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Dedicated to Academician Štefan Schwarz on the occasion of his 80th birthday

SEMIGROUPS AND HILBERT'S FIFTH PROBLEM

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(Communicated by Tibor Katriňák)

ABSTRACT. We revisit the full content of Hilbert's Fifth problem which asks whether topological groups on manifolds are automatically analytical. It is not too well know that this same problem has history in the case of topological semigroup, too, and this history can be traced back to A b e l. We explain what is known in this regard and lead up to contemporary problems in the Lie theory of semigroups.

In the year 1900, at the International Congress of Mathematicians in Paris, D a v i d H i l b e r t formulated 23 Problems which, on the basis provided by the achievements of 19th century mathematics, would decisively influence the course of the history of mathematics in the 20th century. Among these, one of the most familiar is the fifth; it deals with the *transformation groups* introduced during the preceding decades by S o p h u s L i e who died in 1899. It is commonly known that essential aspects of H i l b e r t's Fifth Problem were solved by papers published by G l e a s o n, and M o n t g o m e r y († March 15, 1992) and Z i p p i n in the year 1952. What is less commonly known is the fact that H i l b e r t's Fifth Problem has other aspects, among which there is a semigroup theoretical one. It is this aspect on which I shall focus in this essay [1)] which I am honored to dedicate to Š t e f a n S c h w a r z, who among a generation of pioneers of semigroup theory such as A. H. C l i f f o r d, P. D u b r e i l, E. H i l l e, V. V. V a g n e r, A. D. W a l l a c e promoted the algebraic and the topological theory of semigroups so significantly.

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[1)] I presented some of these observations in a lecture at the University of Erlangen on July 6, 1993

The second article in the very first volume of Crelle's Journal in the year 1826 is entitled „Untersuchung der Funktionen zweier unabhängig veränderlicher Größen x und y, wie $f(x,y)$, welche die Eigenschaft haben, daß $f(z, f(x,y))$ eine symmetrische Funktion von z, x und y ist".

Its author is N. H. Abel, who communicates the following result: *Hat eine Funktion die im Titel genannte Eigenschaft, so gibt es eine Funktion ψ derart, daß $\psi f(x,y) = \psi(x) + \psi(y)$ gilt.* While it is not specifically stated that ψ is invertible, the discourse in the paper makes it clear that this is meant. We note right away that any of the functions f described in Abel's proposition defines on $\mathbb{R}$ the structure of a commutative semigroup.

Let us have a hard look at this statement and take $S =]4, \infty[$. This set is homeomorphic to $\mathbb{R}$, and therefore any example we inspect on this space is, by transport of structure, an example on $\mathbb{R}$. We write $a \wedge b = \min\{a, b\}$ and define $f\colon S \times S \to S$ by $f(x,y) = \big((x \wedge 6) + y\big) \wedge 12$. Then for any choice of elements $x, y, z \in S$ we observe $z \wedge 6 > 4$ and $f(x,y) > 8$, and thus $f\big(z, f(x,y)\big) = 12$. Thus f is certainly an example of one of Abel's functions. But we also note that $f(5,7) = (5+7) \wedge 12 = 12$, yet $f(7,5) = (6+5) \wedge 12 = 11 \neq f(5,7)$. Thus f is not commutative, and thus cannot satisfy the conclusion of Abel's proposition. Continuity can't be the problem since our function f is certainly continuous.

A closer inspection of Abel's paper shows that early on he claims that $f\big(z, f(x,y)\big) = f\big(z, f(y,x)\big)$ can hold only if $f(x,y) = f(y,x)$ holds. This is not a legitimate conclusion as our example shows. But let us simply go ahead and impose, in addition to Abel's explicit hypotheses, the assumption of commutativity of f. Then it is certainly correct to say that the functions $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ considered by Abel are, in modern parlance, precisely the commutative semigroup multiplications on $\mathbb{R}$. Then, assuming the invertibility of ψ and the continuity of f (which is also implicit in Abel's argument) we may restate Abel's proposition in the following form:

ABEL'S THEOREM. (Preliminary version) *An abelian topological semigroup on $\mathbb{R}$ is isomorphic to the group $(\mathbb{R}, +)$ or one of its open connected subsemigroups.*

Indeed, the intervals homeomorphic to $\mathbb{R}$ are exactly the nonempty open intervals of $\mathbb{R}$. Hence apart from $\mathbb{R}$ we have to allow the intervals $(]a, \infty[, +)$, $0 \leq a$, and all those arising from these by reflection $x \mapsto -x$ about the origin.

