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In papers [1] and [2] three lattices belonging to every semigroup were studied by R. Šulka. Two of them are distributive. The aim of this note is to consider the distributivity of the remaining lattice.

Let S be a semigroup. By $F(S)$ (respectively $I(S)$) we denote the set of all periodic (respectively aperiodic) elements of S. Clearly $I(S) = S \backslash F(S)$. The union of all periodic subgroups of S is denoted by $G(S)$. Put $K(S) = F(S) \backslash G(S)$. Terminology and notation not defined here may be found in [3].

By $\mathscr{P}(S)$ we denote the lattice of all subsets of a semigroup S with respect to the inclusion $\subseteqq$. The symbol N stands for the set of all positive integers. Following [1] we put $N_2(M) = \{x \in S; x^n \in M$ for infinitely many $n \in N\}$ and $N_3(M) = \{x \in S; x^n \in M$ for some $n \in N\}$ for every $M \subseteqq S$. Further let $\mathscr{N}_{23}(S) = \{M \in \mathscr{P}(S);\ N_2(M) = N_3(M)\}$. From the paper [1] it follows that $\langle \mathscr{N}_{23}(S), \subseteqq \rangle$ is a complete lattice and a complete upper subsemilattice of $\langle \mathscr{P}(S), \subseteqq \rangle$. It need not be a lower subsemilattice of $\langle \mathscr{P}(S), \subseteqq \rangle$.

Using the results (Theorem 5 and Theorem 6) of [1] we obtain:

Proposition 1. *Let S be a semigroup and $M \subseteqq S$. Then $M \in \mathscr{N}_{23}(S)$ if and only if for every $x \in M$ there exists a positive integer $m > 1$ such that $x^m \in M$.*

This implies:

Proposition 2. *Every union of subsemigroups of a semigroup S belongs to $\mathscr{N}_{23}(S)$.*

Proposition 3. *Let S be a semigroup and $M \in \mathscr{N}_{23}(S)$. If $x \in M \cap F(S)$, then there exists a positive integer m such that $x^m \in M \cap G(S)$.*

Proposition 4. *Let a be an element of a semigroup S. Then $\{a\} \in \mathscr{N}_{23}(S)$ if and only if $a \in G(S)$.*

Proposition 5. *Let S be a semigroup. Then $\mathscr{P}(G(S))$ is a complete sublattice of $\mathscr{N}_{23}(S)$.*

Proposition 6. *Let S be a semigroup and $M \subseteqq S$. Then M is an atom in the lattice $\mathscr{N}_{23}(S)$ if and only if $M = \{a\}$ for some $a \in G(S)$.*

Proof. Suppose that M is an atom in the lattice $\mathscr{N}_{23}(S)$. Then $M \neq \emptyset$.

Choose $a \in M$. It follows from Proposition 1 that there exists a sequence of positive integers $m_i > 1$ such that $m_i < m_{i+1}$, $m_i | m_{i+1}$ and $a^{m_i} \in M$ for all $i \in N$. According to Proposition 1 we have $Q = \{a^{m_i};\ i \in N\} \in \mathcal{N}_{23}(S)$. Since $Q \subseteqq M$ and M is an atom, we obtain $a \in M = Q$ and so $a = a^{m_k}$ for some $k \in N$. This means that $a \in G(S)$. By Proposition 4 we have $\{a\} \in \mathcal{N}_{23}(S)$ and so $M = \{a\}$.

Let $M = \{a\}$ for some $a \in G(S)$. Then, by Proposition 4, we have $M \in \mathcal{N}_{23}(S)$. Therefore M is an atom in the lattice $\mathcal{N}_{23}(S)$.

Theorem 1. *Let S be a semigroup. Then the lattice $\mathcal{N}_{23}(S)$ is atomic if and only if S is periodic.*

Proof. Let the lattice $\mathcal{N}_{23}(S)$ be atomic. Assume that $I(S) \neq \emptyset$. Choose $x \in I(S)$ and by $\langle x \rangle$ we denote the subsemigroup of S generated by the aperiodic element x. By hypothesis we have $A \subseteqq \langle x \rangle$ for an atom A in $\mathcal{N}_{23}(S)$. According to Proposition 6 there exists a periodic element $a \in S$ such that $A = \{a\}$ and so $a \in \langle x \rangle$, which is a contradiction. Therefore the semigroup S is periodic.

Suppose that a semigroup S is periodic. Let $\emptyset \neq M \in \mathcal{N}_{23}(S)$. Choose $x \in M$. It follows from Proposition 3 that there exists $m \in N$ such that $a = x^m \in M \cap G(S)$. According to Proposition 4, $A = \{a\}$ in an atom in the lattice $\mathcal{N}_{23}(S)$. We have $A \subseteqq M$. Therefore the lattice $\mathcal{N}_{23}(S)$ is atomic.

Theorem 2. *Let S be a semigroup. Then the following statements are equivalent:*

(i) $\mathcal{N}_{23}(S)$ *is a complete sublattice of* $\mathcal{P}(S)$.

(ii) $\mathcal{N}_{23}(S)$ *is a sublattice of* $\mathcal{P}(S)$.

(iii) $\mathcal{N}_{23}(S)$ *is distributive.*

(iv) $\mathcal{N}_{23}(S)$ *is modular.*

(v) *For every $a \in S$ there exists a positive integer n such that $a^{n+1} \in \{a, a^n\}$.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is evident.

(iv) $\Rightarrow$ (v). Suppose that $\mathcal{N}_{23}(S)$ is modular.

