Tibor Šalát; Jana Tomanová

On the product of divisors of a positive integer

Mathematica Slovaca, Vol. 52 (2002), No. 3, 271--287

Persistent URL: http://dml.cz/dmlcz/129329

Math. Slovaca, **52** (2002), No. 3, 271–287

ON THE PRODUCT OF DIVISORS OF A POSITIVE INTEGER

TIBOR ŠALÁT — JANA TOMANOVÁ

(*Communicated by Stanislav Jakubec*)

ABSTRACT. In this paper we will study properties of the arithmetic functions f and f^*, where $f(n) = \prod\limits_{d|n} d$, $f^*(n) = \frac{1}{n}f(n)$ $(n = 1, 2, \dots)$ and sets of their values.

Introduction and background

The notion of perfect numbers is well known in number theory. A number $n \in \mathbb{N}$ is called *perfect* if $\sigma(n) = 2n$, where $\sigma(n) = \sum\limits_{d|n} d$. Hence n is perfect if it is equal to $\sigma(n) - n$ (the sum of its proper divisors — cf. [14; p. 171–175]).

Only a finite number of perfect numbers is known up this time and no odd perfect number is known.

There exists a multiplicative analogon to the concept of a perfect number. Let f be the arithmetic function, $f(n) = \prod\limits_{d|n} d$. A number $n > 1$, $n \in \mathbb{N}$, can be called a *multiplicatively perfect number* provided that $\frac{f(n)}{n} = n$ (i.e. if it is equal to the product of its proper divisors). It is well known that such numbers are exactly the numbers of the form p^3 $(p \in \mathbb{P})$ and all numbers $p_1 \cdot p_2$, where p_1, p_2 are distinct primes (cf. [14; p. 174, Exercise 2]).

The concept of amicable numbers is also well known. Two numbers $a, b \in \mathbb{N}$ are called *amicable* if $\sigma(a) - a = b$ and $\sigma(b) - b = a$, i.e. if $\sigma(a) = a + b = \sigma(b)$. This concept was familiar already to Pythagoras and his students. They have known the pair 220, 284 of such numbers. At this time more than 400 pairs of amicable numbers is known. It is not known whether the set of all pairs of amicable numbers is finite or not. P. Erdős [1] has shown that the asymptotic density of the set of all amicable numbers is zero.

2000 Mathematics Subject Classification: Primary 11A25, 11B05.
Keywords: product of divisors, average order, normal order, statistical convergence, density.

In analogy with the concept of multiplicatively perfect numbers we can introduce also the concept of multiplicatively amicable numbers. Two numbers $a, b \in \mathbb{N}$ are called *multiplicatively amicable* if $\frac{f(a)}{a} = b$, $\frac{f(b)}{b} = a$, i.e. if $f(a) = ab = f(b)$.

It is easy to see that a, b are multiplicatively amicable exactly in the cases if $a = b = 1$ or $a = b$ and a is a multiplicatively perfect number. Hence the answer to the question about the structure of the set of all pairs of multiplicatively amicable numbers is trivial, but our previous considerations lead to the deeper study of properties of functions f, f^*, $f(n) = \prod_{d|n} d$, $f^*(n) = \frac{1}{n} f(n)$ ($n = 1, 2, \dots$). This study began already in [8] by W. E. Briggs and in [9] by I. Niven. Observe that f can be considered as a multiplicative analogue of the function σ.

In what follows we will use some concepts and notations that we introduce now.

If $A \subseteq \mathbb{N}$, then for $n \in \mathbb{N}$ we put $A(n) = \operatorname{card}(\{1, 2, \dots, n\} \cap A)$. If there exists

$$d(A) = \lim_{n \to \infty} \frac{A(n)}{n},$$

then $d(A) \in [0, 1]$ and $d(A)$ is called the *asymptotic density of the set* A (cf. [6; p. 71]).

Recall the concept of the statistical convergence of sequences (cf. [2], [4], [11], [12], [13]). A sequence $(x_n)_1^\infty$ of real numbers is said to be *statistically convergent to* $L \in R$ provided that for every $\varepsilon > 0$ we have $d(A_\varepsilon) = 0$, where $A_\varepsilon = \{n : |x_n - L| \geq \varepsilon\}$. We write briefly $\lim \operatorname{stat} x_n = L$ and the number L is called the *statistical limit* of the sequence $(x_n)_1^\infty$. If $\lim_{n \to \infty} x_n = L$ in the usual sense, then $\lim \operatorname{stat} x_n = L$ as well, so that the statistical convergence can be considered as a generalization of the usual convergence.

There is a simple connection between the concepts of the normal order of an arithmetical function g and the statistical convergence of $(g(n))_1^\infty$ (cf. [12]).

If $A \subseteq \mathbb{N}$, then we put $R(A) = \{\frac{x}{y} : x, y \in A\}$. The set $R(A)$ is called the ratio set of A (cf. [10]).

If g is an arithmetical function, then we put $S(g, m) = \frac{g(1) + \dots + g(m)}{m}$ ($m \in \mathbb{N}$). A function h is called the *average order of* g if $S(g, m) \sim h(m)$ (cf. [3; p. 263]).

