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ON SOME PROPERTIES OF TRANSFORMATIONS OF A LOGIC

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In the paper the notion of the ergodicity on a logic will be introduced and the different types of the transformations of a logic will be characterized and the recurrence theorems will be proved.

1. Ergodic properties of homomorphisms

Let L be a σ-lattice with the first and the last elements O and 1, respectively and an orthocomplementation $\perp : a \mapsto a^{\perp}$, which satisfies (i) $(a^{\perp})^{\perp} = a$ for all $a \in L$; (ii) if $a < b$, then $b^{\perp} < a^{\perp}$ for all $a, b \in L$; (iii) $a \vee a^{\perp} = 1$ for all $a \in L$. We say that a, b are orthogonal and write $a \perp b$ if $a < b^{\perp}$. We further assume that if $a, b \in L$ and $a < b$, then there exists an element $c \in L$ such that $a \perp c$ and $a \vee c = b$. A σ-lattice satisfying the above axioms will be called a logic (see [1]).

A state is a map m from L into $\langle 0, 1\rangle$ such that $m(1) = 1$ and $m(\bigvee_{i=1}^{\infty} a_i) = \sum_{i=1}^{\infty} m(a_i)$ if $a_i \perp a_j$ for $i \neq j$. A logic is full in the case: (i) if $a \neq b$, there exists a state m such that $m(a) \neq m(b)$; (ii) if $a \neq O$, there exists a state m such that $m(a) = 1$. An observable is a map x from the Borel sets $B(R_1)$ of R_1 into a logic L, which satisfies (i) $x(R_1) = 1$; (ii) $x(E) \perp x(F)$ if $E \cap F = \emptyset$; (iii) $x(\bigcup_{i=1}^{\infty} E_i) = \bigvee_{i=1}^{\infty} x(E_i)$ if $E_i \cap E_j = \emptyset$, $i \neq j$, $E_i \in B(R_1)$.

Let x be an observable and m be a state. Then we shall say that x is

(i) a constant (a constant a. e. $[m]$) if there is a real number λ such that $x(\{\lambda\}) = 1$ $(m(x(\{\lambda\}) = 1)$;

(ii) bounded (bounded a. e. $[m]$) if there is a compact set K with the property $x(K) = 1$ $(m(x(K)) = 1)$.

We denote by $\sigma(x)$ $(\sigma_m(x))$ the smallest closed set E such that $x(E) = 1$ $(m(x(E)) = 1)$.

A homomorphism of a logic L is a map T from L into L such that $TO = O$;

$T(a^\perp) = (Ta)^\perp$ for all $a \in L$; $T(\bigvee_{i-1}^{\infty} a_i) = \bigvee_{i-1}^{\infty} Ta_i$. We shall say that a homo morphism T of a logic L is (i) invariant in a state m if $m(Ta) = m(a)$ for all $a \in L$; (ii) ergodic in a state m if the equality $Ta = a$ implies $m(a) \in \{0, 1\}$.

Let T be a homomorphism of L and x be an observable. We shall say that x is T-invariant if $T(x) = x$, where $(T(x))(E) = T(x(E))$, $E \in B(R_1)$.

Theorem 1.1. *A homomorphism T of a full logic L is ergodic in every state iff the constants are the only T-invariant observables.*

Proof. For sufficiency, let the constants be the only T-invariant observables and let $Ta = a$. We define an observable $q_a : q_a(\{0\}) = a^\perp$, $q_a(\{1\}) = a$. It follows that q_a is T-invariant and hence $q_a(\{1\}) = a$ is either 1 or O. Then $m(a) \in \{0, 1\}$ for all m.

Conversely, let T be ergodic in every state and let $T(x) = x$, hence $m(x(E)) \in \in \{0, 1\}$ for all m. If $O \neq x(E) \neq 1$ for some $E \in B(R_1)$, then there exist two states m_1, m_2 such that $m_1(x(E)) = 1$, $m_2(x(E)^\perp) = 1$. Thus if $m - \frac{1}{2}(m_1 + + m_2)$, we have $m(x(E)) = \frac{1}{2}$. This is a contradiction and hence $x(E)$ is either O or 1. Let us denote

$$\mathscr{C} = \{E \in B(R_1) : E \supset \sigma(x) \text{ or } E \cap \sigma(x) = \emptyset\}.$$

If $a < b$, then either $\langle a, b\rangle$ or (a, b) is in $\mathscr{C}$. But $\mathscr{C}$ is a σ-algebra and hence it equals $B(R_1)$. Hence it follows that there is a $\lambda \in R_1$ such that $x(\{\lambda\}) = 1$.

