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Mathematica Slovaca, Vol. 32 (1982), No. 4, 349--354

Persistent URL: http://dml.cz/dmlcz/129389

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ON THE FUNCTIONAL INDEPENDENCE OF SCALAR INVARIANTS OF CURVATURE FOR DIMENSIONS $n=2, 3, 4$

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1. Introduction

Consider an n-dimensional real vector space R^n, the dual vector space R^{n*}, and denote

$$Q_n = R^n \otimes S_n^1,$$

where S_n^1 is a vector subspace of the tensor product $\otimes^3 R^{n*}$ defined by the Young scheme $(2, 1, 0, \ldots, 0)$. Recall that in the canonical coordinates t_{ijk}, $1 \leqslant i, j, k \leqslant n$ on $\otimes^3 R^{n*}$ the subspace S_n^1 is defined by the equations

$$t_{ijk} + t_{ikj} = 0; \quad t_{ijk} + t_{kij} + t_{jki} = 0,$$

which means that the canonical t_{ijk} coordinates restricted to S_n^1 are antisymmetric in the last two indices and satisfy a set of "Bianchi" identities.

Denote by R^p_{qrs}, $1 \leqslant p, q, r, s \leqslant n$ the canonical coordinates on the vector space Q_n. In agreement with the above relations

$$R^p_{qrs} + R^p_{qsr} = 0 \tag{1}$$

$$R^p_{qrs} + R^p_{sqr} + R^p_{rsq} = 0. \tag{2}$$

In what follows we shall consider the vector space Q_n as a $GL_n(R)$ — module defined by the natural representation of the group $GL_n(R)$. Denote by a^u_v the canonical coordinates on $GL_n(R)$ and introduced the functions b^q_r on $GL_n(R)$ by the relation

$$a^p_q b^q_r = \delta^p_r, \tag{3}$$

where δ^p_r is the Kronecker symbol.

The action of the group $GL_n(R)$ on Q_n is given in canonical coordinates by the formula

$$\bar{R}^p_{qrs} = a^p_t b^u_q b^v_r b^w_s R^t_{uvw}. \tag{4}$$

Let $U \subset Q_n$ be a $GL_n(R)$ — invariant open set. Let f: $U \to R$ be a function. Recall that the function f is called a scalar invariant of weight 0, or shortly an invariant, if for every $q \in U$ and every $a \in GL_n(R)$,

$$f(a \cdot q) = f(q). \tag{5}$$

The aim of this paper is to investigate the existence and the number of functionally independent invariants defined on open sets in Q_n (compare with [1]—[5]).

We note that the vector space Q_n coincides with the type fibre of the space of tensors of the type (1, 3) on an n-dimensional manifold X, in which the so-called curvature tensor, associated with a linear connection on X, takes its values. Having in mind this relation, we shall call the invariants whose domains of definition are subsets of Q_n, the invariants of curvature.

In the canonical coordinates R^p_{qrs} the condition (5) has the form

$$f(\bar{R}^p_{qrs}) = f(R^p_{qrs}). \tag{6}$$

Differentiating this equation with respect to b^j_i at the unit element $e = (\delta^i_j)$ of the group $GL_n(R)$, we get, for the function f, the following system of linear, homogeneous, first order partial differential equations for the function f:

$$\Xi^i_j(f) = 0, \tag{7}$$

where, by (6), (4) and (3),

$$\Xi^i_j = -R^i_{qrs} \frac{\partial}{\partial R^j_{qrs}} + R^p_{jrs} \frac{\partial}{\partial R^p_{irs}} + 2R^p_{qrj} \frac{\partial}{\partial R^p_{qri}}. \tag{8}$$

The classical Frobenius theorem on differential systems enables us to solve the system (7) and to establish the maximal number of functionally independent solutions.

Solving (7) we restrict our attention to the so-called *maximal points*, i.e. to those points of the space Q_n for which the rank of (7) is maximal. Therefore it is necessary to determine the rank of the matrix $\mathbf{M}_n$ formed by the coefficients in (8) with respect to the base $\partial/\partial R^t_{uvw}$.

