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A THEOREM ABOUT CARATHÉODORY'S SUPERPOSITION

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ABSTRACT. Let $\mathbb{R}$ be the set of reals, and let Y be a separable Banach space. Suppose that D is a nonempty open subset of $\mathbb{R}\times Y$ and $f\colon D \to Y$ is a locally bounded function having the sections $f_x(u) = f(x,u)$ equicontinuous and the sections $f^y(v) = f(v,y)$ being derivatives. Then for every continuous function $g\colon I \to Y$ ($I \subset \mathbb{R}$ is an interval and $(x, g(x)) \in D$ for $x \in I$) Carathéodory's superposition $h(u) = f(u, g(u))$ is a derivative. Some applications of this theorem to the ordinary differential equations are shown.

I. The theorem about the Carathéodory superposition

Denote by $\mathbb{R}$ the set of reals. Let Y be a separable Banach space, and let $D \subset \mathbb{R} \times Y$ be a nonempty open set. A function $f\colon A \to Y$ ($A \subset \mathbb{R}$) is *measurable* (in the Lebesgue sense) if $f^{-1}(U)$ is measurable (L) for every open set $U \subset Y$. Observe that the separability of the space Y implies that a function $f\colon A \to Y$ is strongly measurable in the sense of [8]. A *locally Bochner integrable function* $f\colon I \to Y$ ($I \subset \mathbb{R}$ is an interval) is said to be a *derivative at a point* $x \in I$ if

$$\lim_{h\to 0} (1/h) \int_x^{x+h} f(u)\,\mathrm{d}u = f(x) \qquad ([1], [4], [6]).$$

In this article we assume that:

(H) $f\colon D \to Y$ is a locally bounded function such that all its sections $f^y(u) = f(u,y)$ ($u \in \mathbb{R}$, $y \in Y$) are derivatives and all its sections $f_u(y) = f(u,y)$ are equicontinuous at each point $y_0 \in Y$ (i.m. for every $r > 0$ there is $s > 0$ such that for every $y \in Y$ with $\|y - y_0\| < s$ we have for each $u \in \mathbb{R}$, $\|f(u,y) - f(u,y_0)\| < r$.

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REMARK 1. *If $f\colon I \to Y$ (I is a finite interval in $\mathbb{R}$) is measurable and bounded, then f is a Bochner integrable function.*

Proof. Indeed, it is an easy consequence of the well-known Bochner theorem ([8], p. 43, Th. 3.5.2.).

THEOREM 1. *Assume* (H). *Then for every continuous function $g\colon I \to Y$, where I is an interval and $(u, g(u)) \in D$ for $u \in I$, the Carathéodory superposition $h(u) = f(u, g(u))$ is a derivative.*

Proof. Fix a point $x_0 \in I$. Since f is locally bounded, there are r, $M > 0$ such that $\|h(u)\| \le M$ if $|u - x_0| \le r$. Remark that the function h is measurable [11]. We have for $u \in I$

$$\begin{aligned}
&\left\| \int_{x_0}^{u} \Big((h(t) - h(x_0))/(u - x_0) \Big)\, \mathrm{d}t \right\| \\
&= \left\| (1/(u - x_0)) \int_{x_0}^{u} \Big(f(t, g(t)) - f(t, g(x_0)) + f(t, g(x_0)) - f(x_0, g(x_0)) \Big)\, \mathrm{d}t \right\| \\
&\le \left\| (1/(u - x_0)) \int_{x_0}^{u} \Big(f(t, g(t)) - f(t, g(x_0)) \Big)\, \mathrm{d}t \right\| \\
&\qquad + \left\| (1/(u - x_0)) \int_{x_0}^{u} \Big(f(t, g(x_0)) - f(x_0, g(x_0)) \Big)\, \mathrm{d}t \right\| . \qquad (1)
\end{aligned}$$

Since the section $t \mapsto f(t, g(x_0))$ is a derivative at x_0, so

$$\begin{aligned}
&\lim_{u \to x_0} (1/(u - x_0)) \int_{x_0}^{u} \Big(f(t, g(x_0)) - f(x_0, g(x_0)) \Big)\, \mathrm{d}t \\
&\qquad = \lim_{u \to x_0} (1/(u - x_0)) \int_{x_0}^{u} f(t, g(x_0))\, \mathrm{d}t - f(x_0, g(x_0)) = 0 . \qquad (2)
\end{aligned}$$

Fix a positive number e. It follows from the equicontinuity of the sections f_u at $g(x_0)$ that there is $s > 0$ such that $\|f(u, y) - f(u, g(x_0))\| < e$ for every y with $\|y - g(x_0)\| < s$ ($(u, y) \in D$). There is also a number $z > 0$ such that

