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Introduction

This paper is a continuation of [7] where we introduced the notion of the integral for a pre-measure (non-negative, monotone, in an empty set vanishing set function), which is defined on a pre-space. In this paper we shall show that if a pre-measure has some other properties, then the integral has some interesting properties. We shall namely be interested in the study of strong subadditive and strong superadditive measures.

If one has a set function on a ring, then constructing an integral means to define a functional on a subclass of the class of all measurable functions. It is natural to desire that the integral must copy as many properties of the set function as possible. It is also natural to desire that the measure of a set should concide with the integral of the characteristic function of the same set.

Our process of integration claims all these desired properties in case of a pre-, strong super and strong submeasures. However, it is easy to see that in the case of a general submeasure the integration process turns out to be useless, since in this case it need not be subadditive.

§ 0 begins with notes about terminology and notation. We give also the definition of the integral and list some of its interesting properties. We show a useful formula for computing some examples. In § 1 we introduce the notion of strong super- and strong submeasures and give examples. § 2 contains a proof of the fact that the integral copies some properties of the set function with respect to which it has been constructed. In § 3 we present a theory of L_p spaces for our integration theory with respect to a strong subadditive measure. In § 4 we establishe a theorem expressing the values of a strong submeasure integral in terms of some additive measure integral values.

§ 0. Preliminaries

By a *pre-space* we mean a pair $(X, \mathscr{D})$, where $\emptyset \in \mathscr{D}$ and $\mathscr{D} \subset 2^X$. An extended real valued monotone set function μ defined on $\mathscr{D}$ is called a *pre-measure* iff $\mu(\emptyset) = 0$.

A pre-measure μ is called *continuous* if it has the following two properties:

(i) $A_n \nearrow A \supset B (A_n, B \in \mathscr{D})$ implies $\lim_n \mu(A_n) \geqq \mu(B)$.

(ii) $A_n \searrow A \subset B$, $\mu(A_1) < \infty$, $(A_n, B \in \mathscr{D})$ implies $\lim_n \mu(A_n) \leqq \mu(B)$.

The function $f: X \to \langle -\infty, \infty \rangle$ is $\mathscr{D}$-*measurable* or only *measurable* iff the sets $\{x ; f(x) \geqq a\}$ and $\{x ; f(x) \leqq -a\}$ are in $\mathscr{D}$ for every $a > 0$.

We denote by $\mathscr{L}(\mathscr{D})$ the set of all $\mathscr{D}$-measurable functions. $f \in \mathscr{L}(\mathscr{D})$ is called a *simple function* if the range of f is finite.

Now we recall the definition of the integral given in [7].

Let $\mathscr{F}$ be a family of all finite subsets of $\langle -\infty, \infty \rangle$ which contains zero. Let $F \in \mathscr{F}$ with

$$F = \{b_m < b_{m-1} < \ldots < b_0 = 0 = a_0 < a_1 < \ldots < a_n\},$$

and let f *be a* $\mathscr{D}$-measurable function. We put

$$S(f, F) = \sum_{i=1}^{n} (a_i - a_{i-1}) \mu(\{x ; f(x) \geqq a_i\})$$
$$+ \sum_{j=1}^{m} (b_j - b_{j-1}) \mu(\{x ; f(x) \leqq b_j\})$$

if the right-hand side expression contains no expression of the type $\infty - \infty$. Since $\mathscr{F}$ is directed by inclusion, the triple $(S(f, F), \mathscr{F}, \supset)$ is a net. We put

$$\mathscr{I}f = \mathscr{I}_\mu f = \int f \, \mathrm{d}\mu = \lim_{F \in \mathscr{F}} S(f, F)$$

if the limit exists. f is called *integrable* iff $\mathscr{I}_\mu f$ is finite.

We denote by $\mathscr{L}_1 = \mathscr{L}_1(X, \mathscr{D}, \mu)$ the set of all integrable functions.

The main properties of $\mathscr{I}_\mu$ proved in [7] are:

1° $\mathscr{I}_\mu \chi_A = \mu(A)$.