This paper consists of two sections. In the first one we describe fundamental properties of the functions f, f^* and sets F, F^*, $F = f(\mathbb{N}) = \{f(1), f(2), \dots, f(n), \dots\}$, $F^* = f^*(\mathbb{N}) = \{f^*(1), f^*(2), \dots, f^*(n), \dots\}$.

In the second section we deal with average order and normal order of some functions that are connected with the functions f, f^*. The results concerning the normal order will be formulated by using the concept of statistical convergence.

1. Fundamental properties of the functions f, f^* and the sets F, F^*

It was proved in [9] by I. Niven that the function f is an injection of $\mathbb{N}$ into $\mathbb{N}$. A similar result does not hold for f^* since for any two distinct primes p, q we have $f^*(p) = f^*(q) = 1$. The following theorem asserts that the partial function $f^*|_{(\mathbb{N} \setminus \mathbb{P})}$, where $\mathbb{P}$ is the set of all prime numbers, is injective. To prove it we shall need an auxiliary lemma, proved in [8] by W. E. Briggs and in [9] by I. Niven.

Lemma 1.1. *For each $m \in \mathbb{N}$ we have*

$$f(m) = m^{\frac{\tau(m)}{2}}, \tag{0}$$

where $\tau(m) = \sum_{d|m} 1$ (= the number of divisors of m).

Proof. The definition of f yields

$$f^2(m) = \prod_{d|m} d \cdot \prod_{d|m} \frac{m}{d} = m^{\tau(m)}.$$

□

Theorem 1.1. *The partial function $f^*|_{(\mathbb{N} \setminus \mathbb{P})}$ is injective ($\mathbb{P}$ is the set of all prime numbers).*

Proof. Obviously we have $f^*(1) \neq f^*(m)$, for $m \in \mathbb{N} \setminus \mathbb{P}$. Hence it suffices to show that for $m \in \mathbb{N} \setminus \mathbb{P}$, $n \in \mathbb{N} \setminus \mathbb{P}$, $m \neq n$, $m, n > 1$, we have $f^*(m) \neq f^*(n)$.

We proceed indirectly. Let $m, n > 1$, $m \neq n$, $m, n \in \mathbb{N} \setminus \mathbb{P}$ and

$$f^*(m) = f^*(n). \tag{1}$$

Then $m \nmid n$, $n \nmid m$ and the standard forms of m and n contain the same set of primes.

Put $(m, n) = d$. Then $d > 1$ and

$$m = dk, \quad n = dt, \quad (k, t) = 1.$$

By Lemma 1.1 we get from (1)

$$m^{\tau(m)-2} = n^{\tau(n)-2}. \tag{2}$$

Here we have $\tau(m) - 2 > 0$, $\tau(n) - 2 > 0$ since m, n are composite numbers.

As $m \neq n$, it is easy to see from (2) that $\tau(m) \neq \tau(n)$. Hence one of the following cases must occur:

(a) $\tau(m) > \tau(n)$,
(b) $\tau(m) < \tau(n)$.

But we show that both these cases are impossible.

In the case (a) we have $\tau(m) = \tau(n) + s$, $s \geq 1$, $s \in \mathbb{N}$. If we put it in (2) we get

$$d^s \cdot k^{\tau(n)-2+s} = t^{\tau(n)-2}.$$

But from this we get $(k,t) > 1$ — a contradiction.

(b) This possibility can be eliminate in a similar way. □

Lemma 1.1 enables us to realize an analyze of the structure of the set $F - f(\mathbb{N}) = \{f(1), f(2), \dots, f(n), \dots\}$. In what follows we will deal with the question what form the numbers belonging to F have.

Suppose that $a = p_1^{\beta_1} \cdots p_r^{\beta_r}$ (standard form) belongs to F. Hence there is an $m \in \mathbb{N}$ such that

$$a = f(m). \tag{3}$$

Then $m > 1$ and from (0) we see that the standard form of m is

$$m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}. \tag{3'}$$

But then by Lemma 1.1 we obtain from (3), (3')

$$p_1^{2\beta_1} \cdots p_r^{2\beta_r} = p_1^{\tau(m)\alpha_1} \cdots p_r^{\tau(m)\alpha_r}.$$

Comparing exponents we get

$$2\beta_j = \tau(m)\alpha_j \qquad (j = 1, 2, \dots, r). \tag{4}$$

Case $r = 1$.
If $r = 1$, then (see (4))

$$\beta_1 = \frac{\alpha_1(\alpha_1 + 1)}{2}.$$

Hence

$$a = p_1^{\frac{\alpha_1(\alpha_1+1)}{2}}.$$

So we see that if $a \in F$, $a = p_1^{\beta_1}$, then $\beta_1 = \frac{\alpha_1(\alpha_1+1)}{2}$. The converse is also true, $a = f(p_1^{\alpha_1})$.

Hence among the numbers $p_1^{\beta_1}$ the set F contains exactly the numbers of the form $p_1^{\frac{\alpha_1(\alpha_1+1)}{2}}$, $\alpha_1 \in \mathbb{N}$ (i.e. F contains exactly the numbers of the form $p_1^{\frac{\alpha_1(\alpha_1+1)}{2}}$ where the exponents are triangular numbers).

