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Mathematica Slovaca, Vol. 31 (1981), No. 1, 71--78

Persistent URL: http://dml.cz/dmlcz/129345

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THE ORIENTABILITY OF THE DIRECT PRODUCT OF GRAPHS

EVA GEDEONOVÁ

The covering graph $C(P)$ of a partially ordered set P is the graph whose vertices are the elements of P and whose edges are those pairs $\{a, b\}$, $a, b \in P$, for which a covers b or b covers a. The covering graph $C(P)$ of a partially ordered set P with some properties determines certain further properties of P. In some cases if $C(P)$ is a direct product of graphs G_1, G_2, then the partially ordered set P is a direct product of partially ordered sets P_1, P_2 and $C(P_i) = G_i$, $i = 1, 2$. The following example shows a case in which this assertion does not hold. The covering graph of the partially ordered set P of Fig. 1 is the direct product of two twoelemented graphs but the partially ordered set is not a direct product of two twoelemented partially ordered sets.

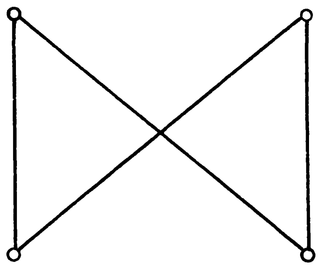

Fig. 1

The direct product $G_1 \times G_2$ of the graphs $G_1 = (V_1, H_1)$, $G_2 = (V_2, H_2)$ is the graph whose vertices are the elements of $V_1 \times V_2$ and whose edges are those pairs $\{(a_1, b_1), (a_2, b_2)\}$ $a_i \in V_1$, $b_i \in V_2$, $i = 1, 2$ satisfying either $a_1 = a_2$ and $\{b_1, b_2\} \in H_2$ or $\{a_1, a_2\} \in H_1$ and $b_1 = b_2$. By a graph isomorphism of graphs $G = (V, H)$ and $G' = (V', H')$ we mean a bijection $f\colon V \to V'$ of vertex sets such that $\{a, b\} \in H$ iff $\{f(a), f(b)\} \in H'$ for all $a, b \in V$. For vertices a and b of a graph G a path from a to b of length n is a sequence $a = c_0, c_1, \ldots, c_n = b$ of vertices of G such that succesive pairs in this sequence are joined by an edge of G. Let $d(a, b)$ denote the distance from a to b, i.e. the length of a shortest path from a to b. A graph is connected if for all its vertices a, b there holds that a, b are connected by a path. Note that every graph isomorphism f of connected graphs is a distance isomorphism (i.e. $d(a, b) = d(f(a), f(b))$). A partially ordered set P is locally finite if for every $a, b \in P$, $a < b$ there is a finite maximal chain between a and b. If a locally finite partially ordered set P has the least element and all maximal chains in P between fixed endpoints have the same order, then we say that P is graded. In this case we define the height $h(a)$ of an element of P as the order of a maximal chain from the least element of P to a, minus one. For elements a and b, $a > b$, of a partially ordered set P we write $a \rhd b$ or $b \lhd a$ (a covers b or b is covered by a) if $a \geqslant c > b$ implies $a = c$ for every element $c \in P$.

In the whole paper G_1, G_2 are graphs, P is a partially ordered set and if f: $G_1 \times G_2 \to C(P)$ is a graph isomorphism, then we denote elements of P by $f(a, b)$, where $a \in G_1$, $b \in G_2$. This is correct since f is a bijection. Note that if $d(x, y) < \infty$, $x, y \in G_1 \times G_2$, then $d(x, y) = d(f(x), f(y))$.

Theorem 1. *Let f: $G_1 \times G_2 \to C(P)$ be a graph isomorphism. The there exist partially ordered sets P_1, P_2 such that G_i is graph isomorphic to $C(P_i)$, $i = 1, 2$.*

Proof. Let $a_0 \in G_1$, $b_0 \in G_2$, If

$$P_1 = \{f(a, b_0), a \in G_1\}, \quad P_2 = \{f(a_0, b), b \in G_2\},$$

then $P_i \subset P$, $i = 1, 2$, hence P_i are partially ordered and it is easy to see that G_i and $C(P_i)$ are graph isomorphic.