2° $\mathscr{I}_\mu f = \sup \{\mathscr{I}_\mu g ; g \leqq f, g \text{ is simple}\}$ *for* $f \in \mathscr{L}^+(\mathscr{D})$

3° $\mathscr{I}_\mu$ *is a monotone functional.*

4° $\mathscr{I}_\mu$ *is additive in a horizontal sense, i.e. if* $a \geqq 0$, *then*

$$\mathscr{I}_\mu f = \mathscr{I}_\mu (f \wedge a) + \mathscr{I}_\mu (f - f \wedge a)$$

if one of the right-hand side expressions is finite.

5° *For* $f \in \mathscr{L}_1$ $\mathscr{I}f = \mathscr{I}f^+ - \mathscr{I}f^-$.

6° $\mathscr{I}(\alpha f) = \alpha \mathscr{I}f$.

7° f, $|f| \in \mathscr{L}(\mathscr{D})$ *and* $|f| \in \mathscr{L}_1$ *implies* $f \in \mathscr{L}_1$.

8° *If* μ *is continuous,* $f_n \in \mathscr{L}(\mathscr{D})$, $f_n \nearrow f \in \mathscr{L}(\mathscr{D})$ *and* $\mathscr{I}f_n \leqq c < \infty$, *then* $\mathscr{I}f_n \nearrow \mathscr{I}f$.

9° *If* μ *is continuous,* $f_n, f \in \mathscr{L}(\mathscr{D})$, $g \in \mathscr{L}_1$, $|f_n| \leqq g$, $f_n \to f$ *and* $\mathscr{D}$ *is a* σ-*lattice, then*

$$\lim_n \mathscr{I}f_n = \mathscr{I}f.$$

10° *Fatou's lemma: If* μ *is continuous,* f_n, $g \in \mathscr{L}_1(\mathscr{D})$ *with* $f_n \geqq g$, $\liminf_n \mathscr{I} f_n \leqq c$, *and* $\mathscr{D}$ *is a* σ*-lattice, then the function* f *defined by* $f(x) = \liminf_n f_n(x)$ *is integrable and* $\mathscr{I} f \leqq \liminf_n \mathscr{I} f_n$.

Remark. Let $f \in \mathscr{L}_1^+(X, \mathscr{D}, \mu)$ and let λ be the Lebesgue measure on R; then

$$\mathscr{I}_\mu f = \int g \, \mathrm{d}\lambda ,$$

where on the right-hand side there is an ordinary Lebesgue integral and $g(t) = \mu(\{x; f(x) \geqq t\})$.

Proof. Suppose first that f is bounded by A. Then $g(t)$ vanishes on $\langle A, \infty)$. Let $\varepsilon > 0$. Choose $F_0 = \{0 = a_0 < a_1 < \ldots < a_k\}$ with $a_k \leqq A$ and such that for $F \supset F_0$ there holds: $|\mathscr{I}_\mu f - S(f, F)| < \varepsilon/2$. Choose such a $\delta > 0$ that for every partition Δ of the interval $\langle 0, A\rangle$ with the norm less then δ there holds $\left|\int_0^A g(t) \, \mathrm{d}t - \Sigma(g, \Delta)\right| < \varepsilon/2$, where $\Sigma(g, \Delta)$ is a Riemann integral sum of g with respect to the partition Δ. Let $F \supset F_0$ be such that $F \cap \langle 0, A\rangle$ is a partition of $\langle 0, A \rangle$ with the norm less then δ. Then

$$\left|\mathscr{I}_\mu f - \int_0^A g(t) \, \mathrm{d}t\right| = \left|\mathscr{I}_\mu f - S(f, F) + S(f, F) - \int_0^A g(t) \, \mathrm{d}t\right|$$

$$= \left|\mathscr{I}_\mu f - S(f, F) + \Sigma(g, F \cap \langle 0, A\rangle) - \int_0^A g(t) \, \mathrm{d}t\right| < \varepsilon$$

because $S(f, F) = \Sigma(g, F \cap \langle 0, A \rangle)$. And so we get $\mathscr{I}_\mu f = \int_0^A g(t) \, \mathrm{d}t = \int g \, \mathrm{d}\lambda$.