Case $r \geq 2$.
We get from (4)

$$\beta_j = \frac{\alpha_j(\alpha_1 + 1) \cdots (\alpha_r + 1)}{2} \qquad (j = 1, 2, \dots, r). \tag{5}$$

Since $\alpha_j + 1 \geq 2$ $(j = 1, 2, \ldots, r)$, from (5) the following estimate for β_j can be deduced:

$$\beta_j \geq 2(r-1) \qquad (j = 1, 2, \ldots, r). \tag{6}$$

Put $h(m) = \min_{1 \leq j \leq r} \alpha_j$. Then we see from (6) that in the case $r \geq 2$ we have $h(m) \geq 2$.

Denote by M the set of all n's from $\mathbb{N}$ having the standard form

$$n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}, \qquad r \geq 2, \ \ h(n) \geq 2. \tag{$*$}$$

We show that the numbers from M satisfying $(\alpha_1, \ldots, \alpha_r) = 1$ do not belong to F. If namely n has the form

$$n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}, \qquad r \geq 2 \quad \text{and} \quad (\alpha_1, \ldots, \alpha_r) = 1,$$

then n cannot belong to F. In the contrary case, there is an $m \in \mathbb{N}$, $m = p_1^{\beta_1} \cdots p_r^{\beta_r}$ (standard form), such that $f(m) = n$. But then Lemma 1.1 yields

$$p_1^{2\alpha_1} \cdots p_r^{2\alpha_r} = p_1^{\tau(m)\beta_1} \cdots p_r^{\tau(m)\beta_r}.$$

Comparing exponents we obtain the following system of equations

$$\beta_i(\beta_1 + 1) \cdots (\beta_r + 1) = 2\alpha_i \qquad (i = 1, 2, \ldots, r). \tag{S}$$

It seems that here two possibilities are:

(a) There exists an i such that β_i is odd.

(b) All β_i $(i = 1, 2, \ldots, r)$ are even.

The case (b) is impossible since in this case we have $(\alpha_1, \ldots, \alpha_r) > 1$.

In the case (a) let e.g. β_j be odd, $1 \leq j \leq r$. Then $\frac{\beta_j + 1}{2}$ is an integer and owing to $r \geq 2$ we see that the greatest common divisor of $\alpha_1, \ldots, \alpha_r$ is at least $\beta_v + 1 \geq 2$, where $v \in \{1, 2, \ldots, r\}$, $v \neq j$.

So we have shown that the following theorem holds:

THEOREM 1.2. *For the set F the following inclusions hold:*

$$\{1\} \cup \left\{p^{\frac{\alpha(\alpha+1)}{2}} : p \in \mathbb{P}, \ \alpha \in \mathbb{N}\right\} \subseteq F \subseteq \{1\} \cup \left\{p^{\frac{\alpha(\alpha+1)}{2}} : p \in \mathbb{P}, \ \alpha \in \mathbb{N}\right\} \cup G \tag{7}$$

where G is the set of all $n \in \mathbb{N}$ having the standard form $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $r \geq 2$, $(\alpha_1, \ldots, \alpha_r) > 1$.

Remark. The set $\mathbb{P}$ of all primes is included in $\{p^{\frac{\alpha(\alpha+1)}{2}} : p \in \mathbb{P}\}$ and therefore it is a subset of F.

The inclusions proved in the foregoing theorem enable us to show that F is a set of zero density.

THEOREM 1.3. *We have* $d(F) = 0$.

P r o o f . The proof will be based on [5; p. 254, Theorem 11.7].
This theorem says:

If $A \subseteq \mathbb{N}$, *then for* p *prime we denote by* A_p *the set of all* $a \in A$ *such that* $p \mid a$ *and* $p^2 \nmid a$. *If* $(q_n)_1^\infty$ *is a sequence of primes* $q_1 < q_2 < \cdots$ *with* $\sum_{n=1}^{\infty} q_n^{-1} = +\infty$ *and* $d(A_{q_n}) = 0$ $(n = 1, 2, \dots)$, *then* $d(A) = 0$.

We will apply this theorem to the set on the right-hand side of (7). We denote this set by A. For $(q_n)_1^\infty$ we choose the sequence of all primes. For fixed $p \in \mathbb{P}$ the set $A_p = \{p\}$ and so $d(A_p) = 0$ (for each $p \in \mathbb{P}$). Hence by [5; Theorem 11.7] we get $d(A) = 0$ and owing to (7) also $d(F) = 0$. □

Theorem 1.3 can be strengthened.

THEOREM 1.4. *For* $x > 1$ *we have*

$$F(x) = \pi(x) + O\big(x^{\frac{2}{3}}\big),$$

where $\pi(x)$ *denotes, as usual, the number of primes not exceeding* x.