q.e.d

Theorem 1.2. *A homomorphism T of a logic L (L is arbitrary) is ergodic in a state m iff the constants a. e. $[m]$ are the only T-invariant observables bounded a. e. $[m]$.*

Proof. Let $T(x) = x$, x be bounded a. e. $[m]$ and let T be ergodic in m, then $m(x(E)) \in \{0, 1\}$ for all $E \in B(R_1)$. If we denote $a = \inf \sigma_m(x)$, $b = \sup \sigma_m(x)$, we shall have $m(x(\langle a, b\rangle)) = 1$ and by application of the Weierstrass method of dividing repeatedly the bounded interval $\langle a, b\rangle$ into halves we shall obtain a sequence $\{\langle a_n, b_n\rangle\}$ of intervals such that $\langle a, b\rangle \supset \langle a_1, b_1\rangle \supset \supset \langle a_2, b_2\rangle \supset \ldots$ and $m(x(\langle a_n, b_n\rangle)) = 1$ for $n = 1, 2, \ldots$. Hence there is a $\lambda \in R_1$ such that $\{\lambda\} = \bigcap_{n-1}^{\infty} \langle a_n, b_n\rangle$ and consequently $m(x(\{\lambda\})) = 1$.

The sufficient condition is trivial. q.e.d.

Corollary 1.2.1. *A homomorphism T of a logic L is ergodic in a state m iff the constants a. e. $[m]$ are the only T-invariant observables (not necessarily bounded).*

Proof. Only necessity. For $\sigma_m(x)$ we have $\sigma_m(x) = \bigcup_{n=-\infty}^{\infty} (\sigma_m(x) \cap \langle n, n + 1)) = 1$. The set $E = \sigma_m(x) \cap \langle n, n + 1)$ is bounded and as above there is a $\lambda \in R_1$ such that $m(x(\{\lambda\})) = 1$.

q.e.d.

Remark 1. Theorem 1.2. will be valid if the assumption of the boundedness a. e. $[m]$ of x is omitted, provided that $x \in O_p(m) = \{x: |\int \lambda^p m(x(d\lambda))| < \infty\}$ for $1 \leqslant p < \infty$. In fact, if $Ta = a$, then the observable q_a is in $O_p(m)$ and $\int \lambda^p m(q_a(d\lambda)) = m(a) \in \{0, 1\}$. On the other hand, the necessity is easily seen from Corollary 1.2.1.

Remark 2. Let L be now a logic in the sense [5], that is, L is not a lattice in general. Then the Theorems 1.1., 1.2., the Corollary 1.2.1. and the Remark 1 will be valid, too.

Lemma 1.3. *An automorphism T of a logic L is ergodic in a state m iff $m(\bigvee_{j=-\infty}^{\infty} T^j a) = 1$ holds for each $a \in L$, $m(a) > 0$.*

Proof. The sufficiency is trivial. On the other hand let $m(a) > 0$, then for $b = \bigvee_{j=-\infty}^{\infty} T^j a$ we have $m(b) > 0$. But $Tb = b$ and hence $m(b) = 1$.

q.e.d.

If we use Wigner's and Gleason's theorems (see [1]) about the representation of automorphisms and the states, respectively, in the case of a logic of all closed subspaces of a Hilbert space H we shall give an interesting example which is a generalization of a known proposition in the ergodic theory (see [2] p. 34).

Let $L = L(H)$ be the logic of all closed subspaces of H and $(.,.)$ be the inner product on H. Since there is a one-to-one correspondence between the closed subspaces M of H and their projectors P^M, we shall write M for an element as well as for its projector. Let U be a unitary operator on H and φ be a unit vector in H. Then $T_U: M \mapsto UMU^{-1}$, $M \in L(H)$, is an automorphism of $L(H)$ and $m_\varphi : M \mapsto (M\varphi, \varphi)$, $M \in L(H)$ is a state of $L(H)$.

Example. Let U be a unitary operator on a Hilbert space H and $P = \{\xi \in H : U\xi = \xi\} \neq 0$. Then an automorphism $T_U(.) = U(.)U^{-1}$, is invariant in a state m_φ, $\varphi \in P$, $||\varphi|| = 1$, where $m_\varphi(M) = (M\varphi, \varphi)$, $M \in L(H)$.

If dim $P = 1$ then, moreover, T_U is ergodic in a state m_φ, Conversely, if for each $\varphi \in P$, $||\varphi|| = 1$, T_U is ergodic in a state m_φ, then dim $P = 1$.