If now f is not necessarily bounded, then

$$\mathscr{I}_\mu f = \lim_{A \to \infty} \mathscr{I}_\mu (f \wedge A) = \lim_{A \to \infty} \int_0^A g(t) \, \mathrm{d}t = \int g \, \mathrm{d}\lambda .$$

The formula just proved may be very usefull for computing examples. Let $\mu_1 = \lambda^k$, $\mu_2 = \sqrt[k]{\lambda}$, $\mu_3 = \operatorname{arctg} \lambda$ and let f be an identity map on $\langle 0, 1\rangle$; then

$$\mathscr{I}_{\mu_1} f = \int_0^1 \mu_1(\{x; x \geqq t\}) \, \mathrm{d}\lambda(t) = \int_0^1 (1-t)^k \, \mathrm{d}t = 1/(k+1).$$

$$\mathscr{I}_{\mu_2} f = \int_0^1 \sqrt[k]{1-t} \, \mathrm{d}t = k/(k+1),$$

$$\mathscr{I}_{\mu_3} f = \pi/4 - \ln \sqrt{2}.$$

§ 1. Non additive measures

Let $(X, \mathscr{C})$ be a pre-space and let $\mathscr{C}$ be a lattice.
A pre-measure μ defined on $\mathscr{C}$ is

a) *subadditive* iff

$$\mu(A \cup B) \leqq \mu(A) + \mu(B),$$

b) *strongly subadditive* iff

$$\mu(A \cap B) + \mu(A \cup B) \leqq \mu(A) + \mu(B)$$

c) *strongly superadditive* iff

$$\mu(A \cap B) + \mu(A \cup B) \geqq \mu(A) + \mu(B)$$

d) *additive* iff

$$\mu(A \cap B) + \mu(A \cup B) = \mu(A) + \mu(B).$$

A strongly subadditive (superadditive) pre-measure is a *strong sub-measure (super-measure).*

We give a list of interesting examples of pre-, sub- and super-mearusres. For this we need the following.

Lemma 1. *If a, b, c, d are non-negative real numbers with $c \leqq a$, $b \leqq d$, $c + d \leqq a + b (c + d \geqq a + b)$ and f: $\bar{R}^+ \to \bar{R}^+$ is an increasing concave (convex) function, then*

$$f(c) + f(d) \leqq f(a) + f(b) \; (f(c) + f(d) \geqq f(a) + f(b)).$$

Proof. Denote $g(x) = f(x) + f(c + d - x)$. $(0 \leqq x \leqq c + d)$. Then g is a concave function with $g(c + y) = g(d - y)$ for $y \in \langle c, d \rangle$. Since f is concave, $\lambda x_2 + (1 - \lambda)(c + d - x_1) = c + d - x_2$ and $(1 - \lambda)x_1 + \lambda(c + d - x_2) = x_2$ we get that g is increasing on $\langle c, (c + d)/2 \rangle$. And so we have

$$g(c) = g(d) = \min \{g(x); x \in \langle c, d \rangle\};$$

since $c + d - a \leqq b$ and f is increasing, we get

$$f(c) + f(d) = g(c) \leqq f(a) + f(c + d - a) \leqq f(a) + f(b).$$

The proof of the second assertion is similar.

Example 2. Let $\mathscr{C}$ be a lattice and μ be a strong submeasure (supermeasure) on $\mathscr{C}$. Further let f: $\bar{R}^+ \to \bar{R}^+$ be an increasing concave (convex) function with $f(0) = 0$. Then the set function ν on $\mathscr{C}$ defined by $\nu(A) = f(\mu(A))$, for $A \in \mathscr{C}$ is a strong submeasure (supermeasure). The proof of this is a conclusion of Lemma 1. Moreover if μ and f are continuous and μ is finite or $f(\infty) = \infty$, then ν is also continuous.

Example 3. From the foregoing example it follows that if μ is a countably additive measure on a ring $\mathscr{R}$ and k is a positive integer, then the set functions $\sqrt[k]{\mu}$, arctg μ and $\nu(A) = \mu(A)/(1+\mu(A))$ for finite ν are continuous strong submeasures. The set functions μ^k and $\nu(A) = \exp \mu(A) - 1$ are continuous strong supermeasures.

Example 4. Let μ be a countably additive measure on a ring $\mathscr{R}$. We put for $A \subset X$ $\mu^*(A) = \inf \{\lim_n \mu(A_n);\ A_n \in \mathscr{R}.\ A_n \cap A \nearrow A\}$.

Then μ^* is a strong submeasure which is not necessarily continuous. Clearly μ^* is an extension of μ. By III. § 1., D 18 of [3] μ^* is a Choquet $\mathscr{R}$-capacity on X.