COROLLARY. *We have*

$$\lim_{x \to \infty} \frac{F(x)}{\pi(x)} = 1.$$

P r o o f o f T h e o r e m 1.4 . If p is a prime number, $p \le x$, then $f(p) = p \le x$ and for $p > x$, $f(p) > x$. Hence among the numbers $f(1)$, $f(p)$ $(p \in \mathbb{P})$ of the set F exactly the numbers $f(1)$ and $f(p)$ $(p \le x)$ are counted in $F(x)$ and their number is $1 + \pi(x)$.

Now, if n is a composite number, then it has a proper divisor $d \ge \sqrt{n}$, thus

$$f(n) \ge n\sqrt{n} = n^{\frac{3}{2}}.$$

Therefore if $n > x^{\frac{2}{3}}$, then $f(n) \ge n^{\frac{3}{2}} > x$. Hence if n is composite, then only $f(n)$ with $n \le x^{\frac{2}{3}}$ can be counted in $F(x)$. From this our theorem follows immediately. □

Similar result can be proved also for the set

$$F^* = f^*(\mathbb{N}) = \{f^*(1), f^*(2), \dots, f^*(n), \dots\}.$$

THEOREM 1.5. *For every* $\varepsilon \in \big(0, \frac{1}{2}\big)$ *we have*

$$F^*(x) = x^{\frac{1}{1+\varepsilon}} + o(x) \qquad (x > 1).$$

COROLLARY. *We have* $d(F^*) = 0$.

Proof of Theorem 1.5. Let $x > 1$, $x \in \mathbb{N}$, and $0 < \varepsilon < \frac{1}{2}$. Put

$$A_\varepsilon = \{n : f^*(n) \le n^{1+\varepsilon}\}.$$

In the first place we investigate the structure of the set A_ε. If $n \in A_\varepsilon$, then by definition of f^* we get

$$d_1 d_2 \cdots d_{\tau(n)} \le n^{2+\varepsilon}, \tag{8}$$

$$\frac{n}{d_1}\frac{n}{d_2} \cdots \frac{n}{d_{\tau(n)}} \le n^{2+\varepsilon}, \tag{8'}$$

where $1 = d_1 < d_2 \cdots < d_{\tau(n)} = n$ are all divisors of n. Multiplying (8), (8') we obtain

$$n^{\tau(n)} \le n^{4+2\varepsilon} < n^5.$$

Hence $\tau(n) \le 4$. By definition of $\tau(n)$ we see that the set A_ε is contained in the union of the sets M_1, M_2, M_3, where

$$M_1 = \{1\} \cup \mathbb{P},$$
$$M_2 = \{p \cdot p' : p, p' \in \mathbb{P}, \ p \ne p'\},$$
$$M_3 = \{p^3 : p \in \mathbb{P}\}.$$

But each of these sets has the asymptotic density 0 (cf. [3; p 368, Theorem 437]). From this we see that the number of numbers $f^*(n)$ with $f^*(n) \le x$ for $n \in A_\varepsilon$ is $o(x)$ $(x \to \infty)$.

The rest values $f^*(n) \le x$ (for $n \notin A_\varepsilon$) satisfy the inequalities $f^*(n) > n^{1+\varepsilon}$ and so

$$x \ge f^*(n) > n^{1+\varepsilon}.$$

From this $n < x^{\frac{1}{1+\varepsilon}}$ and so the number of these values is $\le [x^{\frac{1}{1+\varepsilon}}]$. The theorem follows. □

In what follows we shall deal with the ratio sets of the sets F, F^* .

THEOREM 1.6. *The ratio set* $R(F)$ *of the set* F *is dense in* $(0, +\infty)$.

Proof. By the definition of f we have $R(F) \supseteq R(P)$. But $R(P)$ is dense in $(0, +\infty)$ (cf. [14; p. 155]). The density of $R(F)$ follows. □

THEOREM 1.6*. *The ratio set* $R(F^*)$ *of the set* F^* *is dense in* $(0, +\infty)$.

Proof. Observe that for $p, q \in \mathbb{P}$ we have

$$\frac{f^*(p^2)}{f^*(q^2)} = \frac{p}{q}.$$

The theorem follows similarly as before. □

In connection with Theorem 1.6 and 1.6*, the natural question arises whether the set $R(F)$ or $R(F^*)$ can coincide with the set Q^+ of all positive rational numbers. We give a negative answer to this question.

THEOREM 1.7. *The set $R(F)$ contains no number of the form $p \cdot q$, where $p, q \in \mathbb{P}$, $p \neq q$.*

Proof. We proceed indirectly. Suppose that there exist $p, q \in \mathbb{P}$, $p \neq q$ and $m, n \in \mathbb{N}$ such that

$$\frac{f(m)}{f(n)} = pq\,,$$

$$f(m) = pqf(n)\,. \tag{9}$$

Obviously $m \neq n$ and m, n are composite numbers. Further $m \nmid n$. We will show that also $n \nmid m$.

Suppose in the contrary that $n \mid m$. Then $m = nd$, $d > 1$. By Lemma 1.1 we get from (9)

$$(nd)^{\tau(nd)} = (pq)^2 \cdot n^{\tau(n)},$$

$$n^{\tau(nd)-\tau(n)} \cdot d^{\tau(nd)} = (pq)^2\,. \tag{10}$$

Since $\tau(nd) > \tau(n)$ and $\tau(nd) > 2$, at least one of the primes p, q occurs on the left-hand side of (10) with exponent greater than 2. But this is a contradiction to the fundamental theorem of arithmetic ([3; p. 3, Theorem 2]).

