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ON THE FAMILY OF FUNCTIONS WITH A CLOSURE OF ITS GRAPH OF MEASURE ZERO

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ABSTRACT. In this paper I consider the family S of all functions $f\colon \mathbb{R} \to \mathbb{R}$ with a closure of its graph of Lebesgue measure zero. In the first part I make known a full characterization of the family S. In the second part of this paper I observe that in like manner it is possible to characterize the family S_1 of functions $f\colon \mathbb{R} \to \mathbb{R}$ with a closure of its graph of the first category.

I

Let $\mathbb{R}$ be a real line. We will consider functions $f\colon \mathbb{R} \to \mathbb{R}$. Let $G(f)$ be the graph of f and $\mathrm{Cl}(H)$ be the closure of a set H. Symbols m and m_2 denote Lebesgue measure on $\mathbb{R}$ and $\mathbb{R}^2$, respectively.

Denote by S the family $\{f\colon \mathbb{R} \to \mathbb{R} :\ m_2(\mathrm{Cl}\,(G(f))) = 0\}$.

Let $K^+(f,x)$ and $K^-(f,x)$ be the right and the left cluster sets respectively for each x of the domain of f. Moreover, let

$$K^+(f,-\infty) = \{y \in \overline{\mathbb{R}};\ \text{there exists } x_n \to -\infty \text{ such that } \lim_{n\to\infty} f(x_n) = y\}$$

and

$$K^-(f,\infty) = \{y \in \overline{\mathbb{R}};\ \text{there exists } x_n \to \infty \text{ such that } \lim_{n\to\infty} f(x_n) = y\}.$$

Put $K(f,x) = K^+(f,x) \cup K^-(f,x)$ and $S(f,x) = K(f,x) \cup \{f(x)\}$.

Remark 1.1. $S(f,x)\setminus\{-\infty;+\infty\} = \big[\mathrm{Cl}\,(G(f))\big]_x$ for each $x \in \mathbb{R}$, where $[H]_x$ denotes the section of a set $H \subset \mathbb{R}^2$, i.e. the set

$$\{y \in \mathbb{R} :\ (x,y) \in H\}.$$

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Denote by $C(f)$ and $D(f)$ the set of all continuity points and the set of all discontinuity points of f, respectively.

Let $S(f) = \{x \in \mathbb{R} : m(K(f,x)) > 0\}$.

THEOREM 1.1. $f \in S \iff m(S(f)) = 0$.

Proof. If $f \in S$, $m_2(\mathrm{Cl}\,(G(f))) = 0$. Then from the Fubini theorem ([10, p. 142, Th. 4]), for almost each $x \in \mathbb{R}$ there is

$$m([\mathrm{Cl}\,(G(f))]_x) = 0.$$

Hence and from Remark 1.1, $m(S(f)) = 0$.

On the other hand, since $\mathrm{Cl}(G(f))$ is a measurable set and $m(S(f)) = 0$, almost each set $[\mathrm{Cl}\,(G(f))]_x$ is of the measure zero; from the Fubini theorem ([10, p. 142, Th. 4]) $f \in S$ and the proof is finished.

The family of all Darboux functions we will denote by D, and let D^* be the family of all f such that for almost each $x \in D(f)$ the set $S(f,x)$ is some interval. It is clear that $D^* \supsetneqq D$.

Since $x \in C(f)$ if and only if $[\mathrm{Cl}\,(G(f))]_x = \{f(x)\}$, we have at once

Remark 1.2. If $f \in D^*$, then $f \in S$ if and only if $m(D(f)) = 0$.

Denote by C_{ae} the family of all almost everywhere continuous functions f. From Theorem 1.1 we have:

COROLLARY 1.1. $C_{ae} \subset S$.

Remark 1.3. There exist functions $f \in S \setminus C_{ae}$; for example – a function of Dirichlet's.