But wait: The operations $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, given by $f(x,y) = x \wedge y$, or by $f(x,y) = x \vee y = \max\{x,y\}$, or by $f(\mathbb{R} \times \mathbb{R}) = \{r\}$, $r \in \mathbb{R}$, are all commutative topological semigroup multiplications on $\mathbb{R}$, and none of these is isomorphic to one of Abel's multiplications. Perhaps f was supposed to be differentiable?

Indeed, in his argument A b e l proceeds without further ado to differentiate as soon as the need arises. But then we consider $f\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $f(x,y)=xy$, a semigroup multiplication which is even (real) analytic. It again fails to be one of those listed by A b e l since it has two idempotents, and A b e l's multiplications have at most one. This example is even mentioned by A b e l himself: he says that the functional equation $\psi(xy)=\psi(x)+\psi(y)$ is solved by $\psi(z)=a\log z$.

At the latest, at this point one realizes one of the basic difficulties hampering all early works in this area: the lack of precision of the very concept of *function*. No domain and codomain is ever specified. Perhaps in elementary analysis this was not so essential. But the issue of domain and codomain begins to be of paramount significance if "arbitrary" or even "wanted" functions are substituted into themselves. This deficiency in a precise definition of a *function* remained a handicap, 60 years later, in S o p h u s L i e's definition and discussion of his "transformation groups"

$$x'_j = f_j(x_1,\dots,x_n;a_1,\dots,a_m)$$

and the postulated substitutions

$$f_j\big(f_1(x_1,\dots,x_n;a_1,\dots,a_m),\dots,f_n(x_1,\dots,x_n;a_1,\dots,a_m);b_1,\dots,b_m\big) = f_j(x_1,\dots,x_n;c_1,\dots,c_m)\,.$$

It was B. R i e m a n n who recognized with full clarity the significance of a domain and codomain of a function, as is exemplified by the *Riemann Mapping Theorem* or the very invention of the *Riemann surfaces*. (I was alerted to this observation by my colleague D e t l e f L a u g w i t z.) But Riemann's insights apparently remained without influence on L i e.

A semigroup S is *cancellable* or *cancellative* if

$$(\forall x,a,b\in S)\qquad (xa=xb \implies a=b)\quad\text{and}\quad (ax=bx\implies a=b)\,.$$

Today we know by and large all topological semigroup multiplications on $\mathbb{R}$. They exist in great abundance (see e.g. [10; p. 206 ff.], and [26]). Most of them are not cancellable.

A b e l, too, in the very first step of his argument had assumed that $f(x,a)=f(x,b)$ implied $a=b$. Thus we should restate the proposition formulated in 1826 by A b e l in Crelle **1** sharply as follows:

ABEL'S THEOREM. *An abelian cancellable topological semigroup on $\mathbb{R}$ is isomorphic to the group $(\mathbb{R},+)$ or one of its open connected subsemigroups.*

This causes us to formulate the issue opened by A b e l as follows:

Abel's Problem. *Determine all cancellable topological semigroup structures on a connected topological manifold.*

Although not noticed always and everywhere, A b e l's considerations have an influence to this very day. Exactly on the turn of the century the following question is being raised:

> **Hilbert 5, Part 2.** Überhaupt werden wir auf das weite und nicht uninteressante Feld der Funktionalgleichungen geführt, die bisher meist nur unter Voraussetzung der Differenzierbarkeit der auftretenden Funktionen untersucht worden sind. Insbesondere die von A b e l mit so vielem Scharfsinn behandelten Funktionalgleichungen ... weisen an sich nicht auf, was zur Forderung der Differenzierbarkeit der auftretenden Funktionen zwingt ... In allen Fällen erhebt sich daher die Frage, *inwieweit etwa die Aussagen, die wir im Falle der Annahme differenzierbarer Funktionen machen können, unter geeigneten Modifikationen ohne diese Voraussetzung gültig sind.*