Hence $m \nmid n$, $n \nmid m$. Put $(m, n) = v$. If a prime p divides n, then by (9) p divides also $f(m)$ and so $p \mid m$. Hence $v > 1$ and by definition of v we get

$$m = vt\,, \quad n = vk\,, \quad (k, t) = 1\,. \tag{11}$$

By Lemma 1.1 we obtain from (9)

$$(vt)^{\tau(m)} = (pq)^2 (vk)^{\tau(n)}\,. \tag{12}$$

If $\tau(m) = \tau(n)$, then according to (12), $(k, t) > 1$ — contrary to (11).

Hence $\tau(m) > \tau(n)$ or $\tau(n) > \tau(m)$. We show that both these cases lead to contradiction.

Let $\tau(m) > \tau(n)$. Then by (12) we get

$$v^{\tau(m)-\tau(n)} t^{\tau(m)} = (pq)^2 k^{\tau(n)}. \tag{13}$$

Since $(k, t) = 1$, the number

$$v^{\tau(m)-\tau(n)} k^{-\tau(n)}$$

is an integer > 0. Moreover, $\tau(m) > 2$ (as m is composite). Thus at least one of the primes p, q occurs on the left-hand side of (13) with exponent greater than 2 — a contradiction to the fundamental theorem of arithmetic.

Let $\tau(m) < \tau(n)$. Then from (12) we have

$$t^{\tau(m)} = v^{\tau(n)-\tau(m)} k^{\tau(n)} (pq)^2$$

and this yields $(k, t) > 1$ — a contradiction to (11). This ends the proof. □

The previous result can be easily generalized, and so we obtain the following theorem.

THEOREM 1.7'. *No square-free number* $q_1 q_2 \cdots q_r$ *with* $r \geq 2$ *belongs to* $R(F)$.

Similar results can be stated also for the set $R(F^*)$. Their proofs are similar and so they can be omitted.

THEOREM 1.8. *Let* p_1, p_2, p_3 *be mutually distinct primes. Then the number* $p_1 \cdot p_2 \cdot p_3$ *does not belong to* $R(F^*)$.

Remark. If $p_1 \neq p_2$ are two primes, then $p_1 \cdot p_2$ belongs to $R(F^*)$. It suffices to put $m = p_1 \cdot p_2$. Then

$$\frac{f^*(m)}{f^*(1)} = p_1 \cdot p_2 \in R(F^*)\,.$$

Theorem 1.8 can be generalized.

THEOREM 1.8'. *No square-free number* $q_1 q_2 \cdots q_r$ *with* $r \geq 3$ *belongs to* $R(F^*)$.

2. Average order of f and f^* and statistical convergence of some related sequences

We can obtain a good information about behaviour and properties of the functions f, f^* also by investigating the average order and normal order of $\log f$, $\log f^*$ and $\log\log f$, $\log\log f^*$, respectively and some sequences connected with these functions. The results concerning the normal order will be formulated using the concept of statistical convergence (cf. [2], [4], [11], [12], [13]).

For the following use, we note that the concept of statistical convergence can be extended for such sequences that are not defined for all $n \in \mathbb{N}$ but only for "almost" all $n \in \mathbb{N}$. This means that we have a sequence $(x_s)_{s\in S}$, where s runs over all positive integers s belonging to a set $S \subseteq \mathbb{N}$, where $d(S) = 1$. Hence if $S = \{s_1 < s_2 < \cdots\}$, then $(x_s)_{s\in S}$ stands for the sequence $\left(x_{s_n}\right)_{n=1}^{\infty}$. Then $\lim\limits_{s\in S}\operatorname{stat} x_s = L$ means that for each $\varepsilon > 0$ we have $d(A_\varepsilon) = 0$, where

$$A_\varepsilon = \left\{s \in S:\ |x_s - L| \geq \varepsilon\right\}.$$

Similarly we say that the statement $V(n)$ *holds almost everywhere in* $\mathbb{N}$ (or for almost all $n \in \mathbb{N}$) provided that the set $S = \left\{n \in \mathbb{N}:\ V(n) \text{ holds}\right\}$ has the density $d(S) = 1$.

For the further use we introduce three auxiliary results.

LEMMA 2.1. *Put*

$$H(n) = \sum_{k=1}^{n} \tau(k) \qquad (n = 1, 2, \dots).$$

Then

$$H_n = n \log n + (2\gamma - 1)n + O(\sqrt{n}\,),$$

where γ is the Euler constant.

Proof. See [3; p. 264, Theorem 320]. □

LEMMA 2.2. *If $\varepsilon \in (0,1)$, then for almost all $n \in \mathbb{N}$ we have*

$$2^{(1-\varepsilon)\log\log n} < \tau(n) < 2^{(1+\varepsilon)\log\log n}.$$

Proof. See [3; p. 359, Theorem 432]. □

Finally we recall the following result from the theory of statistical convergence.