Example 5. Let μ be defined on $\mathscr{C}$ (a lattice of sets) as follows $\mu(\emptyset) = 0$ and $\mu(A) = 1$ for $\emptyset \neq A \in \mathscr{C}$. Then μ is a strong submeasure on $\mathscr{C}$.

We note that all examples of submeasures and strong supermeasures are also pre-measures.

A special case of strong submeasures are the so called maxitive measures [5]. A pre-measure m defined on a lattice $\mathscr{C}$ will be called a maxitive measure iff

$$m(A \cup B) = \max \{m(A), m(B)\}.$$

The following examples show that a maxitive measure may be considered as an indicator of the size of a set.

Example 6. Let $X = R$ and let

$$m(E) = \sup \{|x\ ;\ x \in E\} \quad \text{for} \quad E \subset R.$$

More generally, let (X, ϱ) be a metric space and

$$m(E) = \sup \{d(x, x_0);\ x \in E\} \ (\sup \emptyset = 0)$$

Example 7. Let (X, ϱ) be a metric space, $\mathscr{C} = 2^X$ and $m(E)$ be the Hausdorff dimension of E.

§ 2. Properties of the integral in special cases

In this section the family $\mathscr{C}$ is assumed to be a σ-lattice.

Theorem 8. *Let f and g be non-negative measurable functions. Then*

$$\mathscr{I}(f \wedge g) + \mathscr{I}(f \vee g) \leqq \mathscr{I}f + \mathscr{I}g$$

if μ is a strong submeasure,

$$\mathscr{I}(f \wedge g) + \mathscr{I}(f \vee g) \geqq \mathscr{I}f + \mathscr{I}g$$

if μ is a strong supermeasure.

Proof. Let $F \in \mathscr{F}$ with $F=\{a_0=0<a_1<a_2<\ldots<a_n\}$. Put

$$A_i=\{x\,;\,f(x)\geqq a_i\} \text{ and } B_i=\{x\,;\,g(x)\geqq a_i\}.$$

Since μ is strongly subadditive we get

$$\mu(\{x\,;\,(f\wedge g)\,(x)\geqq a_i\})+\mu(\{x\,;\,(f\vee g)\,(x)\geqq a_i\})$$
$$=\mu(A_i\cap B_i)+\mu(A_i\cup B_i)\leqq\mu(A_i)+\mu(B_i).$$

After multiplying by (a_i-a_{i-1}) and summing over i we get

$$S(f\wedge g,F)+S(f\vee g,F)\leqq\sum_{i=1}^{n}(a_i-a_{i-1})\,(\mu(A_i)+\mu(B_i))$$
$$=S(f,F)+S(g,F).$$

If $\mathscr{I}f+\mathscr{I}g=\infty$, then the assertion is trivial. Let $\mathscr{I}f+\mathscr{I}g<\infty$. Let $\varepsilon>0$. Choose $F_0\in\mathscr{F}$ with $|S(f,F)\;-\;\mathscr{I}f|<\varepsilon$ and $|S(g,F)-\mathscr{I}g|<\varepsilon$ for $F\supset F_0$.

Then

$$\mathscr{I}f+\mathscr{I}g>S(f,F)+S(g,F)-2\varepsilon$$
$$\geqq S(f\wedge g,F)+S(f\vee g,F)-2\varepsilon.$$

Since ε was arbitrary, we get

$$\mathscr{I}f+\mathscr{I}g\geqq\mathscr{I}(f\wedge g)+\mathscr{I}(f\vee g).$$

The proof for a strong supermeasure is similar.

Corollary 9. *If f and g are non-negative measurable functions and μ is a strong submeasure, then*

$$\mathscr{I}(f\vee g)\leqq\mathscr{I}f+\mathscr{I}g.$$

We shall now establish that $\mathscr{I}$ has similar properties on non $-$ negative $\mathscr{C}$ $-$ measurable functions as μ has on $\mathscr{C}$.

Lemma 10. *Let $A_1\supset A_2\supset\ldots\supset A_n$ and A be measurable sets and let μ be a strong submeasure on $\mathscr{C}$; then*

$$\sum_{i=1}^{n+1}\mu((A\cap A_{i-1})\cup A_i)\leqq\sum_{i=1}^{n+1}\mu(A_i)+\mu(A),$$

where $A_0=A$ and $A_{n+1}=\emptyset$. If μ is a strong supermeasure on $\mathscr{C}$, then the opposite inequality holds.