It was D a v i d H i l b e r t, who expressed these sentences on the occasion of the International Congress of Mathematicians in the year 1900. Here he placed before the mathematical public his famous 23 problems. On the one hand, this lecture was a stock-taking of the situation of mathematics at the turn of the century, i.e., of the achievements of the 19th century in this field. On the other it proclaimed a program for the mathematics of the 20th century in many facets. It is impossible to overestimate the influence which this address exerted on the development of mathematics in this century. Our quote of the problem concerning A b e l's functional equations is a portion of the famous Fifth Problem. However, this portion is much less known than the first part.

> **Hilbert 5, Part 1.** L i e hat bekanntlich mit Hinzuziehung des Begriffs der kontinuierlichen Transformationsgruppe ein System von Axiomen für die Geometrie aufgestellt und auf Grund seiner Theorie der Transformationsgruppen bewiesen, daß dieses System von Axiomen zum Aufbau der Geometrie hinreicht. Da L i e jedoch bei Begründung seiner Theorie stets annimmt, daß die die Gruppe definierenden Funktionen *differenziert* werden können, so bleibt in den L i eschen Entwicklungen unerörtert, ob die Annahme der Differenzierbarkeit bei der Frage nach den Axiomen der Geometrie tatsächlich unvermeidlich ist oder nicht vielmehr als eine Folge des Gruppenbegriffes und der übrigen geometrischen Axiome erscheint. Diese Überlegungen ... legen uns die allgemeine Frage nahe *inwieweit der L i esche Begriff der kontinuierlichen Transformationsgruppe auch ohne Annahme der Differenzierbarkeit der Funktionen unserer Untersuchung zugänglich ist.*

In modern parlance, a [real] L i e group is a group on a [real] analytic manifold, whose operations are [real] analytic. At the time of Hilbert's proclamation, the classification of simple Lie groups was well under way. Lie himself, but also Engel and Killing had realized, that such a classification was primarily a problem of linear algebra. A global determination of these groups was achieved by Poincaré, Elie Cartan, and Hermann Weyl.

The first part of Hilbert's Fifth Problem is a problem on transformation groups. When specialized to the action of a group on itself by translation, it may be formulated in the form of a question as follows: *Is every topological group on a topological manifold a Lie group?* In view of an eventual classification of connected Lie groups it seems permissible to formulate Hilbert's bold grasp of the problem as follows:

HILBERT'S FIFTH PROBLEM FOR GROUPS, GENERAL VERSION. *Determine all topological group structures on a connected topological manifold.*

When formulated in this way, our version of Abel's Problem is even more comprehensive. Hilbert's comprehensive formulation yields a key to both problems:

HILBERT'S FIFTH PROBLEM. *Investigate the circumstances under which the solutions of functional equations on topological manifolds are automatically differentiable or even analytic.*

I think that at the time Hilbert formulated his problems the difference between a topological and a differentiable manifold was adequately understood. Weierstrass' functions were well known. It was known since 1906, that Koch's snow flake curve ([14]) was topologically equivalent to the circle, and geometric intuition alone sufficed to convince anyone that it could not inherit a differentiable structure from the plane in which it was embedded. It was, however, premature for a precise understanding of topological groups on connected topological spaces which were not euclidean manifolds. There are indications in the formulation of Hilbert's Fifth Problem. Subsequent to his description of Lie's formalism he states

> **Hilbert 5: "Infinite" groups.** Auch für unendliche Gruppen ist, wie ich glaube, die Untersuchung der entsprechenden Frage von Interesse.