LEMMA 2.3. *A sequence $(x_n)_1^\infty$ converges statistically to L ($\operatorname{lim\,stat} x_n = L$) if and only if there is a set*

$$M = \{m_1 < m_2 < \dots < m_n < \dots\} \subseteq \mathbb{N}$$

such that $d(M) = 1$ and $\lim\limits_{k\to\infty} x_{m_k} = L$.

Proof. See [11; Lemma 1.1]. □

Remark. It can be easily checked that Lemma 2.3 remains valid also for generalized statistical convergence we mentioned at the beginning of this section. If namely the sequence (x_n) is defined only for $n \in \mathbb{N} \setminus H$, $d(H) = 0$, then the set M mentioned in Lemma 2.3 can be chosen in the form $M = (\mathbb{N} \setminus H) \setminus H_0$, where $d(H_0) = 0$. But then again we have $d(M) = 1$.

In the first place we shall deal with statistical convergence of some sequences connected wit the sequence $\big(\log\log f(n)\big)_{n=2}^\infty$.

THEOREM 2.1. *We have*

$$\operatorname{lim\,stat} \frac{\log\log f(n)}{\log\log n} = 1 + \log 2$$

($\log 2$ *is the natural logarithm of* 2).

COROLLARY. *There exists a set*

$$M = \{m_1 < m_2 < \cdots < m_n < \dots\} \subseteq \mathbb{N}$$

such that $d(M) = 1$ *and*

$$\lim_{k\to\infty} \frac{\log\log f(m_k)}{\log\log m_k} = 1 + \log 2 .$$

(*see Lemma* 2.3 *and Remark after it*).

Proof of Theorem 2.1. By Lemma 1.1 we have $f(n) = n^{\frac{\tau(n)}{2}}$. Taking logarithms we obtain

$$\log f(n) = \frac{\tau(n)}{2} \log n \qquad (n > \mathrm{e})$$

and again by the same way

$$\log\log f(n) = \log \frac{1}{2} + \log\tau(n) + \log\log n \qquad (n > \mathrm{e}^{\mathrm{e}}). \tag{14}$$

Let $\varepsilon \in (0,1)$. According to Lemma 2.2 the inequalities

$$(1-\varepsilon)\log 2 \log\log n < \log\tau(n) < (1+\varepsilon)\log 2\log\log n$$

hold for almost all n's. From this we infer

$$\operatorname{lim\,stat} \frac{\log\tau(n)}{\log\log n} = \log 2 . \tag{15}$$

By a little arrangement of (14) we get

$$\frac{\log\log f(n)}{\log\log n} = 1 + \frac{\log\tau(n)}{\log\log n} + \frac{\log\frac{1}{2}}{\log\log n} \qquad (n > \mathrm{e}^{\mathrm{e}}).$$

It is well known that a sum of a finite number of statistically convergent sequences is again a statistically convergent sequence whose statistical limit is equal to the sum of statistical limits of those sequences (cf. [2], [11], [13]). Owing to this fact we get from (15), (16)

$$\operatorname{lim\,stat} \frac{\log\log f(n)}{\log\log n} = 1 + \log 2 .$$

□

A question arises whether an analogous result holds also for the function f^*. Observe that $f^*(n) = 1$ exactly if $n = 1$ or $n = p$ is a prime. For these values of n we have $\log f^*(n) = 0$ and therefore $\log\log f^*(n)$ does not exist. But $d(\{1\} \cup \mathbb{P}) = 0$ and so the sequence

$$\left(\frac{\log\log f^*(n)}{\log\log n}\right)_{n\in S},$$

$S = \mathbb{N} \setminus (\{1\} \cup \mathbb{P})$ is defined almost everywhere in $\mathbb{N}$ and we can speak about its statistical convergence.

By Lemma 1.1 we have

$$f^*(n) = n^{\frac{\tau(n)}{2}-1}$$

for $n \in S$, $n > \mathrm{e}$. Thus

$$\log\log f^*(n) = \log\left(\frac{\tau(n)}{2} - 1\right) + \log\log n \tag{17}$$

(for $n \in S^*$, $S^* = \{n > \mathrm{e}^{\mathrm{e}} : n \in S\}$). Then $d(S^*) = 1$ and (17) holds almost everywhere in $\mathbb{N}$.

THEOREM 2.2. *We have*

$$\lim\operatorname{stat} \frac{\log\log f^*(n)}{\log\log n} = 1 + \log 2 .$$

COROLLARY. *There exists a set $M = \{m_1 < m_2 < \dots < m_n < \dots\} \subseteq \mathbb{N}$ such that $d(M) = 1$ and*

$$\lim_{k\to\infty} \frac{\log\log f^*(m_k)}{\log\log m_k} = 1 + \log 2 .$$

Proof of Theorem 2.2. Let $\varepsilon \in (0, \frac{1}{2})$. Then by Lemma 2.2 we have (for almost all $n \in \mathbb{N}$)

$$2^{(1-\varepsilon)\log\log n - 1} - 1 < \frac{\tau(n)}{2} - 1 < 2^{(1+\varepsilon)\log\log n - 1} - 1 .$$