THEOREM 1.2. *The set C_{ae} is closed and nowhere dense in the family S with the uniform convergence metric*

$$\rho(f,g) = \min\{1, \sup_{x\in\mathbb{R}} |f(x) - g(x)|\}.$$

Proof. It is clear that in the sense of the metric ρ the family C_{ae} is closed. We shall show that the set C_{ae} is nowhere dense in S. To this effect fix $f \in C_{ae}$, $\varepsilon > 0$ and let d be a function of Dirichlelt's.

We put $g(x) = f(x) + \frac{\varepsilon}{2} d(x)$. Then g fulfils all required conditions. Really, $g \notin C_{ae}$ and since

$$[\mathrm{Cl}\,(G(g))]_x \subset \left\{0, \tfrac{\varepsilon}{2}\right\} + [\mathrm{Cl}\,(G(f))]_x,$$

$m\big(\big[\mathrm{Cl}\,(G(g))\big]_x\big) = 0$ for almost every $x \in \mathbb{R}$, where for each set A, B we have $A + B = \{x + y : \ x \in A \text{ and } y \in B\}$. Hence $g \in S$.

From Remark 1.2 and Corollary 1.1 we have

COROLLARY 1.2. $S \cap D^* = C_{ae} \cap D^*$ *and* $S \cap D = C_{ae} \cap D$.

Pointwise convergence.

Let A be a family of functions and let $B_1(A)$ be the set of all limits of sequences $(f_n)_{n=1}^{\infty}$ of functions from A convergent at every point.

Since every simple function $s\colon \mathbb{R} \to \mathbb{R}$ belongs to S and every $f\colon \mathbb{R} \to \mathbb{R}$ is the limit of sequence of simple functions s, $B_1(S)$ is the family of all functions $f\colon \mathbb{R} \to \mathbb{R}$.

Denote by C the family of all continuous functions f.

THEOREM 1.3. *The following equality* $B_1(C_{ae}) = B_1(S \cap D)$ *is true.*

P r o o f. Since $S \cap D \subset C_{ae}$, $B_1(C_{ae}) \supset B_1(S \cap D)$. On the contrary, it is known that if $f \in B_1(C_{ae})$, then there exists $g \in B_1(C)$ and there is an F_σ-set A of the measure zero such that

$$\big\{x \in \mathbb{R} : \ f(x) \neq g(x)\big\} \subset A \qquad \text{(s. [11])}.$$

Since A is an F_σ, $A = \bigcup_{n=1}^{\infty} A_n$, where $A_1 \subset A_2 \subset \dots \subset A_n \subset \dots$ and all sets A_n $(n = 1, 2, \dots)$ are closed.

Since $g \in B_1(C)$, there exists a sequence of continuous functions $(g_n)_{n=1}^{\infty}$ such that $\lim_{n\to\infty} g_n = g$ and $|g_n| < n$ for $n = 1, 2, \dots$.

In the first step, for each component (a, b) of the complement of the set A_1, there exists ([2], [6]) function $h^1_{a,b}\colon (a, b) \to \mathbb{R}$ such that:

(a$_1$) $h^1_{a,b} \in B_1(C) \cap C_{ae} \cap D$;

(b$_1$) the set $\big\{x \in (a, b) : \ h^1_{a,b}(x) \neq 0\big\} \subset \mathbb{R} \setminus A$ is of the first category and of measure zero;

(c$_1$) $\overline{\lim}_{x\to a^+} h^1_{a,b}(x) = \overline{\lim}_{x\to b^-} h^1_{a,b}(x) = +\infty$,
$\underline{\lim}_{x\to a^+} h^1_{a,b}(x) = \underline{\lim}_{x\to b^-} h^1_{a,b}(x) = -\infty$;

(d$_1$) $h^1_{a,b} + g_1\big|_{(a,\,b)}$ has Darboux property.