Proof. If $\mu(A)=\infty$, then the assertion is trivial. Suppose that $\mu(A)$ is finite.
Let $n=2$; then using the fact that μ is strongly subadditive we get

$$\mu(A\cup A_1)+\mu((A\cap A_1)\cup A_2)+\mu(A\cap A_2)\leqq$$
$$\leqq\mu(A)+\mu(A_1)-\mu(A\cap A_1)+\mu(A\cap A_1)+\mu(A_2)-\mu(A\cap A_1\cap A_2)+\mu(A\cap A_2)\leqq$$
$$\leqq\mu(A)+\mu(A_1)+\mu(A_2).$$

For $n>2$ the assertion can be similarly proved by induction. The proof of the opposite inequality for a strong supermeasure is similar.

Lemma 11. *Let f and g be non-negative simple functions $g=c\chi_A$ and μ be a strong submeaure. Then*

$$\mathcal{I}(f+g)\leqq \mathcal{I}f+\mathcal{I}g .$$

If μ is a strong supermeasure, then

$$\mathcal{I}(f+g)\geqq \mathcal{I}f+\mathcal{I}g .$$

Proof. Let μ be a strong sub-measure. If $\mu(A)=\infty$, then the proof is trivial. Let $\mu(A)<\infty$. Let $\{a_1, \dots, a_n\}$ $0=a_0<a_1\dots<a_n$ be the range of f.

Denote $A_i=\{x;\ f(x)\geqq a_i\}$ $i=1,2,\dots,n$ $A_0=A$, $A_{n+1}=\emptyset$ $c_{n+1}=0$ and $c_i=a_i-a_{i-1}$. Suppose first that $c\leqq c_i$ $i=1,2,\dots,n$; then

$$f+g=\sum_{i=1}^{n+1}[c\cdot\chi_{(A\cap A_{i-1})\cup A_i}+(c_i-c)\chi_{A_i}]$$

and

$$A\cup A_1\supset A_1\supset(A\cap A_1)\cup A_2\supset A_2\supset\dots\supset(A\cap A_{n-1})\cup A_n\supset A_n\supset A\cap A_n$$

From Corollary 15 of [7] we have

$$\begin{aligned}\mathcal{I}(f+g)&=\sum_{i=1}^{n+1}[c\mu((A\cap A_{i-1}\cup A_i)+(c_i-c)\mu(A_i)]\\&=\sum_{i=1}^{n+1}c_i\mu(A_i)+c\left[\sum_{i=1}^{n+1}\mu((A\cap A_{i-1})\cup A_i)-\sum_{i=1}^{n+1}\mu(A_i)\right]\leqq\\&\leqq\mathcal{I}f+\mathcal{I}g ,\end{aligned}$$

where the last inequality follows from Lemma 10. We have proved the lemma for c with $0<c\leqq c_i$ $i=1,2,\dots,n$. Let now c be an arbitrary positive number and let $c=m\cdot b$, where m is a natural number and

$$b\leqq\min\{c_i;\ i=1,2,\dots,n\} .$$

Denote

$$g_j=b\cdot\chi_A\quad j=1,2,\dots,m .$$

Then

$$\begin{aligned}\mathcal{I}(f+g)&=\mathcal{I}(f+g_1+g_2+\dots+g_n)\leqq\\&\leqq\mathcal{I}(f+g_1+\dots+g_{m-1})+\mathcal{I}g_m\leqq\\&\leqq\mathcal{I}(f+g_1+\dots+g_{m-2})+\mathcal{I}g_{m-1}+\mathcal{I}g_m\leqq\\&\leqq\mathcal{I}f+\mathcal{I}g_1+\dots+\mathcal{I}g_m\\&=\mathcal{I}f+\mathcal{I}g .\end{aligned}$$

Here we have used the first part of this proof. In the case of a strong supermeasure the proof is the same.