Lemma 1. (Kotzig [2]). *Let a graph G be a direct product of the graphs G_1, G_2, let $a, b \in G_1$, $c, d \in G_2$. Then*

$$d((a, c), (b, d)) = d(a, b) + d(c, d).$$

Definition 1. *Let f: $G_1 \times G_2 \to C(P)$ be a graph isomorphism. We say that f has the property $\square$ if for every* $a_1, a_2 \in G_1$, $b_1, b_2 \in G_2$, $d(a_1, a_2) = d(b_1, b_2) = 1$,

$$f(a_1, b_1) \lhd f(a_2, b_1) \text{ implies } f(a_1, b_2) \lhd f(a_2, b_2)$$

and

$$f(a_1, b_1) \lhd f(a_1, b_2) \text{ implies } f(a_2, b_1) \lhd f(a_2, b_2).$$

Lemma 2. *Every graph isomorphism f: $G_1 \times G_2 \to C(L)$, where L is a lattice, has the property $\square$.*

Proof. $d(a_1, a_2) = d(b_1, b_2) = 1$ and $f(a_1, b_1) \lhd f(a_2, b_1)$. Since by Lemma 1 $d(f(a_2, b_1), f(a_2, b_2)) = 1$, $d(f(a_2, b_2), f(a_1, b_2)) = 1$, $d(f(a_1, b_2), f(a_1, b_1)) = 1$ and L is a lattice, there must be $f(a_1, b_2) \lhd f(a_2, b_2)$.

In the same way the second implication of Definition 1 can be proved.

Lemma 3. *Every graph isomorphism f: $G_1 \times G_2 \to C(P)$, where P is a graded partially ordered set, has the property $\square$.*

Proof Let $d(a_1, a_2) = d(b_1, b_2) = 1$, $a_1\ a_2 \in G_1$, $b_1, b_2 \in G_2$ and let

$$f(a_1, b_1) \lhd f(a_2, b_1). \tag{1}$$

Since $d(f(a_1, b_2), f(a\ , b_2)) = 1$ there is either

$$f(a_1, b_2) \lhd f(a_2, b_2) \text{ or } f(a_2, b_2) \lhd f(a_1, b_2). \tag{2}$$

Let us suppose that

$$f(a_2, b_2) \lhd f(a_1, b_2). \tag{3}$$

Since $d(f(a_1, b_1), f(a_1, b_2)) = d(f(a_2, b_1), f(a_2, b_2)) = 1$ and P is a partially ordered set, it is easy to check that

$$f(a_1, b_1) \lhd f(a_1, b_2) \text{ and } f(a_2, b_2) \lhd f(a_2 b_1). \tag{4}$$

From (3) and (4) it follows that

$$h(f(a_1, b_1)) = h(f(a_2, b_2)) \text{ and } h(f(a_2, b_1)) = h(f(a_1, b_2)) \tag{5}$$

Let the least element of the partially ordered set P be $f(a_0, b_0)$. Since $h(f(a, b)) = d(f(a, b), f(a_0, b_0)) = d((a, b), (a_0, b_0))$, by Lemma 1 and by (5) we have

$$d(a_1, a_0) + d(b_1, b_0) = d(a_2, a_0) + d(b_2, b_0),$$
$$d(a_2, a_0) + d(b_1, b_0) = d(a_1, a_0) + d(b_2, b_0).$$

From these equalities it follows that

$$d(a_1, a_0) - d(a_2, a_0) = d(a_2, a_0) - d(a_1, a_0).$$

Hence we have

$$d(a_1, a_0) = d(a_2, a_0). \tag{6}$$

Moreover, on the basis of $d(a_1, a_2) = 1$ we have either $f(a_1, b_0) \lhd f(a_2, b_0)$ or $f(a_2, b_0) \lhd f(a_1, b_0)$. If $f(a_1, b_0) \lhd f(a_2, b_0)$, then by Lemma 1 $d(a_2, a_0) = h(f(a_2, b_0)) = h(f(a_1, b_0)) + 1 = d(a_1, a_0) + 1$, which contradicts (6). Analogously $f(a_2, b_0) \lhd f(a_1, b_0)$ leads to contradiction, too.

Since supposition (3) does not hold, from (2) there follows the assertion. The second implication of Definition 1 can be proved in the same way.