Let

$$f_1(x) = \begin{cases} g_1(x) + h^1_{a,b}(x) & \text{for } x \in (a, b), \text{ where } (a, b) \text{ is a component of } \mathbb{R}\setminus A_1, \\ f(x) & \text{for } x \in A_1. \end{cases}$$

Generally, in the nth step, for each component (c,d) of the set $\mathbb{R}\setminus A_n$ we find ([2], [6]) the function $h^n_{c,d}\colon (c,d)\to\mathbb{R}$ such that:

(a_n) $h^n_{c,d}\in B_1(C)\cap C_{ae}\cap D$;

(b_n) the set $\{x\in(c,d):\ h^n_{c,d}(x)\neq 0\}\subset\mathbb{R}\setminus A$ is of the first category of measure zero and disjoint with all sets of the form $\{x\in(*,\#):\ h^i_{*,\#}(x)\neq 0\}$ for $i<n$, chosen before;

(c_n) $\overline{\lim\limits_{x\to c^+}}\, h^n_{c,d}(x)=\overline{\lim\limits_{x\to d^-}}\, h^n_{c,d}(x)=+\infty$,
$\underline{\lim\limits_{x\to c^+}}\, h^n_{c,d}(x)=\underline{\lim\limits_{x\to d^-}}\, h^n_{c,d}(x)=-\infty$;

(d_n) $h^n_{c,d}+g_n\big|_{(c,d)}$ has Darboux property.

Put

$$f_n(x)=\begin{cases} g_n(x)+h^n_{c,d}(x) & \text{for } x\in(c,d),\\ & \text{where } (c,d) \text{ is a component of } \mathbb{R}\setminus A_n,\\ f(x) & \text{for } x\in A_n.\end{cases}$$

It is easy to verify that for each $n=1,2,\dots, f_n\in S\cap D$ and $\lim\limits_{n\to\infty} f_n=f$.

Uniform convergence.

Since every bounded function is the limit of a uniformly convergent sequence of simple functions, the uniform closure of S contains all bounded functions.

It is known that the limit of a uniformly convergent sequence of functions from $D\cap B_1(C)$ is a function from $D\cap B_1(C)$ ([1]) and the limit of a uniformly convergent sequence of functions from C_{ae} is a function from C_{ae}. Hence the family of functions $S\cap B_1(C)\cap D=D\cap B_1(C)\cap C_{ae}$ is uniformly closed. It is known that the family $B_1\big(B_1(C)\big)\cap D$ is not uniformly closed ([15]). Appropriate is then the question, are families $S\cap D^*$ and $S\cap D$ uniformly closed?

THEOREM 1.4. *The limit of a uniformly convergent sequence of functions from $S\cap D^*$ is a function from $S\cap D^*$.*

P r o o f. Let $(f_n)_n$ be some sequence of functions from $S\cap D^*$ uniformly convergent to f. Since $S\cap D^*\subset C_{ae}$ and the limit of a uniformly convergent sequence of functions from C_{ae} is a function from C_{ae}, $f\in C_{ae}\subset S$. We prove that $f\in D^*$.

Let $x\in\mathbb{R}$ be any point such that $S(f_n,x)$ is some interval for each $n=1,2,\dots$. We will show that $S(f,x)$ is some interval, too.

Suppose that $S(f,x)$ is not any interval. Since $S(f,x)$ is closed, there exists an interval (a, b) such that $S(f,x)\cap(a,b)=\emptyset$ and $(-\infty,a)\cap S(f,x)\neq\emptyset$, $(b,\infty)\cap S(f,x)\neq\emptyset$.

Establish $\varepsilon > 0$ such that $(a+\varepsilon,\, b-\varepsilon)$ is an open interval. Since $(f_n)_{n=1}^{\infty}$ is uniformly convergent, there exists f_m such that $|f_m(u) - f(u)| < \frac{\varepsilon}{2}$ for each $u \in \mathbb{R}$. Then $S(f_m, x) \cap (a+\varepsilon,\, b-\varepsilon) = \emptyset$ and $(-\infty, a+\varepsilon) \cap S(f_m, x) \neq \emptyset$, $(b-\varepsilon, \infty) \cap S(f_m, x) \neq \emptyset$.