Proposition 12. *Let f and g be non-negative simple functions and μ be a strong submeasure; then*

$$\mathcal{I}(f+g) \leqq \mathcal{I}g + \mathcal{I}g .$$

Moreover if μ is a strong supermeasure, then

$$\mathcal{I}(f+g) \geqq \mathcal{I}f + \mathcal{I}g .$$

Proof. Let $\{b_1, \ldots, b_n\}$ with $0 = b_0 < b_1 < \ldots < b_n$ be the range of g. Denote $g_i = (b_i - b_{i-1}) . \chi\{x ; g(x) \geqq b_i\}$ for $i = 1, 2, \ldots, n$; then

$$g = \sum_{i=1}^{n} g_i \text{ and } \mathcal{I}g = \sum_{i=1}^{n} \mathcal{I}g_i .$$

Further

$$\begin{aligned}\mathcal{I}(f+g) &= \mathcal{I}(f+g_1+\ldots+g_n) \leqq \mathcal{I}(f+g_1+\ldots+g_{n-1}) + \mathcal{I}g_n \\ &\leqq \mathcal{I}(f+g_1+\ldots+g_{n-2}) + \mathcal{I}g_{n-1} + \mathcal{I}g_n \\ &\leqq \mathcal{I}f + \mathcal{I}g_1 + \ldots + \mathcal{I}g_n \\ &= \mathcal{I}\text{f} + \mathcal{I}\text{g},\end{aligned}$$

using Lemma 11. The case of a strong supermeasure is similar.

Theorem 13. *Let f and g be measurable non-negative functions; if μ is a strong submeasure, then*

$$\mathcal{I}(f+g) \leqq \mathcal{I}f + \mathcal{I}g ,$$

if μ is a strong supermeasure, then

$$\mathcal{I}(f+g) \geqq \mathcal{I}f + \mathcal{I}g .$$

Proof. Let μ be a strong submeasure. If $\mathcal{I}f$ or $\mathcal{I}g$ is infinite, then the assertion is clear. Let $\mathcal{I}f$ and $\mathcal{I}g$ be finite; then since

$$f + g \leqq 2 \cdot (f \vee g)$$

from the monotonicity of $\mathcal{I}$ and Corollary 9 we have

$$\mathcal{I}(f+g) \leqq 2. \ \mathcal{I}(f \vee g) \leqq 2. \ (\mathcal{I}(f) + \mathcal{I}(g)) < \infty .$$

Take a simple function $h \leqq f+g$. Clearly $\mu(S_h) < \infty$. Let $F = \{a_0, a_1, \ldots, a_n\} \in \mathcal{F}$ ($a_0 = 0 < a_1 < \ldots < a_n$) be such that $\max\{a_i - a_{i-1}\} < \varepsilon/2$, $\mathcal{I}f - S(f, F) < \varepsilon/2$ and $\mathcal{I}g - S(g, F) < \varepsilon/2$. Let $(f+g)_F(x) = a_i$, $f_F(x) = a_j$ and $g_F(x) = a_k$; then $a_i \leqq f(x) + g(x)$, $a_j \leqq f(x) < a_{j+1}$ and $a_k \leqq g(x) < a_{k+1}$. Since $\max\{a_i - a_{i-1}\} < \varepsilon/2$, we get

$$h(x) \leqq h_F(x) \leqq (f+g)_F(x) = a_i \leqq f(x) + g(x) \leqq a_{j+1} + a_{k+1}$$
$$\leqq a_j + a_k + \varepsilon = f_F(x) + g_F(x) + \varepsilon ,$$

and so

$$h(x) \leqq f_F(x) + g_F(x) + \varepsilon \chi_{S_h}(x) .$$

Hence

$$\mathscr{I} h \leqq \mathscr{I}(f_F + g_F) + \varepsilon \mu(S_h) \leqq \mathscr{I} f_F + \mathscr{I} g_F + \varepsilon \cdot \mu(S_h)$$
$$\leqq \mathscr{I} f + \mathscr{I} g + \varepsilon \cdot \mu(S_h) .$$

Since $\mu(S_h) < \infty$, we have

$$\mathscr{I} h \leqq \mathscr{I} f + \mathscr{I} g .$$

The proof now follows by 2°.