Lemma 4. *Let a graph isomorphism f: $G_1 \times G_2 \to C(P)$ have the property $\square$. Then for any $a_1, a_2 \in G_1$, $b_1, b_2 \in G_2$, $d(a_1, a_2) < \infty$, $d(b_1, b_2) < \infty$*

$$f(a_1, b_1) \lhd f(a_2, b_1) \textit{ implies } f(a_1, b_2) \lhd f(a_2, b_2)$$

and

$$f(a_1, b_1) \lhd f(a_1, b_2) \textit{ implies } f(a_2, b_1) \lhd f(a_2, b_2).$$

Proof. We prove the first implication. If $d(b_1, b_2) = 1$, then the assertion follows by Definition 1, because $f(a_1, b_1) \lhd f(a_2, b_1)$ implies $d(a_1, a_2) = 1$. For the second part of the induction we assume that the assertion is true for $d(b_1, b_2)=k$. If $d(b_1, b_2)=k+1$, then there exists a path

$$b_1=c_0, c_1, \ldots, c_{k+1}=b_2, c_i \in G_2, 0 \leqslant i \leqslant k+1.$$

Since $d(b_1, c_k)=k$, by the induction hypothesis $f(a_1, c_k) < f(a_2, c_k)$. But $d(c_k, c_{k+1}) = d(a_1, a_2)=1$ and the graph isomorphism f has the property □, which yields $f(a_1, b_2) < f(a_2, b_2)$.

The second statement of Lemma 4 can be proved analogosly.

Lemma 5. *Let a graph isomorphism f: $G_1 \times G_2 \to C(P)$ have the property* □ *Let*

$$f(a_1, b_1) \lhd f(a_2, b_2) \lhd \ldots \lhd f(a_{k-1}, b_{k-1}) \lhd f(a_{k-1}, b_k), \tag{7}$$

$k \geqslant 3$ *be a chain in the partially ordered set P. Then in P there exists a chain*

$$f(a_1, b_1)=c_1 \lhd c_2 \lhd \ldots \lhd c_k=f(a_{k-1}, b_k)$$

such that $c_2=f(a_1, d)$ *for some* $d \in G_2$.

Proof. Let $k=3$. In the case $a_1=a_2$ there is nothing to prove. If $a_1 \neq a_2$, then $f(a_1, b_1) \lhd f(a_2, b_2)$ implies $b_1=b_2$. Applying now the preceding Lemma we infer that

$$f(a_1, b_1) \lhd f(a_2, b_1) \text{ implies } f(a_1, b_3) \lhd f(a_2, b_3)$$

and

$$f(a_2, b_3) \lhd f(a_2, b_3) \text{ implies } f(a_1, b_2) \lhd f(a_1, b_3),$$

which means

$$f(a_1, b_1)=f(a_1, b_2) \lhd f(a_1, b_3) \lhd f(a_2, b_3).$$

Assume, as usual, for the second part of the induction that the statement of the Lemma is true for $k=n-1$. If $k=n$, then the last three elements of the chain (7) are

$$f(a_{n-2}, b_{n-2}) \lhd f(a_{n-1}, b_{n-1}) \lhd f(a_{n-1}, b_n).$$

If $a_{n-2}=a_{n-1}$, then by the induction hypothesis there exists a chain with the required property. If $a_{n-2} \neq a_{n-1}$, then $b_{n-2}=b_{n-1}$ and analogously as in the first part of this proof it is easy to show that

$$f(a_{n-2}, b_{n-2})=f(a_{n-2}, b_{n-1}) \lhd f(a_{n-2}, b_n) \lhd f(a_{n-1}, b_n).$$

The chain $f(a_1, b_1) \lhd \ldots \lhd f(a_{n-2}, b_{n-2}) \lhd f(a_{n-2}, b_n)$ is of the length $n-2$. Using the induction hypothesis, we can find a chain

$$f(a_1, b_1)=c_1 \lhd c_2 \lhd \ldots \lhd c_{n-1}=f(a_{n-2}, b_n) \lhd f(a_{n-1}, b_n)$$

such that $c_2=f(a_1, d)$ and the induction is completed.

Lemma 6. *Let a graph isomorphism* f: $G_1 \times G_2 \to C(P)$ *have the property* $\square$. *If*

$$f(a_1, b_1) \lhd (a_2, b_2) \lhd \ldots \lhd f(a_{n-1}, b_{n-1}) \lhd f(a_n, b_n) = f(a_1, b_n), \tag{8}$$

$n > 1$, then $a_i = a_1$ for all i, $1 \leqslant i \leqslant n$.