First we shall show that S is periodic. Assume that $I(S) \neq \emptyset$. Choose $x \in I(S)$. For an arbitrary $i \in N$ let us put $m_i = 2^i$ and $n_i = 3^{i-1}$. It follows from Proposition 1 that A, B, $C \in \mathcal{N}_{23}(S)$, where $A = \{x^{m_i};\ i \in N\}$, $B = \{x^{n_i};\ i \in N\}$ and $C = A \cup \{x\}$. Since x is an aperiodic element of S, we have in the lattice $\mathcal{N}_{23}(S)$, by Proposition 1, $A \vee (B \wedge C) = A \cup \emptyset = A \neq C = (A \cup B) \cap C = (A \vee B) \wedge C$. Hence the lattice $\mathcal{N}_{23}(S)$ is not modular, a contradiction. Therefore S is periodic.

Let a be a periodic element of S. Then there exists $n \in N$ such that $e = a^n$ is an idempotent. If $a \in G(S)$, then $ae = a$. Suppose that $a \in K(S) = F(S) \backslash G(S)$. Then it follows from Proposition 4 that $\{a\} \notin \mathcal{N}_{23}(S)$. We shall show that $ae = e$. Assume that $b = ae \neq e$. Clearly b and e belong to a periodic subgroup of S. This implies that $e \neq a \neq b$. According to Proposition 1 and Proposition 4, we have A, B, $C \in \mathcal{N}_{23}(S)$, where $A = \{e\}$, $B = \{a, b\}$ and $C = \{a, e\}$. It follows from

Proposition 1 that in the lattice $\mathcal{N}_{23}(S)$ we have $A \vee (B \wedge C) = A \cup \emptyset = A \neq C = (A \cup B) \cap C = (A \vee B) \wedge C$. Hence the lattice $\mathcal{N}_{23}(S)$ is not modular, a contradiction. Then we have $ae = e$. This gives in both cases $a^{n+1} \in \{a, a^n\}$. Therefore the semigroup S satisfies the conditions (v).

(v) $\Rightarrow$ (i). Suppose that S has the property (v). To prove that $\mathcal{N}_{23}(S)$ is a complete sublattice of $\mathcal{P}(S)$ it suffices to show that $\mathcal{N}_{23}(S)$ is a complete lower subsemilattice of $\mathcal{P}(S)$. Let $M_j \in \mathcal{N}_{23}(S)$ for $j \in J \neq \emptyset$. We shall show that $M = \bigcap_{j \in J} M_j \in \mathcal{N}_{23}(S)$. Let $x \in M$. According to (v), there exists $n \in N$ such that $x^{n+1} \in \{x, x^n\}$. It is easy to show that $e = x^n$ is an idempotent of S and so $x \in F(S)$. If $x \in G(S)$, then $x^{n+1} = xe = x \in M$. If $x \in K(S)$, then $xe \neq x$ and so $xe = x^{n+1} = x^n = e$. In this case we have according to Proposition 3 $x^{n_j} \in M_j \cap G(S)$ for some $n_j \in N$, because $x \in M_j (j \in J)$. Thus for every $j \in J$ we have $x^{n_j} = x^{n_j}e = \ldots = xe = e$. Therefore $x^{2n} = e \in M$. It follows from Proposition 1 that $M \in \mathcal{N}_{23}(S)$.

Note. If the lattice $\mathcal{N}_{23}(S)$ is modular, then S is a periodic semigroup and so $\mathcal{N}_{23}(S)$ is atomic.

Theorem 3. *Let S be a semigroup. Then the following statements are equivalent*:

(i) $\mathcal{N}_{23}(S) = \mathcal{P}(S)$.

(ii) $\mathcal{N}_{23}(S)$ *is boolean.*

(iii) *Every element of $\mathcal{N}_{23}(S)$ is the least upper bound of some set of atoms.*

(iv) *S is a union of periodic groups.*

Proof. (i) $\Rightarrow$ (ii). It is clear.

(ii) $\Rightarrow$ (iv). Suppose that $\mathcal{N}_{23}(S)$ is boolean. Then it follows from Theorem 2 that S is periodic. Assume that there exists an element $a \in K(S)$. According to (v) of Theorem 2, we have $a^{n+1} = a^n$. Evidently $e = a^n$ is an idempotent of S and $ae = e \neq a$. Put $E = \{e\}$. It follows from Proposition 4 that $E \in \mathcal{N}_{23}(S)$. Then, by hypothesis, there exists $X \in \mathcal{N}_{23}(S)$ such that $E \cup X = E \vee X = S$ and $E \wedge X = \emptyset$. This implies that $a \in X$. According to Proposition 3, there exists $m \in N$ such that $a^m \in X \cap G(S)$. Thus we have $a^m = a^m e = \ldots = ae = e$ and so $e \in X$. It follows from Proposition 4 that $\{e\} \subseteqq E \wedge X = \emptyset$, which is a contradiction. Therefore $K(S) = \emptyset$ and so $S = G(S)$.

(iv) $\Rightarrow$ (i). This follows from Proposition 5.

• (i) $\Leftrightarrow$ (iii). Apply Proposition 6.

REFERENCES

[1] ŠULKA, R.: On three lattices that belong to every semigroup. Math. Slovaca 34, 1984, 217—228.
[2] ŠULKA, R.: Nilpotency in semigroups and sublattices of their Booleans. Math. Slovaca (to appear).
[3] CLIFFORD, A. H. aand PRESTON, G. B.: The algebraic theory of Semigroups. Amer. Math. Soc., Providence, R. I. Vol. I (1961), Vol. II (1967).

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ЗАМЕЧАНИЕ О НИЛЬПОТЕНТНОСТИ В ПОЛУГРУППАХ

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Резюме

Недавно Р. Шулка определил с помощью нильпотентности три структуры, элементы которых принадлежат булеану полугруппы. Две из этих структур являются дистрибутивными. В статье даются необходимые и достаточные условия для дистрибутивности третьей структуры.