This is impossible since $S(f_m, x)$ is some interval. Since the set

$$\{x \in \mathbb{R} :\ S(f,x) \text{ is not any interval}\}$$

is of the measure zero, $f \in S \cap D^*$.

THEOREM 1.5. *There exists a sequence $(f_n)_n$ of functions from $S \cap D$ uniformly convergent to $f \notin S \cap D$.*

P r o o f. The proof is a modification of J. S m í t a l 's constructions from [16].

Let $K \subset [0,1]$ be a Cantor ternary set of the measure zero. Let B be a set containing points 0, 1 and all points from K which are one-hand isolated in K. Then $K \setminus B$ is a G_δ-set.

Let $(P_n)_n$ be a sequence of pairwise disjoint nowhere dense non-empty perfect sets such that for each n, $P_n \subset K \setminus B$ and for each open interval I, if $I \cap K \neq \emptyset$, then I contains some sets P_n. For each n, the set of component intervals of $I \setminus P_n$ with natural ordering is similar to the set of rational numbers contained in the closed unit interval $J = [0,1]$; let h_n be a corresponding isomorphism. Define functions g'_n as follows:

If G is some component of $I \setminus P_n$ and $x \in G$, let $g'_n(x) = h_n(G)$;
if $x \in P_n$, let $g_n(x) = \inf\{g'_n(y) :\ y \in I \setminus P_n \text{ and } y > x\}$.

Now put $g_n = g'_n|P_n$. It is easy to see that each g_n maps the set P_n continuously onto J and that g_n takes on each irrational value from J exactly at one point of P_n.

Let ξ be some irrational number in J; for each n, let a_n be the point of P_n for which $g_n(a_n) = \xi$. Define functions f_n and f as follows:

$$f_n(x) = \begin{cases} \max\left\{\xi - \frac{1}{m}, 0\right\} & \text{if } x = a_m \text{ and } m \le n, \\ g_m(x) & \text{if } x \in P_m \text{ and } x \not\equiv a_1, \dots, a_n, \\ 0 & \text{otherwise,} \end{cases}$$

$$f(x) = \begin{cases} \max\left\{\xi - \frac{1}{m}, 0\right\} & \text{if } x = a_m \text{ for some } m, \\ g_m(x) & \text{if } x \in P_m \text{ and } x \neq a_1, a_2, \dots, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to verify that functions f_n and f have all desired properties (for each $x \in J$, $f(x) \neq \xi$).

Transfinite convergence of type Ω.

The notion of the limit of transfinite sequence of real numbers and of the limit of transfinite sequence of real functions was introduced by Sierpiński ([14]). This idea and some results of [14] were developed in some further papers of Kostyrko [7], Šalát [13] and Lipiński [9].

Let Ω denote the first uncountable ordinal number. The transfinite sequence of real numbers $\{b_\xi\}_{\xi<\Omega}$ is said to be convergent and have the limit b if and only if for each $\varepsilon > 0$ there exists an ordinal number $\alpha < \Omega$ such that $|b - b_\xi| < \varepsilon$ for each $\xi > \alpha$. If $\{b_\xi\}$ has a limit b we write $\lim_{\xi\to\Omega} b_\xi = b$.

The transfinite sequence of functions $f_\xi \colon \mathbb{R} \to \mathbb{R}$, $\xi < \Omega$ is said to be convergent and have the limit function f if and only if for each point x we have $\lim_{\xi\to\Omega} f_\xi(x) = f(x)$.