$\|g(u)-g(x_0)\| < s$ for every u with $|u-x_0| < z$ $(u \in I)$. If $|u-x_0| < z$, we have $\|f(t,g(t)) - f(t,g(x_0))\| < e$ for $t \in [x_0,u]$ and

$$\left\|(1/(u-x_0)) \int_{x_0}^{u} \Big(f(t,g(t)) - f(t,g(x_0))\Big)\,\mathrm{d}t\right\|$$

$$\leq (1/|u-x_0|)\left|\int_{x_0}^{u} \|f(t,g(t)) - f(t,g(x_0))\|\,\mathrm{d}t\right|$$

$$\leq (1/|u-x_0|)\left|\int_{x_0}^{u} e\,\mathrm{d}t\right| = (e|u-x_0|)|u-x_0| = e$$

or

$$\lim_{u-x_0}\left\|(1/(u-x_0)) \int_{x_0}^{u} \Big(f(t,g(t)) - f(t,g(x_0))\Big)\,\mathrm{d}t\right\| = 0\,. \tag{3}$$

It follows from (1), (2), (3) that $\lim_{u\to x_0}(1/(u-x_0)) \int_{x_0}^{u} h(t)\,\mathrm{d}t = h(x_0)$ and the proof is complete.

Example 1. There is a function $f\colon [0,1]^2 \to [0,1]$ such that all its sections f_x, and f^y are continuous, $f(0,0) = 0$, and $f(x,x) = 1$ for $x \in (0,1]$ ([5]). Remark that $h(u) = f(u,u)$ is not a derivative.

II. Applications to the differential equations

In this Section we show some application of Theorem 1 to the ordinary differential equations.

1°. Picard theorems.

It follows immediately from Theorem 1:

Remark 2. *Assume* (H). *If* $I \subset \mathbb{R}$ *is an interval and* $g\colon I \to Y$ *is a continuous function such that*

$$g(u) = y_0 + \int_{u_0}^{u} f(t,g(t))\,\mathrm{d}t \qquad \textit{for} \quad u \in I \quad (u_0 \in I),$$

then $g'(u) = f(u,g(u))$ *for* $u \in I$ *and* $g(u_0) = y_0$.

THEOREM 2. *Assume* (H). *If* f *satisfies the local Lipschitz condition with respect to* y *on* D, *then:*

(a) *every solution of the differential equation*

$$y'(u) = f(u, y(u)) \tag{4}$$

has an extension which is a global solution of (4);

(b) *every global solution of* (4) *is defined on an open interval;*

(c) *for every* $(u_0, y_0) \in D$ *there is exactly one global solution of* (4) *which satisfies the condition*

$$y(u_0) = y_0 \, . \tag{5}$$

P r o o f . Because the above Remark 2 holds it suffices to repeat the proof of the classical Picard theorem from [9] (pp. 194–196).

THEOREM 3. *Assume* (H), *where* $D = (a,b) \times Y$. *If for every closed interval* $I \subset (a,b)$ f *satisfies the Lipschitz condition with respect to* y *on* $I \times Y$, *then every solution of* (4) *has some extension on the interval* (a,b). *Moreover the global solution* y *of the equation* (4) *which satisfies* (5) *is the limit of uniformly convergent (on every closed interval* $I \subset (a,b)$*) sequence of the approximations*

$$y_0(u) = u_0 \, , \quad y_n(u) = u_0 + \int_{u_0}^{u} f(t, y_{n-1}(t)) \, \mathrm{d}t \, , \qquad n = 1, 2, \dots \, .$$

P r o o f . Remark 2 enables to repeat the proof of Theorem 2 from [9], (pp. 197–198).

The following Remark 3 shows the range of the generalization of the classical Picard theorems by Theorems 2 and 3.

R e m a r k 3. Assume (H) and denote by L_K the set of all functions f satisfying (H), which satisfy the Lipschitz condition with the constant K in y on D and by C_K – the set of all continuous functions being in L_K. For $g, h \in L_K$ let

$$p(g,h) = \min\Bigl(1, \sup_{(u,y) \in D} \|g(u,y) - h(u,y)\|\Bigr) \, .$$

Remark that L_K is a complete metric space with the distance p and C_K is a closed subset of L_K. We prove that it is nowhere dense in L_K. Let $g\colon \mathbb{R} \to [0,1]$

be an approximately continuous function such that $g^{-1}(0)$ is a dense subset of $\mathbb{R}$ of measure zero ([1] and [12]). Evidently g is not continuous at every point $u \notin g^{-1}(0)$. Moreover g is a derivative ([12], [1], [6], [4]). Fix $f \in C_K$ and $r > 0$. Put

$$h(u,y) = f(u,y) + rg(u)y_0 ,$$

where $(u,y) \in D$ and $y_0 \in Y$ is such that $\|y_0\| = 1$. Then $h \in L_k - C_k$ and $p(f,h) \leq r$. So C_K is nowhere dense in L_K.

2°. Extension theorem.

THEOREM 4. *Assume* (H), *where* $Y = \mathbb{R}^m$. *Every solution* $g\colon I \to \mathbb{R}^m$ *of the equation* (4) *can be extended (as a solution) over a maximal interval of existence* (c,d) *to a global solution* $h\colon (c,d) \to \mathbb{R}^m$. *Moreover if* $\lim_{t\to x}\big(t,h(t)\big) = (x,y_0)$, *then* $(x,y_0) \in \operatorname{Fr} D$, *where* $\operatorname{Fr} D$ *denotes the boundary of* D *and* $x = c$ *or* d.

