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TOPOLOGY OF THE SPACE OF ALL 2-DIMENSIONAL LIE SUBALGEBRAS OF THE LIE ALGEBRA gl(2; R)

IVAN KULICH

The aim of the present paper is to give a global topological characterization of the space $\mathcal{A}$ of all 2-dimensional Lie subalgebras of the Lie algebra $\mathbf{gl}(2;\mathbf{R})$ regarded as a topological subspace of the Grassmannian $G_2(\mathbf{gl}(2;\mathbf{R}))$. We shall use the standard symbols $T^2 = S^1 \times S^1$ and RP^2 for a 2-dimensional torus and a real projective plane, respectively.

The following theorem will be proved:

Theorem. *Let f: $S^1 \to T^2$ and g: $S^1 \to RP^2$ be injective and continuous maps such that $f(S^1) = \{a\} \times S^1$ for some point $a \in S^1$ and $g(S^1)$ is contractible in RP^2. Further let ω be an equivalence relation on the disjoint union $T^2 \cup RP^2$ defined by $x\omega y \Leftrightarrow x = y \vee (\exists z \in S^1\colon x = f(z) \wedge y = g(z))$. Then $\mathcal{A}$ is homeomorphic to the factor space $T^2 \cup RP^2/\omega$.*

Proof. As usually, $\mathbf{gl}(2;\mathbf{R})$ is a Lie algebra of all real matrices of type 2×2 with the standard bracket operation $[U, V] = U \cdot V - V \cdot U$. The matrix

$$U = \begin{pmatrix} u^1 & u^2 \\ u^3 & u^4 \end{pmatrix}$$

will be written also as a row-vector (u^1, u^2, u^3, u^4), a subspace $\langle U, V \rangle$ spanned by $U = (u^1, u^2, u^3, u^4)$ and $V = (v^1, v^2, v^3, v^4)$ will be denoted by

$$\begin{pmatrix} u^1 & u^2 & u^3 & u^4 \\ v^1 & v^2 & v^3 & v^4 \end{pmatrix}.$$

For linearly independent vectors $U, V \in \mathbf{gl}(2;\mathbf{R})$ let us consider the subspace $A = \langle U, V \rangle$. There holds

$$[U, V] = (D_{23}, D_{12} + D_{24}, -D_{13} - D_{34}, -D_{23})$$

where

$$D_{ij} = \det \begin{pmatrix} u^i & u^j \\ v^i & v^j \end{pmatrix}.$$

Therefore $A \in \mathscr{A}$ if and only if there exists a solution (a, b) of

$$\begin{aligned} au^1 + bv^1 &= D_{23} \\ au^2 + bv^2 &= D_{12} + D_{24} \\ au^3 + bv^3 &= -D_{13} - D_{34} \\ au^4 + bv^4 &= -D_{23}. \end{aligned} \tag{1}$$

There is a decomposition $\mathscr{A} = \mathscr{A}_T \cup \mathscr{A}_N$, where $\mathscr{A}_T(\mathscr{A}_N)$ denotes the space of algebras of $\mathscr{A}$ with a trivial (a nontrivial) bracket. An easy calculation gives

$$\mathscr{A}_T = \left\{ \begin{pmatrix} \alpha & \beta & \gamma & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}; \quad (\alpha, \beta, \gamma) \in RP^2 \right\},$$

hence $\mathscr{A}_T \approx RP^2$.

Let now $\langle U, V \rangle \in \mathscr{A}_N$, i. e. (1) has a nontrivial solution. Then, regarding the nullity of the numbers D_{ij}, the investigation of (1) falls into the following five cases:

1. $D_{23} \neq 0$
2. $D_{23} = D_{14} = 0, \; D_{12} \neq 0$
3. $D_{23} = D_{14} = D_{12} = 0, \; D_{34} \neq 0$
4. $D_{23} = D_{14} = D_{12} = D_{34} = 0 \quad D_{13} \neq 0$
5. $D_{23} = D_{14} = D_{12} = D_{34} = D_{13} = 0, \; D_{24} \neq 0.$