THEOREM 1.6. *If transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ of functions from S is convergent to f, then $f \in S$.*

Proof. Since $G(f)$ is a separable space contained in $\mathbb{R}^2$, there exists a set $\{(x_n, f(x_n))\}_{n=1}^{\infty}$ dense in $G(f)$. For every $n = 1, 2, \dots$ there exists a countable transfinite number $\xi_n < \Omega$ such that $f_\xi(x_n) = f(x_n)$ for each $\xi > \xi_n$. Let $\xi < \Omega$ be a transfinite number such that $\xi > \xi_n$ for every $n = 1, 2, \dots$. Hence $(x_n, f_\xi(x_n)) = (x_n, f(x_n)) \in G(f_\xi)$ $(n = 1, 2, \dots)$, and

$$\mathrm{Cl}(G(f_\xi)) \supset \mathrm{Cl}(G(f)).$$

If $m_2(\mathrm{Cl}\,(G(f))) > 0$, then $m_2(\mathrm{Cl}\,(G(f_\xi))) > 0$ and $f_\xi \notin S$. This contradiction supplies the proof.

It is known that the family C_{ae} is transfinitely closed and every function $f \in B_1(C)$ is the limit of a transfinite sequence of functions from $D \cap B_1(C)$ ([3]).

We will show the folowing result:

THEOREM 1.7. *If $f \in C_{ae}$, then f is the limit of a transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ of functions from $C_{ae} \cap D$.*

Proof. Since $D(f)$ is an F_σ-set of the first category and of measure zero, there exists (s. [6, the proof of Th. 3]) a function $g_1 \colon \mathbb{R} \to \mathbb{R}$ such that $g_1 \in D \cap B_1(C) \cap C_{ae}$, $g_1 + f \in D$ and $B_1 = \{x \in \mathbb{R} : g_1(x) \neq 0\}$ is an F_σ-set of measure zero disjoint with $D(f)$. In the first step let $f_1 = g_1 + f$.

Generally (from [6, the proof of Th. 3]), in the ξth step ($\xi < \Omega$) there exists a function $g_\xi : \mathbb{R} \to \mathbb{R}$ such that:

$g_\xi \in D \cap B_1(C) \cap C_{ae}$ and $f + g_\xi \in D$;

$B_\xi = \{x \in \mathbb{R} : g_\xi(x) \neq 0\}$ is an F_σ-set of measure zero and such that

$$\Big(\bigcup_{\eta<\xi} B_\eta\Big) \cap B_\xi = \emptyset \quad \text{and} \quad B_\xi \cap D(f) = \emptyset .$$

Let $f_\xi = f + g_\xi$.

Then we can observe easily that $f_\xi \in D \cap C_{ae}$ for every $\xi < \Omega$ and $\lim_{\xi \to \Omega} f_\xi = f$.

Operations on functions from S.

THEOREM 1.8. *Let $f \in S$ be a locally bounded function. If $g \in C$, and for every compact set F of the measure zero the set $g(F)$ is of the measure zero, then the superposition $g \circ f \in S$.*

Proof.

If $f \in S$, then $m_2(\mathrm{Cl}\,(G(f))) = 0$ and consequently $m([\mathrm{Cl}\,(G(f))]_x) = 0$ for almost everywhere $x \in \mathbb{R}$. Since f is locally bounded, for every $x \in \mathbb{R}$ the set $[\mathrm{Cl}\,(G(f))]_x = S(f,x)$ is compact as a bounded and closed set. The function g is continuous, so

$$[\mathrm{Cl}\,(G(g \circ f))]_x = g([\mathrm{Cl}\,(G(f))]_x), \tag{1}$$

and for $x \notin S(f)$ we have $m([\mathrm{Cl}\,(G(g \circ f))]_x) = 0$. Hence and from Theorem 1.1, $g \circ f \in S$.