It remains in the dark what is meant by "infinite". It is clear that not the cardinality of the underlying set is meant, but the "number of parameters" which permit a description of the group. In the rendering of Lie's formalism above, the "parameters" are $a_1, \ldots, a_n$. Today we would speak of an infinite dimensional group. However, to this very day this opens a vast field which, among other things would have to include the additive groups of all topological vector spaces. Nevertheless we might focus, from modern perspective, at least on *locally compact spaces* without any dimensional restriction, and formulate a problem for which the time had not arrived in 1900, but which, in a certain sense, is implicit in H i l b e r t 's formulation:

HILBERT 5: "INFINITE" GROUPS, MODERN INTERPRETATION. *Determine all topological group structures on a connected locally compact space.*

This problem is more comprehensive than the first part of H i l b e r t 's Fifth Problem. In this formulation it is no longer evident at all what the problem should have to do with analyticity, and it is indeed amazing that it should turn out that such links exist.

Several different strands of mathematical culture are tied to these early works. It is certainly natural that the first part of H i l b e r t 's Fifth Problem should have evoked the development of a theory of topological groups, their representation theory – at any rate it provided continuous momentum to such developments. A b e l 's work on functional equations is claimed as the origin of the general theory of functional equations one of whose foremost promoters is J á n o s A c z é l. There is a very worth while survey article on the state of the second part of H i l b e r t 's Fifth Problem [2]. In respect of what we formulated above as A b e l 's Problem, however, it does not represent the latest state of information.

In the preceding sections we have attempted a certain overview of the problem situation at the end of the 19th century which was presented by H i l b e r t in his Fifth Problem in a visionary fashion. We now turn to a discussion of the present state of knowledge. Obviously it cannot be exhaustive.

In the first half of the century the theory of *compact* topological groups was completed through pioneering contributions by H e r m a n n W e y l, J o h n v o n N e u m a n n and L. S. P o n t r y a g i n. The essential tool is the existence of an invariant probability measure for the Borel sets on a compact group, and the essential structural insights arise from the representation theory (or harmonic analysis, as one likes to say in the context of topological groups). The information that there are sufficiently many irreducible unitary representations, since these have to be finite dimensional, suffices for the following important conclusion for a compact group G:

THEOREM 1. *There are arbitrarily small closed normal subgroups N of G such that G/N is a L i e group.*

In other words: Every compact group G can be "approximated by L i e groups". (Given certain background theories one can say that G is a strict projective limit of compact L i e groups. But for our purposes the present formulations suffice.) One might be induced to conclude that this information settles H i l b e r t 's Fifth Problem for compact groups in the affirmative right away. A closer inspection, however, shows that this is not yet the case. However, a closer penetration into the structure theory of compact Lie groups produces the following result which renders the information in Theorem 1 considerably more precise:

THEOREM 2. *For every neighborhood V of the identity in G there is a compact normal subgroup N and a local Lie group*[2] *U in G which commutes elementwise with N and is such that the map $(n,u) \mapsto nu\colon N \times U \to NU \subseteq V$ is a homeomorphism onto a neighborhood of the identity which is contained in V.*

This theorem allows us a reformulation, whose approach belongs to Lie group theory:

THEOREM 3. *For any neighborhood V of the identity there is a compact normal subgroup $N \subseteq V$, a connected Lie group L, and an injective continuous homomorphism $f\colon L \to G$ such that the function $(n,x) \mapsto nf(x)\colon N \times L \to G$ is a continuous group homomorphism with discrete kernel.*

In particular, G and $N \times L$ are locally isomorphic topological groups.

With this Theorem 3 we can give fairly direct answer to Hilbert's Problem: A topological group which is locally isomorphic to a Lie group is a Lie group. Thus, by the Theorem 3, we may assume that $G = N \times L$ with a Lie group L in such a fashion that there is a euclidean ball neighborhood E of the identity containing N (identified with $N \times \{1\}$). The inclusion $i\colon N \to E \to N \times L$ is null-homotopic since E is contractible. Let $p\colon N \times L \to N$ be the projection. Then $pi\colon N \to N$ is the identity, and is null-homotopic as i is null-homotopic. Thus N is contractible. However, the only contractible compact group is the singleton one. Thus $G = L$. We have obtained a positive answer for Hilbert's Fifth Problem for compact groups.