The number of lines of the matrix $\mathbf{M}_n$ is equal to that of the differential operators Ξ^i_j, i.e. n^2. The relation (1) restricts the number of independent vectors $\partial/\partial R^t_{uvw}$ to $(1/2)n^3(n-1)$ (we take $v < w$). Now consider (2). For $n = 2, 3$, (2) is satisfied identically. For $n \geq 4$, (2) gives us $(1/6)n^2(n^2 - 3n + 2)$ additional independent relations among the functions R^i_{jkl}. A similar consideration may be applied to the vectors $\partial/\partial R^t_{uvw}$. In this way we obtain that the number of linearly independent vectors $\partial/\partial R^t_{uvw}$ is equal to $\dim Q_n$, where

$$\dim Q_n = \begin{cases} \frac{1}{2}n^3(n-1), & n = 2, 3 \\ \frac{1}{2}n^3(n-1) - \frac{1}{6}n^2(n^2 - 3n + 2), & n \geq 4. \end{cases}$$

The following table illustrates the dependence of $\dim Q_n$ on n. Let p (resp. q) denote the number of lines in $\mathbf{M}_n$ (resp. the number of independent relations (2)).

To establish the maximal number of functionally independent invariants we must determine the rank of the matrix $\mathbf{M}_n$. The number of functionally independent invariants which is to be found, is equal to the difference of $\dim Q_n$ and the rank of the matrix $\mathbf{M}_n$ at *maximal points.* In the sequel, we determine the number of independent invariants for $n=2, 3, 4$.

Table 1

n	p	$\frac{1}{2}n^3(n-1)$	q	$\dim Q_n$
2	4	4	0	4
3	9	27	0	27
4	16	96	16	80
5	25	250	50	200
6	36	540	120	420

2. Invariants of curvature for $n=2$

The matrix $\mathbf{M}_2$ is in this case a squared matrix of order 4. It is expressed explicitly in the following table.

Table 2

	$\frac{\partial}{\partial R^1_{112}}$	$\frac{\partial}{\partial R^1_{212}}$	$\frac{\partial}{\partial R^2_{112}}$	$\frac{\partial}{\partial R^2_{212}}$
$\frac{1}{2}\Xi^1_1$	R^1_{112}	0	$2R^2_{112}$	R^2_{212}
$\frac{1}{2}\Xi^2_2$	R^1_{112}	$2R^1_{212}$	0	R^2_{212}
$\frac{1}{2}\Xi^1_2$	R^1_{212}	0	$-R^1_{112}+R^2_{212}$	$-R^1_{212}$
$\frac{1}{2}\Xi^2_1$	$-R^2_{112}$	$R^1_{112}-R^2_{212}$	0	R^2_{112}

To establish the rank of matrix $\mathbf{M}_2$, we calculate its determinant. By simple calculation we get

$$\det \mathbf{M}_2 = (R^1_{112}+R^2_{212}) \begin{vmatrix} 0 & -R^1_{212} & R^2_{112} \\ R^1_{212} & 0 & -R^1_{112}+R^2_{212} \\ -R^2_{112} & R^1_{112}-R^2_{212} & 0 \end{vmatrix} .$$

Hence it follows immediately that det $\mathbf{M}_2 \equiv 0$ and thus rank $\mathbf{M}_2 < 4$. It can be easily seen that there exist some points of Q_2 *(maximal points)* at which rank $\mathbf{M}_2 = 3$. Thus there exists one and only one functionally independent invariant.

It can be shown that such an invariant can be taken for example, as the function

$$f = R^{i}{}_{km} \cdot R^{k}_{piq} \cdot R^{qp} \cdot R^{mi},$$

where R^{pq} denotes the elements of the matrix, inverse to the matrix $(R_{ij}) = (R^{p}_{ipj})$.