Proof. We observe that $a_i \neq a_{i+1}$ for all i, $1 \leqslant i \leqslant n-1$ implies $b_i = b_{i+1}$ for all i. This means that $b_1 = b_n$, which is impossible. We conclude that there exists j, $1 \leqslant j \leqslant n-1$ such that

$$a_j = a_{j+1}. \tag{9}$$

Clearly, the statement of the Lemma is true for $n = 3$. Assume for the second part of the induction that the assertion is true for $k < n$. If $j = n-1$ (see(9)), then $a_{n-1} = a_n = a_1$. The chain

$$f(a_1, b_1) < f(a_2, b_2) < \ldots < f(a_{n-1}, b_{n-1}) = f(a_1, b_{n-1})$$

has $n-1$ elements and by the induction hypothesis $a_i = a_1$ for all i, $1 \leqslant i \leqslant n-1$. Let us suppose that

$$a_j = a_{j+1}, \quad j < n-1.$$

We divide the chain (8) into two parts.

$$f(a_1, b_1) \lhd \ldots \lhd f(a_j, b_j) \lhd f(a_{j+1}, b_{j+1}) = f(a_j, b_{j+1}), \tag{10}$$

$$f(a_{j+2}, b_{j+2}) \lhd \ldots \lhd f(a_n, b_n) = f(a_1, b_n). \tag{11}$$

Applying now the preceding Lemma on the chain (10) we obtain the chain (12) with $j+1$ members

$$f(a_1, b_1) \lhd f(a_1, d) \lhd \ldots \lhd f(a_j, b_{j+1}). \tag{12}$$

The chain

$$f(a_1, d) \lhd \ldots \lhd f(a_j, b_{j+1}) \lhd f(a_{j+2}, b_{j+2}) \lhd \ldots \lhd f(a_n, b_n) = f(a_1, b_n), \tag{13}$$

which we obtain from (12) by omitting the least element and from (11), has $n-1$ elements. Using the induction hypothesis we have $a_1 = a_j = a_{j+2} = \ldots = a_{n-1}$. The chain (10) has $j+1$ elements ($j+1 < n$) and $a_j = a_1$, hence by the inductin hypothesis $a_1 = a_2 = a_3 \ldots = a_{j-1}$.

Lemma 7. *Let a graph isomorphism* f: $G_1 \times G_2 \to C(P)$ *have the property* $\square$, *let* P *have a locally finite length, let* $f(a, b) < f(c, d)$. *Then there exist elements*

$$z_0, z_1, \ldots, z_j \in G_2, \quad z_{j+1}, \ldots, z_n \in G_1$$

such that

$$f(a, b) = f(a, z_0) \lhd f(a, z_1) \lhd \ldots \lhd f(a, z_j) \lhd f(z_{j+1}, d) \lhd \ldots \lhd f(z_n, d) = f(c, d).$$

Pr of We pro ed by ınduction through $n-d(f(a, b)\ f(c, d))$. If $n=1$, the on ıs obvıous For the second part of the induction a sume th t t e statem nt of Lemma holds for $d(f(a, b), f(c, d))=n-1$. If $d(f(a, b), f(c, d))\ \ n$, then th re exists a chaın

$$f(a, b)-f(x_0, y_0)\lhd f(x_1, y_1)\lhd \ldots f(x_n, y_n)=f(c, d) \tag{14}$$

1. If $a=x_1$, then using the induction hypothesis we infer that there exist elements $z_0, \ldots, z_j \in G_2, z_{j+1}, \ldots, z_{n\ 1} \in G_1$ such that

$$f(a, y_1)=f(a, z_0)\lhd \ldots \lhd f(a, z_j)\lhd f(z_{j+1}, d)\lhd \ldots \lhd f(z_{n\ 1}, d)=f(c, d).$$

If we denote the elements of this chain by $c_0, \ldots, c_{n\ 1}$, then $\{f(a, b), c_1, \ldots, c_{n\ 1}\}$ is a chain with the required property.

2. If $a\neq x_1$, and $x_i\neq x_{i+1}$ for all i, $0\leqslant i\leqslant$ $<n-1$, then $y_i=y_{i+1}$ for all $i, 0\leqslant i\leqslant n-1$, hence $b=y_0=y_1\ =\ \ldots-d$. If we denote b as z_0 and x_i'as z_i for all i, $1\leqslant i\leqslant n$, then the chain (14) has the required property.