A second class of groups for which this problem was solved by the fourth decade in this century is that of abelian groups. The duality theory of Pontryagin and van Kampen provides the following structure theorem:

THEOREM 4. *A locally compact connected abelian group is isomorphic to $K \times \mathbb{R}^n$ with a compact group K.*

Thus Hilbert's Fifth Problem for abelian groups is reduced to that for compact groups (such as K), and for these the problem was decided. The compact abelian Lie groups are distinguished among all compact groups by the fact that their character group is finitely generated.

[2] U is called a local Lie group in G if there is a Lie group L, an identity neighborhood $U' \subseteq L$, and a homeomorphism $\phi\colon U' \to U$ satisfying $f(xy) = f(x)f(y)$ for all $x, y, xy \in U'$.

This was the situation before the Second World War and still by the middle of the century. Even for low dimensional topological manifolds nothing was known. Among the cognoscenti it was said that Montgomery invested a lot of effort in the dimensions 2 and 3 and finally cracked these cases. Typically it was unknown whether a locally compact and connected group had to contain an arc. It was realized by A. M. Gleason that this could be proved. When Montgomery heard about this, he instantly recognized the significance for a solution of Hilbert's Fifth Problem. His joint paper with Leo Zippin was submitted to the Annals of Mathematics on March 28, 1952, and Gleason's article on June 13, 1952. Recently, G. D. Mostow pointed out in a lecture [16; p. 11] that "Gleason's arc resulted from his remarkable idea of constructing a *semig-group* of subsets; according to Gleason, that idea came to him while reading Hille's book 'Semi-groups of operators on Hilbert space' – a wonderful instance of unpredictable pregnancies in mathematics". With these contributions, Hilbert's Fifth Problem was resolved, in as much as locally euclidean groups were concerned, by an affirmative answer. The proof was presented by Montgomery and Zippin in a text book in 1955 which instantly became a classic. The considerable technical complication of the proof was never really simplified.

The decade of the fifties is also marked by the first attempts to deal with topological semigroups in a systematic way. In the USA the prime promoter was A. D. Wallace in New Orleans, in Central Europe Štefan Schwarz was the one mathematician who recognized certain basic features of compact semigroups which were to become basic stock in the trade [20], [21], [22], [23]. In particular, he contributed to the understanding of the structure of compact monothetic semigroups, i.e., compact topological semigroups in which the powers of one element are dense. Commutative semigroups in which every singly generated subsemigroups is contained in a compact one allows a partition into what one calls *archimedean components*. In the sixties, the structure theory of compact topological semigroups reached a plateau on which it stayed since; the monograph by Paul Mostert and myself [10] was an attempt to round off this theory and present a summary of what was known then.

Questions on the wider frame of Hilbert's Problem concerning the structure of connected locally compact groups were not settled at once, however. Yet this aspect of the problem was resolved by Hidehiko Yamabe who showed [24], [25] that the Theorem 1, which we formulated for compact groups above, remains intact for all locally compact groups G such that the factor group G/G_0 modulo the connected component of the identity is compact. These groups are called *almost connected.*

This information was all that was needed at the time in order to elucidate the structure of locally compact connected groups. For as early as 1949 the Annals of Mathematics published a seminal paper by Iwasawa [12] in which the basic properties of those locally compact groups were uncovered which could be approximated by connected Lie groups. For instance, the Theorem 2 above was proved for these groups. As a consequence, the Theorems 2 and 3 are available for locally compact and almost connected groups.

Iwasawa also proved the following result, which after Yamabe's theorem can be formulated as follows:

Theorem 5. *In a locally compact almost connected group G, every compact group is contained in a maximal compact group K to which all other maximal compact groups are conjugate, and there are continuous homomorphisms $f_1, \dots, f_n \colon \mathbb{R} \to G$ such that the map*

$$\big(k, (x_1, \dots, x_n)\big) \mapsto k f_1(x_1) \cdots f_n(x_n) \colon K \times \mathbb{R}^n \to G$$

is a homeomorphism.

