcal{M}_\wedge$. *Assume that* (1) *and* (1*) *are regular subdirect decompositions of* A *such that* $\overline{B} = \overline{B^*}$. *Then* $\overline{C} = \overline{C^*}$ *and* $\overline{\varphi} = \overline{\varphi^*}$.

Proof. By applying the notation as above we put

$$\varrho_1 = P(\varphi^1), \qquad \varrho_2 = P(\varphi^2), \qquad \varrho_3 = P(\varphi^{*1}), \qquad \varrho_4 = P(\varphi^{*2}).$$

Let ϱ_0 be the minimal partition on A. Then we have

$$\varrho_1 \wedge \varrho_2 = \varrho_0 = \varrho_3 \wedge \varrho_4 . \tag{2}$$

The relation $\overline{B} = \overline{B^*}$ yields

$$\varrho_1 = \varrho_3 \,. \tag{3}$$

Suppose that $y, z \in A$, $y \,\varrho_2\, z$. There exists $u \in y \wedge_m z$. Since $y[\varrho_2] = z[\varrho_2]$, we obtain

$$z[\varrho_2] = y[\varrho_2] \,. \tag{4}$$

If $t \in A$, $u \leqq t \leqq y$, then $z[\varrho_2] \leqq t[\varrho_2] \leqq y[\varrho_2]$, whence

$$t[\varrho_2] = y[\varrho_2] \,. \tag{5}$$

Because φ^* is regular, there exist $x_0, x_1, \dots, x_n$ in A such that $u = x_0 \leqq x_1 \leqq x_2 \leqq \dots \leqq x_n = y$ and for each $i \in \{0, 1, \dots, n-1\}$ we have either $x_i \,\varrho_3\, x_{i+1}$ or $x_i \,\varrho_4\, x_{i+1}$.

Let $i \in I$ and suppose that $x_i \,\varrho_3\, x_{i+1}$. Hence $x_i \,\varrho_1\, x_{i+1}$. But from (5) we infer that

$$x_i \,\varrho_2\, y \,\varrho_2\, x_{i+1} \,.$$

Hence, in view of (2), we get $x_i = x_{i+1}$. Therefore $u \,\varrho_4\, y$. Similarly, $u \,\varrho_4\, z$. Hence $y \,\varrho_4\, z$ and so $\varrho_2 \leqq \varrho_4$. Analogously we obtain $\varrho_4 \leqq \varrho_2$. Thus $\varrho_2 = \varrho_4$, yielding that $\overline{C} = \overline{C^*}$ and $\overline{\varphi} = \overline{\varphi^*}$. □

In particular, 10.2 is valid in the case when A is a semilattice.

11. Examples

11.1. This example shows that for each semilattice A with $\operatorname{card} A > 1$ there exist subdirect decompositions (1) and (1') such that (under the notation as above)

(i) the subdirect decomposition (1) is regular and (1') fails to be regular;
(ii) $\overline{B} = \overline{B^*}$ but $\overline{C} \neq \overline{C^*}$.

We put $B = A$ and let C be a one-element set, e.g., $C = \{c_0\}$. For each $a \in A$ we set $\varphi(a) = (a, c_0)$. Then (1) is a regular subdirect decomposition of A. Moreover, $P(\varphi^1)$ is the minimal partition on A and $P(\varphi^2)$ is the largest partition on A.

Further, let us put $B^* = C^* = A$ and for each $a \in A$ put $\varphi^*(a) = (a, b)$. Then (1') is a subdirect decomposition of A which fails to be regular. Also, both $P(\varphi^{*1})$ and $P(\varphi^{*2})$ are equal to the minimal partition of A. Hence (ii) is valid.

The following three examples present regular subdirect decompositions of semilattices.

Whenever Y is a partially ordered set and $\emptyset \neq Z \subseteq Y$, then Z is partially ordered (by the induced partial order).

11.2. Let B be a semilattice having more than one element. Put $C = B$, $X = B \times C$. Put

$$A = \{(b,c) \in X : \ b \geqq c\}.$$

We consider the mapping

$$\varphi \colon A \to B \times C$$

such that φ is the identity on A.

11.3. Let A be a linearly ordered set and let $a_0 \in A$ such that a_0 is neither the least element nor the largest element of A. Put

$$B = \{a \in A : \ a \leqq a_0\}, \qquad C = \{a \in A : \ a \geqq a_0\}.$$

We consider the mapping $\varphi \colon A \to B \times C$ which is defined as

$$\varphi(a) = \begin{cases} (a, a_0) & \text{if } a \in B, \\ (a_0, a) & \text{if } a \in C. \end{cases}$$

11.4. Let A be the set of all integers with natural linear order. Put

$$B = \{2a : \ a \in A\}, \qquad C = \{b+1 : \ b \in B\}.$$

We consider the mapping $\varphi \colon A \to B \times C$ which is defined by

$$\varphi(a) = \begin{cases} (a, a+1) & \text{if } a \in B, \\ (a+1, a) & \text{if } a \in C. \end{cases}$$

REFERENCES

[1] BENADO, M.: *Les ensembles partiellement ordonnées et le théorème de Schreier II*, Czechoslovak Math. J. **5** (1955), 308–344.

[2] BIRKHOFF, G.: *Lattice Theory*, Colloquium Publications, Vol. XXV, Amer. Math. Soc., Providence, RI, 1967.

[3] GRÄTZER, G.: *Universal Algebra*, D. Van Nostrand Company, Inc., Princeton, 1968.

[4] GRÄTZER, G.: *General Lattice Theory*, Akademie Verlag, Berlin, 1978.

[5] HALAŠ, R.: *Decompositions of directed sets with zero*, Math. Slovaca **45** (1995), 9–17.

[6] HASHIMOTO, J.: *On direct product decompositions of partially ordered sets*, Ann. of Math. (2) **54** (1951), 315–318.

[7] JAKUBÍK, J.: *Weak product decompositions of discrete lattices*, Czechoslovak Math. J. **21** (1971), 399–412.

[8] JAKUBÍK, J.: *Weak product decompositions of partially ordered sets*, Colloq. Math. **25** (1972), 13–26.

[9] JAKUBÍK, J.—CSONTÓOVÁ, M.: *Convex isomorphisms of directed multilattices*, Math. Bohem. **118** (1993), 359–374.

[10] JAKUBÍK, J.—CSONTÓOVÁ, M.: *Cancellation rule for internal direct product decompositions of a connected partially ordered set*, Math. Bohem. **125** (2000), 115–122.

[11] KOLIBIAR, M.: *Graph isomorphisms of semilattices*. In: Contributions to General Algebra 3, Proc. Vienna Conference, 1984, Verlag Hölder-Pichler-Temspsky, Wien, 1985, pp. 225–235.

[12] PRINGEROVÁ, G.: *Direct product decompositions of a directed set*, Acta Fac. Paed. Univ. Šafárik **17** (1996), 1–8. (Slovak)

Received June 11, 2001

Matematický ústav SAV
Grešákova 6
SK–040 01 Košice
SLOVAKIA
E-mail: kstefan@saske.sk