If $a\neq x_1$ and there exists i, $1\leqslant i\leqslant n-1$ such that $x_i=x_{i+1}$, then we divide the chaın (14) into the chains

$$f(a, b)=f(x_0, y_0)\lhd \ldots \lhd f(x_i, y_i)\lhd f(x_i, y_{i+1}) \tag{15}$$

$$f(x_{i+2}, y_{i+2})\lhd \ldots \lhd f(x_n, y_n)=f(c, d) \tag{16}$$

By Lemma 5 there exists a chain

$$f(a, b)\lhd f(a, t)\lhd \ldots \lhd f(x_i, y_{i+1}), \tag{17}$$

which has the length $i+1$. The elements of the chain (16) and (17) build a chaın of the length n between $f(a, b)$ and $f(c, d)$ and as in the part 1 of this proof we can find a chain between $f(a, b)$, $f(c, d)$ with the required property.

Theorem 2. ([1]). *Let $(M, \leqslant)$ be a quasiordered set. If Θ_1, Θ_2 are equivalences on the set M such that*

(i) $\Theta_1\cap\Theta_2=\omega$, *where ω is the least equivalence on M,*
(ii) $\Theta_1\cup\Theta_2=\iota$, *where ι is the greatest equivalence on M,*
(iii) $\Theta_1\circ\Theta_2=\Theta_2\circ\Theta_1$.
(iv) $c_1\Theta_i d_1$, $c_2\Theta_i d_2$, $d_1\Theta_j d_2$, $i\neq j$, $c_1\leqslant c_2$ *implies* $d_1\leqslant d_2$,
then $(M, \leqslant)$ is isomorphic to the direct product of the quasiordered sets M/Θ_1 and M/Θ_2. ($[a_1]\Theta_i\leqslant[a_2]\Theta_i$ *iff* $b_1\leqslant b_2$ *for some* $b_i\in M$, $b_i\Theta_i a_i$, $i=1, 2$)

Corollary 1. *If $(M, \leqslant)$ is a partially ordered set or a lattice and Θ_1, Θ_2 are equivalences on the set M with the properties* (i), (ii), (iii), (iv) *of Theorem 2, then M/Θ_1, M/Θ_2 are partially ordered sets, respectively lattices.*

Theorem 3. *Let a graph isomorphism f: $G_1\times G_2\rightarrow C(P)$ have the property* $\Box$, *let the partially ordered set P have a locally finite length. Then P is isomorphic to the direct product of the partially ordered sets P_1, P_2 such that $C(P_i)$ is graph isomorphic to G_i, $i=1,2$.*

Proof. Clearly, the relations

$$\Theta_1=\{(f(a,b),f(a,d)),\ a\in G_1,\ b,\ d\in G_2\},$$
$$\Theta_2=\{(f(a,b),f(c,b)),\ a,\ c\in G_1,\ b\in G_2\}$$

are equivalence relations on P. It is easy to see that Θ_1, Θ_2 have the properties (i), (ii), (iii).

Let $c_1, c_2, d_1, d_2\in P$, $c_1=f(a,b)$, $c_2=f(c,d)$ and $f(a,b)\leqslant f(c,d)$. We suppose also $c_1\Theta_1 d_1$, $c_2\Theta_1 d_2$ and $d_1\Theta_2 d_2$. Then $d_1=f(a,x)$, $d_2=f(c,x)$ for some x G_2.

In the case $a=c$ we have $d_1=d_2$. If $a\neq c$, then $f(a,b)<f(c,d)$ and by Lemma 7 there exist elements $z_0, \ldots,, z_j\in G_2$, $z_{j+1}, \ldots, z_n\in G_1$ such that

$$f(a,b)=f(a,z_0)\lhd f(a,z_1)\lhd\ldots\lhd f(a,z_j)\lhd f(z_{j+1},d)\lhd\ldots\lhd f(z_n,d)=f(c,d).$$

If $z_{j+1}\neq a$, then $z_j=d$ and $f(a,d)<f(c,d)$

If $z_{j+1}=a$, then also $f(a,d)<f(c,d)$.