In particular, G is homeomorphic to $K \times \mathbb{R}^n$. As a consequence, all topological characteristics of the group G are completely known, since the structure of K is extremely well understood.

It was above all one part of the information emerging from the solution of Hilbert's Fifth Problem which influenced group theory vitally thereafter, namely, that portion which dealt with the structure theory of locally compact groups. This information permitted the accumulation of much knowledge about the structure of locally compact groups since the sixties. It was Hilbert, who spoke in the connection with Lie groups about the foundations of geometry. The structure theory of locally compact groups plays a fundamental role in the contemporary theory of the foundation of geometry, notably in the theory of locally compact connected projective planes which appears to reach a certain level of completion [7].

What is the status of Abel's Problem which was more comprehensive than Hilbert's?

In the aftermath of the work on Hilbert's Fifth Problem we register an article by R. Jacoby [13] in 1957 in the Annals of Mathematics . In this paper, it was shown that a locally euclidean *local* group was a local Lie group. This result, whose proof is very complicated, appeared to fall into oblivion until in the middle of the seventies, under the direction of Dennison R. Brown of the University of Houston a dissertation was written by R. S. Houston which addressed the question of cancellable topological semigroups on manifolds. This author combined classical semigroup techniques (as they are known in the context of the so called Ore condition, sufficient for embeddability of a semigroup into groups) with Jacoby's result and constructed for each of his semigroups a Lie group in which quotients of elements from the semigroups can be locally embedded. The belated publication of these results appeared in 1987 [4]. A systematic clarification of the situation was achieved by Wolfgang Weiss and myself in 1988 [11] by employing sheaf theoretical methods. On the basis of the results by Brown and Houston we showed the following results:

THEOERM 6. *Let S be a cancellative topological semigroup on a topological manifold. Then the following propositions hold:*

(1) *On S there exists a unique analytic structure, with respect to which the multiplication $S \times S \to S$ is analytic.*

(2) *There exists a canonically determined simply connected Lie group $\widetilde{G}(S)$ and an analytic cancellative semigroup $\widetilde{S}$ with an analytic covering morphism $p\colon \widetilde{S} \to S$ and an analytic homomorphism $f\colon \widetilde{S} \to \widetilde{G}(S)$, which in all points is a local isomorphism of analytic manifolds.*

(3) *$\widetilde{G}(S)$ contains a countable central subgroup G_S which is algebraically isomorphic to the group of covering transformations of the covering $p\colon \widetilde{S} \to S$. If one defines $G(S) = \widetilde{G}(S)/G_S$, there is a commutative diagrams of homomorphisms*

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{f} & \widetilde{G}(S) \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \text{quot}} \\ S & \longrightarrow & G(S) \end{array}$$

in which the map $S \to G(S)$ is the universal homomorphism of S into a topological group.

It is still an open problem whether G_S has to be closed (and then, because of its countable cardinality, has to be discrete). This is of considerable interest because this condition is necessary and sufficient for $G(S)$ to be a Lie group. Apart from this open problem, the Theorem 6 settles Abel's Problem in the affirmative: Cancellative topological connected locally euclidean semigroups are analytic and closely tied to a Lie group. Abel was right: Abel's Theorem is correct as soon as the postulate of cancellability, made implicitly by Abel, is made explicit.

The general circumstances described in the Theorem 6 are interesting in many respects. We illustrate them by an example which is very close to a very classical environment.

We consider the Lie algebras $\mathfrak{g} = \mathrm{sl}(2,\mathbb{R}) \subseteq \mathrm{sl}(2,\mathbb{C}) = \mathfrak{g}_{\mathbb{C}}$. In the algebra $\mathfrak{g}$

we set

$$[x, y, t] = x\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + y\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + t\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} x & y+t \\ y-t & -x \end{pmatrix}.$$