Using some simple estimations we get from this for almost all n's

$$\log\left(\frac{\tau(n)}{2} - 1\right) < (1+\varepsilon)\log 2 \log\log n$$

and

$$\log\left(\frac{\tau(n)}{2} - 1\right) > (1-\varepsilon)\log 2 \log\log n .$$

Thus

$$\lim\operatorname{stat} \frac{\log\left(\frac{\tau(n)}{2} - 1\right)}{\log\log n} = \log 2 .$$

From (17), (18) the theorem follows immediately. □

In connection with Theorem 2.1 and 2.2 the question arises whether these theorems could be strengthened in the following sense:

In [4] a new type of convergence of sequences is introduced (so called $\mathfrak{I}_c$-convergence) which is stronger than the statistical convergence. For its introduction the following well-known fact is used (cf. [7]):

If $A \subseteq \mathbb{N}$ *and* $\sum\limits_{a\in A} a^{-1} < +\infty$, *then* $d(A) = 0$.

A sequence $(x_n)_1^\infty$ is said to be $\mathfrak{I}_c$*-convergent* to L (briefly: $\mathfrak{I}_c\text{-}\lim x_n = L$) if for every $\varepsilon > 0$ we have $\sum\limits_{n\in A_\varepsilon} n^{-1} < +\infty$, where $A_\varepsilon = \{n : |x_n - L| \geq \varepsilon\}$.

According to the mentioned result of [7] if $\mathfrak{I}_c\text{-}\lim x_n = L$, then also $\lim\operatorname{stat} x_n = L$.

The converse in general is not true.

We now ask whether the Theorems 2.1, 2.2 can be strengthened in such a way that the statistical convergence in them is replaced by $\mathfrak{I}_c$-convergence.

We give a negative answer to this question.

THEOREM 2.3. *The statements*

(V_1) $\mathfrak{I}_c\text{-}\lim \frac{\log\log f(n)}{\log\log n} = 1 + \log 2$,

(V_2) $\mathfrak{I}_c\text{-}\lim \frac{\log\log f^*(n)}{\log\log n} = 1 + \log 2$

do not hold.

Proof. Let $p_1 < p_2 < \dots < p_n < \dots$ be the sequence of all prime numbers.

We prove that (V_1) does not hold.

If $n = p_k$, then

$$\frac{\log\log f(p_k)}{\log\log p_k} = \frac{\log\log p_k}{\log\log p_k} = 1 \tag{19}$$

and so for $0 < \varepsilon < \log 2$ the set

$$A_\varepsilon = \left\{n : \left|\frac{\log\log f(n)}{\log\log n} - (1 + \log 2)\right| \geq \varepsilon\right\}$$

contains by (19) all primes so that

$$\sum_{n\in A_\varepsilon} n^{-1} \geq \sum_{k=1}^{\infty} p_k^{-1} = +\infty$$

(cf. [3; p. 16, Theorem 19]). Hence (V_1) does not hold.

We prove that (V_2) does not hold.

Put $n = p_i p_j$, $i \neq j$. Then $f^*(p_i p_j) = p_i p_j$ so that

$$\frac{\log\log f^*(p_i p_j)}{\log\log p_i p_j} = 1\,.$$

Then for $0 < \varepsilon < \log 2$ the set

$$A_\varepsilon^* = \left\{n:\ \left|\tfrac{\log\log f^*(n)}{\log\log n} - (1+\log 2)\right| \geq \varepsilon\right\}$$

contains all numbers $p_i p_j$, $i \neq j$.

Therefore $\sum\limits_{n\in A_\varepsilon^*} n^{-1} = +\infty$ because already the series $\sum\limits_{i=1}^{\infty} \frac{1}{2p_i}$ diverges. Hence (V_2) is not valid. □

We now will deal with the average order of the functions $\log f$ and $\log f^*$. By Lemma 1.1 we have

$$S(\log f, m) = \frac{1}{2m}\cdot T_m\,, \tag{20}$$

where

$$T_m = \sum_{n=1}^{m} \tau(n)\log n \qquad (m = 1, 2, \dots). \tag{21}$$

THEOREM 2.4. *We have*

$$S(\log f, m) = \frac{1}{2}\log^2 m + O(\log m)\,.$$

COROLLARY.

(a) *We have*

$$\lim_{m\to\infty} \frac{S(\log f, m)}{\log^2 m} = \frac{1}{2}\,.$$

(b) *The "average value" of the function* $\log f$ *on the interval* $[1, m]$ *is* $\frac{1}{2}\log^2 m$.

P r o o f o f T h e o r e m 2.4. We will use Abel's partial summation for estimate T_m (see (21)).