Since P has a locally finite length, there exists a finite maximal chain connecting $f(a,d)$ and $f(c,d)$. By Lemma 6 the chain has the following form

$$f(a,d)=f(x_0,d)\lhd f(x_1,d)\lhd\ldots\lhd f(x_n,d)=f(c,d)$$

Since the graph isomorphism has the property □, we have

$$f(a,x)=f(x_0,x)\lhd f(x,x)\lhd\ldots\lhd f(x_n,x)=f(c,x)$$

(see Lemma 4), hence $d_1\leqslant d_2$.

Analogously the second condition of (iv) of Theorem 2 can be proved.

If we denote P/Θ_i as P_i, then from Theorem 2 and Corollary 1 it follows that P is isomorphic to $P_1\times P_2$, P_1, P_2 are partially ordered sets.

Let $a_0\in G_1$, $b_0\in G_2$ be some fixed elements. We show that the mapping

$$g\colon C(P_1)\to C(\{f(a,b_0),\ a\in G_1\})$$

defined by $g([f(a,b)]\Theta_1)=f(a,b_0)$ is a graph isomorphism. If $[f(a,c)]\Theta_1\lhd\lhd[f(b,d)]\Theta_1$, then there exist elements $x, y\in G_2$ such that $f(a,x)\lhd f(b,y)$ (see Theorem 2) But $a\neq b$, hence $x=y$ and $d(a,b)=1$. From $f(a,x)\lhd f(b,x)$ and from the property □ of the graph isomorphism f it follows that $f(a,b_0)\lhd f(b,b_0)$ (see Lemma 4).

If $f(a,b_0)\lhd f(b,b_0)$, then $[f(a,b_0)]\Theta<[f(b,b_0)]\Theta_1$. It is easy to see that $C(\{f(a,b_0),\ a\in G_1\})$ is graph isomorphic to G_1. Hence $C(P/\Theta_1)$ is graph isomorphic to G_1.

Analogously $C(P/\Theta_2)$ and G_2 are graph isomorphic.

Note that the mapping g is also an isomorphism of partially ordered sets P_1 and $\{f(a, b_0),\ a\in G_1\}$.

Corollary 2. *If the supposition of the preceding Theorem are fulfilled and* $a_0\in G_1$, $b_0\in G_2$, *then*

$$A_1=\{f(a, b_0), a\in G_1\}, \quad A_2=\{f(a_0, b), b\in G_2\}$$

are partially ordered sets and A_i *are isomorphic to* P_i.

Theorem 4. *If* f: $G_1\times G_2\to C(P)$ *is a graph isomorphism and* P *is a graded partially ordered set (*P *is a lattice of a locally finite length), then there exist graded partially ordered sets* P_1, P_2 *(lattices* P_1, P_2 *of a locally finite length) such that* P *is isomorphic to* $P_1\times P_2$ *and* G_i *is graph isomorphic to* $C(P_i)$, $i=1, 2$.

Proof. The statement of the Theorem is an easy consequence of Theorem 3, Corollary 1, Lemma 3, respectively Lemma 2.

Theorem 5. *Let a lattice* H *be a direct product of lattices* A_1, A_2 *and let there exist a graph isomorphism* f: $C(H)\to C(L)$, *where* L *is a lattice of a locally finite length. Then* L *is a direct product of lattices* B_1, B_2 *such* $C(A_i)$ *is graph isomorphic to* $C(B_i)$, $i=1, 2$.

Proof. Note that $C(H)=C(A_1)\times C(A_2)$ and apply the preceding Theorem.

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Received December 7, 1978

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ОБ ОРИЕНТИРОВАНИИ ПРЯМОГО ПРОИЗВЕДЕНИЯ ГРАФОВ

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Резюме

Граф, вершины которого являются элементами частично упорядоченного множества P и ребра суть те пары $\{a, b\}$, $a, b\in P$, где a покрывает b или b покрывает a, называется покрывающим графом $C(P)$ частично упорядоченного множества P. Пусть P является решеткой локально конечной длины, или P является частично упорядоченным множеством удовлетворяющим условию Дедекинда, тогда верно следующие уртверждение: Если G_1, G_2 графы $i f$: $G_1\times G_2\to C(P)$ изоморфизм графов, то $P=P_1\times P_2$ P_1, P_2 являются частично упорядоченными множествами и графы G_i и $C(P)$ суть изоморфны, $i=1, 2$.