The determinant

$$L(x, y, t) \stackrel{\text{def}}{=} \det[x, y, t] = \begin{vmatrix} x & y+t \\ y-t & -x \end{vmatrix} = -x^2 - y^2 + t^2$$

is a Lorentzian form on $\mathfrak{g}$ which is invariant under inner automorphisms. The set

$$W = \{[x, y, t] : \ L(x, y, z) \geq 0\,, \ \ t \geq 0\}$$

is a Lorentzian cone which is invariant under inner automorphisms. In the 6-dimensional real Lie algebra $\mathfrak{g}_{\mathbb{C}}$ the set $\mathfrak{g} \oplus iW$ is a wedge which is invariant under inner automorphisms generated by $\mathfrak{g}$. We consider the groups $G = \mathrm{Sl}(2, \mathbb{R}) \subseteq \mathrm{Sl}(2, \mathbb{C}) = G_{\mathbb{C}}$ of dimensions 3 and 6, respectively. There is a theorem which is relevant to our situation and of which we know far reaching generalizations today.

THEOREM. (G. I. Ol'shanskiĭ) *The subset $H = G\exp(i \cdot W)$ is a closed subsemigroup with a nonvoid interior S (which satisfies $SH = HS \subseteq S$).*

The map $(g, X) \mapsto g\exp(i \cdot X)\colon G \times W \to S$ is a diffeomorphism which induces an isomorphism of analytic manifolds $G \times \mathrm{interior}(W) \to S$.

The group $\mathrm{Sl}(2, \mathbb{C})$ acts on the Riemann sphere $\mathbb{S}^2 = \mathbb{C} \cup \{\infty\}$ according to the following definition:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}\,.$$

Let N denote the northern hemisphere of $\mathbb{S}^2$. We have

$$H = \{g \in \mathrm{Sl}(2, \mathbb{C}) : \ g \cdot N \subseteq N\}\,,$$
$$S = \{g \in \mathrm{Sl}(2, \mathbb{C}) : \ g \cdot N \subseteq \mathrm{Int}\, N\}\,.$$

The group $G = \mathrm{Sl}(2, \mathbb{R})$ is homeomorphic to $\mathbb{R}^2 \times \mathbb{S}^1$. Its fundamental group is $\mathbb{Z}$. The Lorentzian cone W is homeomorphic to a closed half space $\mathbb{R}^+ \times \mathbb{R}^2$, $\mathbb{R}^+ = [0, \infty[$. According to our theorem, S is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}^4 \times \mathbb{R}^+$ while S is homeomorphic to $\mathbb{S}^1 \times \mathbb{R}^5$. Hence there is a simply connected covering monoid $\widetilde{H}$ of H which is homeomorphic to $\mathbb{R}^5 \times \mathbb{R}^+$. The interior T of $\widetilde{H}$ is the simply connected covering semigroup of S. It is cancellative and homeomorphic to $\mathbb{R}^6$. The group $\mathrm{Sl}(2,\mathbb{C})$ is homeomorphic to $\mathrm{SU}(2)\times\mathbb{R}^3 \sim \mathbb{S}^3\times\mathbb{R}^3$. Accordingly it is simply connected. The Theorem 6 applies to T. The canonically associated simply connected Lie group $\widetilde{G}(\widetilde{S})$ is $\mathrm{Sl}(2,\mathbb{C})$, and the covering homomorphism $p\colon \widetilde{T}\to T$ mentioned in the Theorem 6 (3) is the identical self-map of T. Indeed, we have a homomorphism $\widetilde{T} = T \to S \to \mathrm{Sl}(2,\mathbb{C})$ which is a local homeomorphism. Its corestriction to the image S is even a covering morphism. One can show [8] that none of the semigroups $\widetilde{H}$ and T is algebraically embeddable into any group, let alone analytically embeddable into a Lie group.

Semigroups of the type $G\exp(iW)$ arise in the context of unitary representation theory. One can ask the question whether unitary representations of a group, like the universal covering $\widetilde{G}$ of $\mathrm{Sl}(2,\mathbb{R})$ in our example, can be "holomorphically extended". Answers to such questions are highly significant for the representation theory of such groups [18].

The Abel-Hilbert-Problem has led us to analytic semigroups which illustrate the phenomenon that a consistent continuation of the program initiated by Hilbert's Fifth Problem breaks the boundaries of group theory and has significant consequences in modern representation theory [8], [9].

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