3. Invariants of curvature for $n = 3$

The matrix $\mathbf{M}_3$ is of type 9/27. To find its rank, we use the determinant from its squared submatrix $\mathbf{M}_3$ of order 9 formed by the coefficients at $\partial/\partial R^{1}_{qrs}(r<s)$. Substitute in $\mathbf{M}_3$

$$R^{1}_{112} = R^{1}_{212} = R^{1}_{312} = R^{1}_{323} = R^{2}_{323} = R^{3}_{113} = R^{3}_{323} = 1,$$

and zeros for the other R^{p}_{qrs}. It can be easily found by a simple calculation (e.g. by a progressive decreasing of the order) that det $\mathbf{M}_3 \neq 0$. It means that $\mathbf{M}_3$ is regular and rank $\mathbf{M}_3 = 9$ on a certain open set containig the considered point. The number of functionally independent invariants at *maximal points* of the set Q_n is therefore equal to 18.

4. Invariants of curvature for $n = 4$

From (8) we get 16 operators Ξ^{i}_{j} with 96 expressions $\partial/\partial R^{i}_{uvw}$ whose coefficients will be linear combinations of 96 components of the functions R^{p}_{qrs}. In addition there are fulfilled the "Bianchi" identities (2) representing in this case 16 independent relations among 48 variables. Thus 16 suitably chosen variables can be expressed by means of the remaining 32 ones, for example, as follows:

$$R^{p}_{213} = R^{p}{}_{3} + R^{p}_{312}; \quad R^{p}_{214} = R^{p}_{124} + R^{p}_{412}$$

$$R^{p}_{314} = R^{p}_{134} + R^{p}_{413}; \quad R^{p}_{324} = R^{p}_{234} + R^{p}_{423}.$$

Applying the chain rule for partial derivation of composed functions to equation (2), we can easily derive the relation

$$\frac{\partial}{\partial R^{p}_{qrs}} = \frac{\partial}{\partial R^{p}_{srq}} + \frac{\partial}{\partial R^{p}_{rqs}},$$

by means of which, we can reduce the number of columns of the matrix $\mathbf{M}_4$ to 80.

Thus the matrix $\mathbf{M}_4$ is of the type 16/80. It will be shown that there exist some points at which the rank of $\mathbf{M}_4$ is equal to 16. Consider the squared matrix $\mathbf{M}_4$ of order 16 formed by the coefficients at the operators $\partial/\partial R^1_{112}, \partial/\partial R^1_{113}, \ldots, \partial/\partial R^1_{413}$. It can be easily found that, e.g., at the point, where

$$R^1_{112} = R^1_{212} = R^1_{224} = R^1_{323} = R^1_{413} = R^1_{434} = R^2_{434} = R^2_{413} = R^3_{413} = R^4_{413} = 1$$

and the remaining coordinates R^p_{qrs} are zeros, we have det $\bar{\mathbf{M}}_4 \neq 0$. Thus at the considered point rank $\mathbf{M}_4 = 16$. Consequently, the number of functionally independent invariants of curvature at *maximal points* of the set Q_4 is 64.

5. Conclusion

Let us summarize the results of this paper. Consider the $GL_n(R)$ — module Q_n and canonical coordinates R^i_{jkl} on Q_n. The relation between the dimension of the vector space Q_n and the number r_n of functionally independent (scalar) invariants of the group $GL_n(R)$ defined on open sets in Q_n is given, for $n = 2, 3, 4$, by the following table.

Table 3

n	dim Q_n	rank $\mathbf{M}_n$ *at maximal points*	r_n
2	4	3	1
3	27	9	18
4	80	16	64

From the above it follows, among others, that for $n = 2$ (resp. $n = 3$, $n = 4$) there exists precisely $r_2 = 1$ (resp. $r_3 = 16$, $r_4 = 64$) functionally independent scalar invariant of curvature.

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Received November 10, 1980

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ФУНКЦИОНАЛЬНАЯ НЕЗАВИСИМОСТЬ СКАЛЯРНЫХ ИНВАРИАНТОВ КРИВИЗНЫ ДЛЯ РАЗМЕРНОСТЕЙ $n = 2, 3, 4$

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Резюме

В работе рассматривается проблема определения числа функционально независимых скалярных инвариантов тензора кривизны R^r_{qrs} линейной связности на n-мерном дифференцируемом многообразии, где $n = 2, 3, 4$. Показано, что для $n = 2$ ($= 3, 4$) существует точно $r_n = 1$ (16, 64) функционально независимых инвариантов.