Proof. For invariancy: $m_\varphi(T_U M) = (UMU^{-1}\varphi, \varphi) = (MU^{-1}\varphi, U^{-1}\varphi) = (M\varphi, \varphi) = m_\varphi(M)$. Now let dim $P = 1$ and $T_U M = M$, that is $UMU^{-1} = M$, $UM = MU$. If φ is a unit vector in P, then $UM\varphi = MU\varphi = M\varphi$,

i. e. $M\varphi \in P$ and $M\varphi = \alpha\varphi$. But $\alpha^2\varphi = M^2\varphi = M\varphi = \alpha\varphi$, hence $\alpha \in \{0, 1\}$ and consequently $m_\varphi(M) = \alpha \in \{0, 1\}$.

Conversely, let T_U be ergodic in all m_φ, $\varphi \in P$, $||\varphi|| = 1$ and let $\dim P > 1$ then there exist two orthonormal vectors φ_1, φ_2 in P. Hence if $\varphi = \frac{\sqrt{2}}{2}(\varphi_1 + \varphi_2)$ and M is a subspace generated by φ_1, then $\varphi \in P$, $||\varphi|| = 1$ and $UM = MU$ because if $\xi = \alpha\varphi_1 + y$, $y \perp \varphi_1$, then $UM\xi = \alpha\varphi_1$, $MU\xi = \alpha\varphi_1 + MUy$. But $(\varphi_1, Uy) = (U^*\varphi_1, y) = 0$ and hence $MU\xi = \alpha\varphi_1$. Finally $m_\varphi(M) = \frac{1}{2}$ and it is a contradiction with our assumption and hence $\dim P = 1$.

q.e.d

2. Characterizing some types of transformations of a logic

For every two elements $a, b \in L$ we shall write $a - b = a \wedge b^\perp$.

Theorem 2.1. (*Recurrence theorem*) *Let T be a homomorphism of L and let T be invariant in a state m. Then for all $a \in L$ we have*

$$m(a - \bigvee_{j=1}^{\infty} T^j a) = 0. \tag{1}$$

Proof. Let $b = a - \bigvee_{j=1}^{\infty} T^j a$, then $\{T^j b\}_{j=0}^{\infty}$ are orthogonal elements of L and therefore $m(\bigvee_{j=0}^{\infty} T^j b) = \sum_{j=0}^{\infty} m(T^j b) = \sum_{j=0}^{\infty} m(b) < 1$. Hence $m(b) = 0$.

q.e.d.

A logic L is said to satisfy the finite chain condition (f.c.c.) if $\{a_m\} \subset L$ with $a_1 > a_2 > \ldots$ implies that there exists N such that $a_n = a_N$ for $n > N$ (see [3]). A logic L is σ-continuous if for $\{a_n\} \subset L$ with $a_1 < a_2 < \ldots$ we have $a \wedge (\bigvee_{n=1}^{\infty} a_n) = \bigvee_{n=1}^{\infty} (a \wedge a_n)$ for all $a \in L$. It is easy to see that if L satisfies f.c.c. then it is σ-continuous. For $\{a_n\} \subset L$ let $\limsup a_n = \bigwedge_{n=1}^{\infty} \bigvee_{j=n}^{\infty} a_j$.

Theorem 2.2. (*Strong recurrence theorem*) *Let L be σ-continuous and T be a homomorphism invariant in a state m. Then for all $a \in L$ we have*

$$m(a - \limsup T^j a) = 0. \tag{2}$$

Proof. Let us put $b = a - \limsup T^j a$, then $b = a \wedge \bigvee_{n=1}^{\infty} (\bigvee_{j=n}^{\infty} T^j a)^{\perp} = \bigvee_{n=1}^{\infty} (a \wedge (\bigvee_{j=n}^{\infty} T^j a)^{\perp}) = \bigvee_{n=1}^{\infty} (a - \bigvee_{j=n}^{\infty} T^j a) = \bigvee_{n=1}^{\infty} b_n$ where $b_n = a - \bigvee_{j=n}^{\infty} T^j a$, $n = 1, 2, \ldots$. Applying Theorem 2.1. to a map $\Pi = T^n$ we get for $b_n^* = a - \bigvee_{j=1}^{\infty} \Pi^j a$, $m(b_n^*) = 0$. But $b_n < b_n^*$, therefore $m(b_n) = 0$, $n = 1, 2, \ldots$ and $m(b) = \lim_n m(b_n) = 0$.

q.e.d.

In the rest of this paper according to [4] some types of transformations will be introduced and relations among them will be shown.