If μ is a strong supermeasure, then the case $\mathscr{I}(f+g) = \infty$ is clear. Let $\mathscr{I}(f+g) < \infty$; then by the mononicity of $\mathscr{I}$ $\mathscr{I} f$ and $\mathscr{I} g$ are finite. Let $\tilde{f}$, $\tilde{g}$ are measurable simple functions with $\tilde{f} \leqq f$ and $\tilde{g} \leqq g$; then $\mathscr{I} \tilde{f} + \mathscr{I} \tilde{g} \leqq \mathscr{I}(\tilde{f} + \tilde{g}) \leqq \mathscr{I}(f+g)$ and so by 2°

$$\mathscr{I} f + \mathscr{I} g \leqq \mathscr{I}(f+g) .$$

Corollary 14. *Let f be a measurable function. Let μ be a strong submeasure; then f is integrable iff $|f|$ is integrable.*

Proof. Let f be integrable; then

$$\mathscr{I}|f| = \mathscr{I}(f^+ + f^-) \leqq \mathscr{I} f^+ + \mathscr{I} f^- < \infty .$$

And so $|f|$ is also integrable. The opposite implication follows by 7°.

The results of this paragraph involve the validity of the following theorem.

Theorem 15. *Let μ be a measure on $\mathscr{C}$. Let f and g be non-negative $\mathscr{C}$-measurable functions, then*

$$\mathscr{I}(f+g) = \mathscr{I}(f \wedge g) + \mathscr{I}(f \vee g) = \mathscr{I} f + \mathscr{I} g .$$

§ 3. Function spaces

In this paragraph we shall assume that $\mathscr{D} = \mathscr{S}$ is a σ-ring of the subsets of X and μ is a continuous strong submeasure on $\mathscr{S}$.

A property P pertaining to points of X is said to hold almost everywhere (a.e.) iff the set of all x for which P does not hold is from $\mathscr{S}$ and is of μ measure zero.

For example $f \leqq g$ a.e. means that

$$E = \{x \,;\, f(x) > g(x)\}$$

is from $\mathscr{S}$ and $\mu(E) = 0$.

In all our convergence theorems (see [7]) we may the pointwise convergence change by convergence a.e. To illustrate this fact we give now the variant of 8°. We shall need some lemmas.

Lemma 16. *If $E \in \mathscr{S}$, $\mu(E)=0$ and f is a measurable function, then*

$$\mathscr{I}_\mu(f\chi_E)=0.$$

Proof. Let $a>0$; then

$$\{x\,;\, f\chi_E \geqq a\} \subset E,$$

and so $S(f, F^+)=0$, for every F in $\mathscr{F}$.. And so we get $\mathscr{I}_\mu(f^+\chi_E)=0$.

Similarly $\mathscr{I}_\mu(f^-\chi_E)=0$. From 5° we get

$$\mathscr{I}_\mu(f\chi_E)=\mathscr{I}_\mu(f^+\chi_E)-\mathscr{I}_\mu(f^-\chi_E)=0.$$

Proposition 17. *Let f be an integrable function, let g be a measurable function and let $f=g$ a.e.; then*

$$\mathscr{I}f=\mathscr{I}g$$

and so g is also integrable.

Proof. Let $E=\{x\,;\, f(x)\neq g(x)\}$; then $E\in\mathscr{S}$ and $\mu(E)=0$. Since

$$\begin{aligned}\mathscr{I}f^+ &= \mathscr{I}(f^+\chi_E+f^+\chi_{E'}) \leqq \mathscr{I}(f^+\chi_E)+\mathscr{I}(f^+\chi_{E'})\\ &= \mathscr{I}(f^+\chi_{E'}) \leqq \mathscr{I}f^+,\end{aligned}$$

we get

$$\mathscr{I}f^+=\mathscr{I}(f^+\chi_{E'}).$$

Similarly

$$\mathscr{I}f^-=\mathscr{I}(f^-\chi_{E'}).$$

From 5° we get

$$\begin{aligned}\mathscr{I}f=\mathscr{I}f^+-\mathscr{I}f^- &= \mathscr{I}(f^+\chi_{E'})-\mathscr{I}(f^-\chi_{E'})=\mathscr{I}(f\chi_{E'})=\\ &= \mathscr{I}(gX_{E'})=\mathscr{I}g\,.\end{aligned}$$

The following result is sometimes called the Theorem of Beppo—Levi.