By Lemma 2.1 we have

$$H_n = \sum_{k=1}^{n} \tau(k) = n\log n + O(n)\,. \tag{22}$$

Using Abel's summation we get

$$\begin{aligned} T_m &= \tau(1)\log 1 + \tau(2)\log 2 + \dots + \tau(m)\log m \\ &= H_1 \log 1 + (H_2 - H_1)\log 2 + \dots + (H_m - H_{m-1})\log m \\ &= H_m \log m + \{H_1 \log 2 + H_2(\log 2 - \log 3) + \dots \\ &\qquad \dots + H_{m-1}(\log(m-1) - \log m)\} \\ &= H_m \log m + V_m , \end{aligned} \tag{23}$$

where

$$V_m = -\left\{\log 2 + H_2 \log \frac{3}{2} + \dots + H_{m-1} \log \frac{m}{m-1}\right\}. \tag{24}$$

Using (22) we get

$$H_m \log m = m \log^2 m + O(m \log m). \tag{25}$$

Further

$$H_k \cdot \log \frac{k+1}{k} \le H_k \cdot \frac{1}{k},$$

and so (see (22), (24))

$$|V_m| \le \sum_{k=2}^{m-1} \frac{1}{k}\big(k \log k + O(k)\big) = \sum_{k=2}^{m-1} \log k + O(m).$$

Observe that

$$\sum_{k=2}^{m-1} \log k \le \int_1^m \log t \;\mathrm{d}t = m \log m - m.$$

Hence

$$V_m = O(m \log m). \tag{26}$$

But then according to (23), (24), (25), (26) we obtain

$$T_m = m \log^2 m + O(m \log m).$$

Putting it into (20) we get

$$\begin{aligned} S(\log f, m) &= \frac{1}{2m}\big(m \log^2 m + O(m \log m)\big) \\ &= \frac{1}{2} \log^2 m + O(\log m). \end{aligned}$$

$\square$

The previous theorem enables us to determine the average order of $\log f^*$ in an easy way.

THEOREM 2.5. *We have*

$$S(\log f^*, m) = \frac{1}{2}\log^2 m + O(\log m)\,.$$

COROLLARY. *We have*

$$\lim_{m\to\infty} \frac{S(\log f^*, m)}{\log^2 m} = \frac{1}{2}\,.$$

Proof of Theorem 2.5. By the definition of f^* we get

$$\begin{aligned}\sum_{n=1}^{m} \log f^*(n) &= \sum_{n=1}^{m} \log \frac{f(n)}{n} \\ &= \sum_{n=1}^{m} \log f(n) - \sum_{n=1}^{m} \log n\,.\end{aligned} \tag{27}$$

Further $\sum\limits_{n=1}^{m} \log n = \log m!$ and so by Stirling's formula (cf. [6; p. 192])

$$m! = \sqrt{2\pi}\cdot m^{m+\frac{1}{2}}\cdot \mathrm{e}^{-m}\big(1+o(1)\big)\,.$$

Thus using this formula we obtain from (27)

$$\sum_{n=1}^{m} \log f^*(n) = \sum_{n=1}^{m} \log f(n) + m\log m + O(m)\,.$$

Hence

$$S(\log f^*, m) = S(\log f, m) + O(\log m)\,.$$

From this the assertion follows immediately. □

REFERENCES

[1] ERDŐS, P.: *On amicable numbers*, Publ. Math. Debrecen **4** (1955), 108–111.

[2] FAST, H.: *Sur la convergence statistique*, Colloq. Math. **2** (1951), 241–244.

[3] HARDY, G. H.—WRIGHT, E. M.: *An Introduction to the Theory of Numbers*, Clarendon Press, Oxford, 1954.

[4] KOSTYRKO, P.—MAČAJ, M.—ŠALÁT, T.: *Statistical convergence and $\mathfrak{I}$-convergence* (To appear).

[5] NIVEN, I.—ZUCKERMAN, H. S.: *An Introduction to the Theory of Numbers*, John Wiley, New York-London-Sydney, 1967.

[6] OSTMANN, H. H.: *Additive Zahlentheorie I*, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956.

[7] POWEL, B. J.—ŠALÁT, T.: *Convergence of subseries of the harmonic series and asymptotic densities of sets of positive integers*, Publ. Inst. Math. (Beograd) (N.S.) **50** (1991), 60–70.

[8] *Problem E 1058 [1953, 188]*. Proposed by Perisastri, M. Solution by Briggs, W. E., Amer. Math. Monthly **60** (1953), 628–629.

[9] *Problem E 2946 [1982, 333]*. Proposed by Simion, R. and Schmidt, F. W. Solution by Niven, I., Amer. Math. Monthly **91** (1984), 650.

[10] ŠALÁT, T.: *On ratio sets of sets of natural numbers*, Acta Arith. **15** (1969), 273–278.

[11] ŠALÁT, T.: *On statistically convergent sequences of real numbers*, Math. Slovaca **30** (1980), 139–150.

[12] SCHINZEL, A.—ŠALÁT, T.: *Remarks on maximum and minimum exponents in factoring*, Math. Slovaca **44** (1994), 505–514.

[13] SCHOENBERG, I. J.: *The integrability of certain functions and related summability methods*, Amer. Math. Monthly **66** (1959), 361–375.

[14] SIERPIŃSKI, W.: *Elementary Theory of Numbers*, PWN, Warszawa, 1964.

Received May 21, 2001

Department of Algebra and Number Theory
Comenius University
Mlynská dolina
SK-842 15 Bratislava
SLOVAKIA

E-mail: tomanova@fmph.uniba.sk