Let T be a transformation $L \to L$ and m be a state. Then we shall say that T is

(i) incompressible in a state m: if $a \in L$, $a < Ta$ implies $m(Ta - a) = 0$;

(ii) conservative in a state m: if $a \in L$, $a \perp T^n a$, $n = 1, 2, \ldots$ implies $m(a) = 0$;

(iii) weakly conservative in a state m: if $a \in L$, $\{T^n a\}_{n=0}^{\infty}$ is a sequence of mutually orthogonal elements of L, then $m(a) = 0$;

(iv) recurrent in a state m: if $a \in L$, then $m(a - \bigvee_{n=1}^{\infty} T^n a) = 0$;

(v) strongly recurrent in a state m: if $a \in L$, then $m(a - \limsup T^n a) = 0$.

Remark 3. If T is a homomorphism of L invariant in a state m, then T is conservative in m.

Theorem 2.3. *Let L be σ-continuous, then (v) implies (iv).*

Proof. Let $a \in L$, then $a - \limsup T^n a = a - \bigwedge_{n=1}^{\infty} \bigvee_{k=n}^{\infty} T^k a = \bigvee_{n=1}^{\infty} (a - \bigvee_{k=n}^{\infty} T^k a) > a - (\bigvee_{k=1}^{\infty} T^k a)$ and hence $m(a - \bigvee_{n=1}^{\infty} T^n a) = 0$.

q.e.d.

Theorem 2.4. *Let T be a monotonic transformation, that is $Ta < Tb$ if $a < b$, m be a state, then (v) implies (iv), and (ii) and (iv) are equivalent.*

Proof. Let $a \in L$ and let (v) hold, then $a - \bigvee_{n=1}^{\infty} T^n a = (a - \bigvee_{n=1}^{\infty} T^n a) - \limsup T^k (a - \bigvee_{n=1}^{\infty} T^n a)$. Indeed, if b is the element on the right-hand side, then $b < a - \bigvee_{n=1}^{\infty} T^n a$. Since $\limsup T^k (a - \bigvee_{n=1}^{\infty} T^n a) < \limsup T^k a < \bigvee_{k=1}^{\infty} T^k a$, we have $b > (a - \bigvee_{n=1}^{\infty} T^n a) - \bigvee_{n=1}^{\infty} T^n a = a - \bigvee_{n=1}^{\infty} T^n a$ and therefore $m(a - \bigvee_{n=1}^{\infty} T^n a) = m(b) = 0$.

Let now (ii) hold. Then if $a \in L$, let $b = a - \bigvee_{n=1}^{\infty} T^n a$. For each $m = 1, 2, \ldots$ we get $T^m(a - \bigvee_{n=1}^{\infty} T^n a) < T^m a < a^{\perp} \vee \bigvee_{n=1}^{\infty} T^n a = (a - \bigvee_{n=1}^{\infty} T^n a)^{\perp}$, therefore $T^m b \perp b$ and hence $m(b) = 0$. On the other hand let (iv) be valid, then if $a \perp T^n a$, $n = 1, 2, \ldots$, we have $a \perp \bigvee_{n=1}^{\infty} T^n a$ and $a - \bigvee_{n=1}^{\infty} T^n a = a$. Therefore (ii) holds, too.

q.e.d.

We shall be able to say something more if we assume the following properties of T

(3) $$T(\bigvee_{n=1}^{\infty} a_n) = \bigvee_{n=1}^{\infty} T a_n \quad \text{for } \{a_n\} \subset L$$

(4) $$T(a^{\perp}) > (Ta)^{\perp} \qquad \text{for all } a \in L.$$

If $\{a_\lambda\}_{\lambda \in \Lambda}$ is a system of orthogonal elements from L, there is a Boolean σ-algebra $A \subset L$ which contains the given system (see [1]). Therefore the distributive law holds for the orthogonal elements of L.

Theorem 2.5. *Let T be a transformation $L \to L$ with the properties* (3), (4) *and m be a state, then* (*i*) *implies* (*iii*).

Proof. Let $a \in L$ and (i) hold. If $\{T^n a\}_{n=0}^{\infty}$ are orthogonal elements of L, then for $b = (\bigvee_{n=0}^{\infty} T^n a)^{\perp}$ we have $T((\bigvee_{n=0}^{\infty} T^n a)^{\perp}) > (T(\bigvee_{n=0}^{\infty} T^n a))^{\perp} = \bigvee_{n=0}^{\infty} \{T^n a \wedge (\bigvee_{m=1}^{\infty} T^m a)^{\perp}\} = a \wedge (\bigvee_{m=1}^{\infty} T^m a)^{\perp} = a$. We conclude finally that $m(a) \leqslant m(Tb - b) = 0$.

q.e.d

Lemma 2.6. *Let T be a homomorphism of L and m be a state, then* (*ii*) – (*iv*) *are equivalent.*

Proof. The equivalency of (ii) and (iv) has been proved, Theorem 2.4., and (ii) and (iii) are equivalent as can easily be seen.

q.e.d.

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