Proof. Remark 2 enables to repeat the proof of Theorem 3.1 from [7] (p. 13).

3°. Carathéodory equations.

REMARK 4. *Assume* (H), *where* $Y = \mathbb{R}^m$. *If an absolutely continuous function* $g\colon I \to \mathbb{R}^m$ *satisfies the differential equation* (4) *almost everywhere on the interval* I, *then* g *satisfies* (4) *everywhere on* I. *So every Carathéodory solution of* (4) *is a solution (in the ordinary sense) of this equation.*

Proof. This remark is an immediate consequence of Remark 2.

As an easy consequence of the above remark and Theorem 1 from [3, p. 7], or Theorem 2 from [3, p. 8] we obtain the following two results:

THEOREM 5. *Assume* (H), *where* $Y = \mathbb{R}^m$ *and* $D = [t_0, t_0 + a] \times \{y \in \mathbb{R}^m\colon |y - y_0| < b\}$ $(a, b > 0)$. *Suppose that there is an integrable function* $k\colon [t_0, t_0 + a] \to \mathbb{R}$ *such that* $|f(t,y)| \leq k(t)$ *for every* $(t,y) \in D$. *Let*

$$g(u) = \int_{t_0}^{u} k(s)\,\mathrm{d}s \qquad \text{for} \quad t_0 \leq u \leq t_0 + a .$$

Then for every d *such that* $0 < d \leq a$ *and* $g(t_0 + d) \leq b$ *there is a solution* y *of* (4) *satisfying* (5) *and defined on the interval* $[t_0, t_0 + d]$.

THEOREM 6. *Assume* (H), *where* $Y = \mathbb{R}^m$. *Let* $(u_0, y_0) \in D$. *If there is an integrable function* $k\colon \operatorname{Pr} D \to \mathbb{R}$ ($\operatorname{Pr} D$ *denotes the projection of* D *on* $\mathbb{R}$) *such that* $\|f(u,y) - f(u,x)\| \le k(u)|x-y|$ *for* (u,x), $(u,y) \in D$, *then the equation* (4) *has at most one solution* y *in* D *such that* (5).

From Remark 4 and the Theorem proved by De Blasi and Myjak in [2] we get:

THEOREM 7. *Assume* (H), *where* $Y = \mathbb{R}^m$ *and* $D = [0,1] \times U$ *and* U *is the open ball in* $\mathbb{R}^m$ *with center* y_0 *and radius* $r_0 > 0$. *Let* $J = [0,T]$, *where* $0 < T \le 1$ *be such that* $\int_0^T \big(k(t)+1\big)\,\mathrm{d}t < r_0$ *and* $k\colon [0,1] \to \mathbb{R}$ *be an integrable function such that* $\|f(t,y)\| \le k(t)$ *for each* $(t,y) \in D$. *Then the set of all solutions* y *of* (4) *satisfying the condition* $y(0) = y_0$ *and defined on* J *is an* R_δ*-set in the space* $C(J, \mathbb{R}^m)$ *of all continuous functions from* J *to* $\mathbb{R}^m$ *with the norm of uniform convergence.*

Recall that a subset of metric space is called an R_δ-set if it is the intersection of a decreasing sequence of (nonempty) compact absolute retracts.

Now, for a bounded $X \subset Y$ denote by $\alpha(X)$ the greatest lower bound of such numbers $r > 0$ that X can be covered by a finite number of sets with the diameter not larger than r. We shall call a *Kamke function* every function $w\colon [0,a] \times \mathbb{R}^+ \to \mathbb{R}^+$ such that all sections w_t are continuous, all sections w^y are measurable, $w(t,0) = 0$ for $t \in [0,a]$ and $y(t) = 0$ is the only continuous solution of the inequality

$$y(t) \le \int_0^t w\big(s, y(s)\big)\,\mathrm{d}s$$

satisfying the condition $y(0) = 0$.

From Remark 4 and Pianigiani's Theorem [10] there follows immediately the following theorem:

THEOREM 8. *Let* D *be the rectangle* $0 \le t \le a$, $\|y - y_0\| < b$. *Assume* (H) *and suppose that* $\|f\| \le M > 0$ *and for each bounded set* $A \subset Y$ *for almost every* $t \in I$, *there holds* $\lim\limits_{\delta \to 0} \alpha\big(f(I_{t,\delta}, A)\big) \le w\big(t, \alpha(A)\big)$, *where* $I = [0,\beta]$, $\beta = \min(a, b/M)$, $I_{t,\delta} = (t-\delta, t+\delta)$. *Then there exists at least one solution of the Cauchy problem* $y(t) = f\big(t, y(t)\big)$, $y(0) = y_0$ *defined on* $[0,\beta]$.

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