THEOREM 1.9. *Let $g \in C$ and for every closed set F of the measure zero the set $g(F)$ is of the measure zero and sets $K^+(g,-\infty)$, $K^-(g,\infty)$ are of the measure zero. Then $g \circ f \in S$ for each $f \in S$.*

Proof. The proof of our theorem is similar to the proof of Theorem 1.8. Suffice to observe only that the condition (1) in the proof of Theorem 1.8 can be replaced by the condition

$$[\mathrm{Cl}\,(G(g \circ f))]_x \subset g([\mathrm{Cl}\,(G(f))]_x) \cup K^+(g,-\infty) \cup K^-(g,\infty) \tag{2}$$

and the proof is finished.

Now we will consider an algebraic structure of S.

THEOREM 1.10. *Let the function $g \in S$ be such that the set*

$$E(g) = \{x \in \mathbb{R} : \ S(g,x) \ \textit{is uncountable or containing} \ \infty \ \textit{or} \ -\infty\}$$

is of the measure zero. Then $f + g \in S$ for each $f \in S$.

Proof. It will do to observe that for each $x \in \mathbb{R} \setminus E(g)$ we have

$$\left[\mathrm{Cl}\left(G(f+g)\right)\right]_x \subset \bigcup_{t \in [\mathrm{Cl}(G(g))]_x} \left[\mathrm{Cl}\left(G(f)\right)\right]_x + t,$$

where $A+t = \{x+t : \ x \in A\}$. The last set on the right-hand side is of the measure zero as the set $\left[\mathrm{Cl}\left(G(g)\right)\right]_x$ is countable and the measure of $\left[\mathrm{Cl}\left(G(f)\right)\right]_x$ is also zero. The proof is finished.

THEOREM 1.11. *If the Continuum-hypothesis is true, then there exist functions $f, g \in S$ such that $f + g \notin S$.*

Proof. Let $K \subset [0,1]$ be the ternary Cantor set. It is known that K is of the measure zero and $K + K = [0,2]$ (s. [8, p. 50, ex. 13]).

Let $\{A_\xi\}_{\xi<\Omega}$ be a family of pairwise disjoint c-dense in $\mathbb{R}$ sets such that $\bigcup\limits_{\xi<\Omega} A_\xi = \mathbb{R}$ (s. [15]). If the Continuum-hypothesis is true, we can arrange all numbers from K in a transfinite sequence $\{c_\xi\}_{\xi<\Omega}$ with $c_\alpha \neq c_\beta$ for $\alpha \neq \beta$, $\alpha, \beta < \Omega$. Define a function $f : \mathbb{R} \to \mathbb{R}$ as $f(x) = c_\xi$ for each $x_\xi \in A_\xi$ $(\xi < \Omega)$.

Next, from Sierpiński's theorem ([15]), each of the sets A_ξ $(\xi < \Omega)$ can be decomposed into continuum of pairwise disjoint, c-dense in $\mathbb{R}$ sets A_ξ^η $(\eta < \Omega)$ so that $A_\xi = \bigcup\limits_{\eta<\Omega} A_\xi^\eta$.

For each $\xi < \Omega$ define a function $g_\xi : A_\xi \to K$ as $g_\xi(x) = c_\eta$ for every $x \in A_\xi^\eta$ $(\eta < \Omega)$.

For each $x \in \mathbb{R}$ put $g(x) = g_\xi(x)$ if $x \in A_\xi$ $(\xi < \Omega)$. Then for each $x \in \mathbb{R}$ we have

$$\left[\mathrm{Cl}\left(G(f)\right)\right]_x = \left[\mathrm{Cl}\left(G(g)\right)\right]_x = K,$$

what shows that $f, g \in S$. Since $\left[\mathrm{Cl}\left(G(f+g)\right)\right]_x = K + K = [0,2]$ for every $x \in \mathbb{R}$, $f + g \notin S$.

THEOREM 1.12. *If the Continuum-hypothesis is true, then there exist functions $h, k \in S$ such that $h \cdot k \notin S$.*

Proof. Let $f, g : \mathbb{R} \to \mathbb{R}$ be functions constructed in the proof of Theorem 1.11. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be the map of the form $\varphi(x) = e^x$ for every $x \in \mathbb{R}$.