Corresponding to these cases we obtain five systems of subalgebras

$$\mathscr{S}_1 = \left\{ \begin{pmatrix} u^1 & 1 & 0 & u^4 \\ u^4 & 0 & (u^4 - u^1)^2 & u^1 \end{pmatrix}; \quad u^4 \neq u^1 \right\}$$

$$\mathscr{S}_2 = \left\{ \begin{pmatrix} 1 & 0 & 0 & u^4 \\ 0 & 1 & 0 & 0 \end{pmatrix}; \quad u^4 \neq 1 \right\}$$

$$\mathscr{S}_3 = \left\{ \begin{pmatrix} 0 & 0 & 1 & 0 \\ v^1 & 0 & 0 & 1 \end{pmatrix}; \quad v^1 \neq 1 \right\}$$

$$\mathscr{S}_4 = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\}$$

$$\mathscr{S}_5 = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

Here each of the sets $\mathscr{S}_1$, $\mathscr{S}_2$ and $\mathscr{S}_3$ is described in some parametrization-form and the system $\{\mathscr{S}_1, \ldots, S_5\}$ is a disjoint decomposition of $\mathscr{A}_N$. The sets $\mathscr{S}_1$, $\mathscr{S}_2$ and $\mathscr{S}_3$ will be re-parametrized by the substitutions

1. $u^1 = \alpha\beta, \; u^4 = \beta(\alpha - 1), \; \beta \neq 0$
2. $u^4 = \dfrac{\alpha}{\alpha - 1}, \; \alpha \neq 1$

3. $v^1 = \frac{\alpha}{\alpha - 1}, \quad \alpha \neq 1$

respectively into a new form

$$\mathscr{S}_1 = \{M_1(\alpha, \beta);\ \beta \neq 0\}$$
$$\mathscr{S}_i = \{M_i(\alpha);\ \alpha \neq 1\} \quad \text{for} \quad i = 2, 3,$$

where

$$M_1(\alpha, \beta) = \begin{pmatrix} \alpha\beta & 1 & 0 & (\alpha-1)\beta \\ \alpha - 1 & 0 & \beta & \alpha \end{pmatrix}$$

$$M_2(\alpha) = \begin{pmatrix} \alpha - 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and}$$

$$M_3(\alpha) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ \alpha & 0 & 0 & \alpha - 1 \end{pmatrix}.$$

In this notation $\mathscr{S}_2$, $\mathscr{S}_5$ and $\mathscr{S}_4$ can be regarded as subsets of "extended $\mathscr{S}_1$" ($= \mathscr{A}^2_N$) or "extended $\mathscr{S}_3$" ($= \mathscr{A}^1_N$) as follows

$$\mathscr{S}_2 = \{M_1(\alpha, \beta);\ \beta = 0, \alpha \neq 1\}$$
$$\mathscr{S}_5 = \{M_1(\alpha, \beta);\ \beta = 0, \alpha = 1\}$$
$$\mathscr{S}_4 = \{M_3(\alpha);\ \alpha = 1\}$$

where

$$\mathscr{A}^2_N = \{M_1(\alpha, \beta);\ \alpha, \beta \in \mathbf{R}\}$$
$$\mathscr{A}^1_N = \{M_3(\alpha);\ \alpha \in \mathbf{R}\}.$$

Thus, we have obtained the most convenient parametrization of $\mathscr{A}_N$ for our requirements.

The following analysis of $\mathscr{A}$ makes use of the Plücker coordinates (see, e. g., [1]) π: $G_2(\mathbf{gl}(2; \mathbf{R})) \to Q_{3,3} \subset RP^5$,

$$\pi(\langle U, V \rangle) = (D_{12}, D_{13}, D_{14}, D_{34}, D_{42}, D_{23}).$$

There holds that

$$\pi(\mathscr{A}^2_N) = \{(1-\alpha, \alpha\beta^2, \beta(2\alpha - 1), -\beta^2(\alpha - 1), -\alpha, \beta);\ \alpha, \beta \in \mathbf{R}\}$$
$$\pi(\mathscr{A}^1_N) = \{(0, \alpha, 0, 1-\alpha, 0, 0);\ \alpha \in \mathbf{R}\}.$$

Now the CW-complex of the closure of $\pi(\mathscr{A}_N)$ in $Q_{3,3}$ will be described by the use of the functions $\alpha = \operatorname{tg} \varphi$, $\beta = \operatorname{tg} \psi$. Let us denote $I = \left\langle -\frac{\pi}{2}, \frac{\pi}{2} \right\rangle$. In the CW-complex there is one 2-cell e_1, two 1-cells e_2, e_3 and one 0-cell e_4:

$e_1 = \pi(\mathscr{A}_N^2)$
$e_2 = \pi(\mathscr{A}_N^1)$
$e_3 = \{(-1, \beta^2, 2\beta, -\beta^2, -1, 0);\ \beta \in \mathbf{R}\}$
$e_4 = \{(0, -1, 0, 1, 0, 0)\}.$

The characteristic maps are

$$\Phi_1 : I \times I \to \overline{\pi(\mathscr{A}_N)}$$

$$\Phi_2,\ \Phi_3 : I \to \overline{\pi(\mathscr{A}_N)}$$

$$\begin{aligned}
\Phi_1(\varphi, \psi) &= (\cos^2\psi(\cos\varphi - \sin\varphi), \sin\varphi \sin^2\psi, \sin\psi\cos\psi(2\sin\varphi - \cos\varphi),\\
&\quad -\sin^2\psi(\sin\varphi - \cos\varphi), -\sin\varphi\cos^2\psi, \cos\varphi\sin\psi\cos\psi)\\
\Phi_2(\varphi) &= (0, \sin\varphi, 0, \cos\varphi - \sin\varphi, 0, 0)\\
\Phi_3(\psi) &= (-\cos^2\psi, \sin^2\psi, \sin 2\psi, -\sin^2\psi, -\cos^2\psi, 0).
\end{aligned}$$

From the above CW-complex there immediately follows that $\overline{\pi(\mathscr{A}_N)}$ is homeomorphic to the square $I \times I$ on the boundary of which there is a gluing map described by

$$\Phi_1\left(\varphi, \pm\frac{\pi}{2}\right) = \Phi_2(\varphi)$$

$$\Phi_1\left(\pm\frac{\pi}{2}, \psi\right) = \Phi_3(\psi)$$

$$\Phi_2\left(\pm\frac{\pi}{2}\right) = \Phi_3 \pm\frac{\pi}{2}\Big) = e_4.$$

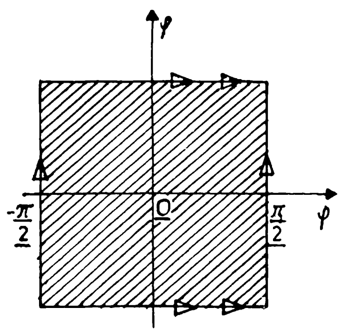

Fig. 1

Thus $\overline{\mathscr{A}_N} \approx \overline{\pi(\mathscr{A}_N)} \approx T^2$. Finally, $e_3 \cup e_4$ is a π-image of

$$\left\{\begin{pmatrix} 2\beta & 1 & -\beta & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix};\ \beta \in \mathbf{R}\right\} \cup \left\{\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}\right\},$$

which is a circle, contractible in $\mathscr{A}_T$.

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ТОПОЛОГИЯ ПРОСТРАНСТВА ВСЕХ ДВУХМЕРНЫХ ПОДАЛГЕБР ЛИ АЛГЕБРЫ ЛИ $\mathbf{gl}(2; \mathbf{R})$

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Резюме

В статье дана глобальная топологическая характеризация пространства $\mathscr{A}$ всех двухмерных подалгебр Ли алгебры Ли $\mathbf{gl}(2; \mathbf{R})$, рассматриваемого как подпространство многообразия Грассманна $G_2(\mathbf{gl}(2; \mathbf{R}))$. Обозначим через $T^2 = S^1 \times S^1$ и RP^2, соответственно, двухмерный тор и вещественную проективную плоскость. Доказывается следующщаъ теорема: Пусть f: $S^1 \to T^2$ и g: $S^1 \to RP^2$ — инъективные и непрерывные отображения, такие, что $f(S^1) = \{a\} \times S^1$ для какой-нибудь точки $a \in S^1$ и $g(S^1)$ стягиваемо в RP^2. Пусть дальше ω — отношение эквивалентности на дизъюнктном объединении $T^2 \cup RP^2$, определенное как

$$x\omega y \Leftrightarrow x = y \vee (\exists z \in S^1 : x = f(z) \wedge y = g(z)).$$

Тогда $\mathscr{A}$ гомеоморфно фактор-пространству $(T^2 \cup RP^2)/\omega$.