Since pairs of functions (φ, f) and (φ, g) fulfil conditions of Theorem 1.9, functions $h = \varphi \circ f$ and $k = \varphi \circ g$ belong to S. If the function $h \cdot k$, where $(h \cdot k)(x) = e^{f(x)+g(x)}$ for each $x \in \mathbb{R}$, belongs to S, then from Theorem 1.9 the function $f+g = \ln(h \cdot k)$ belongs to S, what is not possible. This contradiction finishes the proof.

Remark 1.4. If functions f and g fulfil conditions of Theorem 1.10 and $\varphi(x) = e^x$, $x \in \mathbb{R}$, then pairs of functions (φ, f) and (φ, g) fulfil assumptions of Theorem 1.9 and functions $s = \varphi \circ f$, $t = \varphi \circ g$ belong to S and $s \cdot t$ belongs to S.

THEOREM 1.13. *If $f, g \in S$ then $\max(f,g) \in S$ and $\min(f,g) \in S$.*

Proof. Let $A = \{x \in \mathbb{R} : f(x) \le g(x)\}$ and $B = \{x \in \mathbb{R} : f(x) > g(x)\}$. Then

$$\max(f,g)(x) = \begin{cases} g(x) & \text{for } x \in A, \\ f(x) & \text{for } x \in B, \end{cases} \qquad \min(f,g)(x) = \begin{cases} f(x) & \text{for } x \in A, \\ g(x) & \text{for } x \in B. \end{cases}$$

Since

$$\begin{aligned} &\mathrm{Cl}\big(G(\max(f,g))\big) \subset \mathrm{Cl}\big(G(f)\big) \cup \mathrm{Cl}\big(G(g)\big) \quad \text{and} \\ &\mathrm{Cl}\big(G(\min(f,g))\big) \subset \mathrm{Cl}\big(G(f)\big) \cup \mathrm{Cl}\big(G(g)\big), \end{aligned}$$

for almost each $x \in \mathbb{R}$

$$m\big(\big[\mathrm{Cl}\,\big(G(\max(f,g))\big)\big]_x\big) = m\big(\big[\mathrm{Cl}\,\big(G(\min(f,g))\big)\big]_x\big) = 0,$$

and hence $\max(f,g) \in S$ and $\min(f,g) \in S$.

Remark 1.5. (On the inverse function). If $f \in S$ has the inverse function f^{-1} then $f^{-1} \in S$.

Really, since $G(f^{-1})$ is symmetrical to $G(f)$ with respect to the graph of the function $\alpha(x) = x$, so $\mathrm{Cl}\big(G(f^{-1})\big)$ is symmetrical to $\mathrm{Cl}\big(G(f)\big)$ with respect to the graph of α. Hence, if $f \in S$, then $m_2\big(\mathrm{Cl}\,\big(G(f)\big)\big) = 0 = m_2\big(\mathrm{Cl}\,\big(G(f^{-1})\big)\big)$ and $f^{-1} \in S$.

At the end it is worth making a note that there exists a function $f \colon \mathbb{R} \to \mathbb{R}$ such that $f = f^{-1} \in S \cap C_{ae} \setminus C$ ([4]).

II

Let $S_1 = \{f\colon \mathbb{R} \to \mathbb{R} :\ \mathrm{Cl}(G(f))$ is of the first category$\}$ and for a given function $f\colon \mathbb{R} \to \mathbb{R}$ let

$$S_1(f) = \{x \in \mathbb{R} :\ K(f,x) \text{ is of the second category set}\}.$$

THEOREM 2.1. *$f \in S_1 \iff S_1(f)$ is of the first category set.*

P r o o f . The proof is analogous to the proof of Theorem 1.1, where we make use of Ulam-Kuratovski's theorem ([12, p. 98, Th. 15.1]).

Denote by D_1 the family of all functions $f\colon \mathbb{R} \to \mathbb{R}$ for which the set $\{x \in \mathbb{R} :\ S(f,x)$ is not connected$\}$ is of the first category. Obviously $D \subset D_1$.

Let P denote the family of all pointwise discontinuous functions f, i.e. such that $D(f)$ is of the first category.

It is evident:

R e m a r k 2.1. $P \subset S_1$.

R e m a r k 2.2. If $f \in D_1$, then $f \in S_1 \iff f \in P$.

R e m a r k 2.3. $S_1 \cap D_1 = P \cap D_1$ and $S_1 \cap D = P \cap D$.

For the family S_1 there are true all analogical theorems to theorems about S from the first part of this paper. All these theorems, without an analogous with Theorem 1.3, have similar proofs to proofs of suitable theorems from the first part.

THEOREM 2.2. (s. Th. 1.2) *The family P is closed and nowhere dense in S_1 with the uniform convergence metric*

$$\rho(f,g) = \min\{1, \sup_{x \in \mathbb{R}} |f(x) - g(x)|\}.$$

THEOREM 2.4. (s. Th. 1.4) *The limit of a uniformly convergent sequence of functions from $S_1 \cap D_1$ is a function from $S_1 \cap D_1$.*

THEOREM 2.5. (s. Th. 1.5) *There exists a sequence $(f_n)_n$ of functions from $S_1 \cap D$ uniformly convergent to $f \notin S_1 \cap D$.*

THEOREM 2.6. (s. Th. 1.6) *If a transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ of functions from S_1 is convergent to f, then $f \in S_1$.*

THEOREM 2.7. (s. Th. 1.7) *If $f \in P$, then f is the limit of a transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ of functions from $P \cap D$.*

THEOREM 2.8. (s. Th. 1.8) *Let $f \in S_1$ be a locally bounded function. If $g \in C$ and for every compact of the first category set F, $g(F)$ is of the first category set, then $g \circ f \in S_1$.*

THEOREM 2.9. (s. Th. 1.9) *Let $g \in C$ and for every closed of the first category set F, $g(F)$ is of the first category and $K^+(g,-\infty)$, $K^-(g,\infty)$ are of the first category sets. Then $g \circ f \in S_1$ for each $f \in S_1$.*

THEOREM 2.10. (s. Th. 1.10) *Let $g \in S_1$ be such that*

$$E(g) = \{x \in \mathbb{R}: \ S(g,x) \ \textit{is uncountable or containing} \ \infty \ \textit{or} \ -\infty\}$$

is of the first category set. Then $f+g \in S_1$ for each $f \in S_1$.

THEOREM 2.11. (s. Th. 1.11) *If the Continuum-hypothesis is true, then there exist functions $f,g \in S_1$ such that $f+g \notin S_1$.*

THEOREM 2.12. (s. Th. 12) *If the Continuum-hypothesis is true, then there exist functions $h,k \in S_1$ such that $h \cdot k \notin S_1$.*

Remark 2.4. If function f and g fulfil conditions of Theorem 2.9 and $\varphi(x) = e^x$, $x \in \mathbb{R}$, then pairs of functions (φ, f) and (φ, g) fulfil conditions of Theorem 2.8 and functions $s = \varphi \circ f$, $t = \varphi \circ g$ belong to S_1 and $s \cdot t \in S_1$.

THEOREM 2.13. (s. Th. 1.13) *If $f,g \in S_1$, then $\max(f,g) \in S_1$ and $\min(f,g) \in S_1$.*

Remark 2.5. If $f \in S_1$ has the inverse function f^{-1}, then $f^{-1} \in S_1$.

An analogical theorem to Theorem 1.3 is the following:

THEOREM 2.3. $B_1(P) = B_1(S_1 \cap D)$.

(It was proved in [5].)

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