Theorem 18. *Let $\{f_n\}$ be a sequence of integrable functions a.e. increasing, which converges a.e. to the measurable function f. Let $\mathscr{I}f_n \leqq c < \infty$ $n=1, 2, \ldots$. Then f is integrable and*

$$\mathscr{I}f=\lim_n \mathscr{I}f_n\,.$$

Proof. Let E be such a measurable set that $\mu(E)=0$ and $f_n(x)\nearrow f(x)$ for $x\notin E$; then

$$f_n\chi_{E'}\nearrow f\chi_{E'}.$$

If follows from 8° that

$$\mathscr{I}(f_n\chi_{E'})\nearrow\mathscr{I}(f\chi_{E'}).$$

By the last proposition $\mathscr{I}(f_n\chi_{E'})=\mathscr{I}f_n$ and $\mathscr{I}(f\chi_{E'})=\mathscr{I}f$, and so

$$\mathscr{I}f=\mathscr{I}(f\chi_{E'})=\lim_n \mathscr{I}(f_n\chi_{E'})=\lim_n \mathscr{I}f_n .$$

The Banach spaces $\mathscr{L}_p(X,\mu)$ $1\leqq p\leqq\infty$ for a continuous strong submeasure are defined in the natural way. All classical results; namely Hölder's inequality, Minkowski's inequality and the completeness of $\mathscr{L}_p$ are valid. We give first the proof of the completeness of $\mathscr{L}_1$.

For a function $f\in\mathscr{L}_p$ we write

$$\|f\|_p=\sqrt[p]{\mathscr{I}|f|^p}.$$

The number $\|f\|_p$ is called a *pseudonorm* of f.

Theorem 19. *The linear space $\mathscr{L}_1(X,\mathscr{S},\mu)$ of all integrable functions on $(X,\mathscr{S},\mu)$ is a complete pseudometric space with respect to the pseudometric*

$$\varrho(f,g)=\|f-g\|_1=\|f-g\|.$$

Proof. Let $\{f_n\}$ be a fundamental sequence of integrable functions with $\|f_{n+1}-f_n\|<1/2^n$. Put $f_0=0$ and

$$\varphi_n=\sum_{k=1}^{n}|f_k-f_{k-1}| \quad (\varphi_n(x)=0 \text{ if the } \sum_{k=1}^{n}|f_k(x)-f_{k-1}(x)| \text{ is not defined}).$$

$$\text{Then } \varphi_n\nearrow\varphi=\sum_{k=1}^{\infty}|f_k-f_{k-1}| \text{ and } \mathscr{I}\varphi_n\leqq 1.$$

By the theorem of Beppo—Levi we get that φ is integrable and the sequence $\{\varphi_n\}$ has a limit a.e. clearly $f_n=\sum_{k=1}^{n}(f_k-f_{k-1})$ has a limit a.e. too. We define $f(x)=0$ if $\lim_n f_n(x)$ does not exist and put $f(x)=\lim_n f_n(x)$ in the oposite case. $|f|=\varphi$, and so f is integrable. Choose n_0 with $\|f_n-f_m\|<\varepsilon$ for $n,m\geqq n_0$; then

$$\begin{aligned}\|f-f_p\|&=\mathscr{I}(|f-f_p|)=\mathscr{I}(|\lim_n f_n-f_p|)\leqq\\ &\leqq\mathscr{I}(\lim_n |f_n-f_p|)=\liminf_n \|f_n-f_p\|\leqq\varepsilon .\end{aligned}$$

In the following we give a proof of a result of Mazur [4] (see also [6]).

Theorem 20. *The spaces $\mathscr{L}_p(X,\mathscr{S},\mu)$ $(p\geqq 1)$ are complete pseudometric spaces.*

Proof. Let $p>1$. Let us define a map Φ: $\mathscr{L}_p\to\mathscr{L}_1$ by $\Phi f=|f|^p \operatorname{sign} f$ and a map Ψ: $\mathscr{L}_1\to\mathscr{L}_p$ by $\Psi g=|g|^{1/p}\operatorname{sign} g$.

Since

$$2^{1-p}|x-y|^p \leqq |x\,|x|^{p-1} - y\,|y|^{p-1}| \quad \text{for } x, y \in R,$$

we get

$$|\Psi(g_1) - \Psi(g_2)|^p \leqq 2^{p-1}|g_1 - g_2|$$

and hence

$$\|\Psi(g_1) - \Psi(g_2)\|_p \leqq 2^{1/q}\|g_1 - g_2\|_1^{1/p}$$

for all g_1 and g_2 in $\mathscr{L}_1$. Clearly Ψ is a continuous map.

Since

$$|x\,|x|^{p-1} - y\,|y|^{p-1}| \leqq p\,|x-y|(|x|^{p-1} + |y|^{p-1})$$

yields

$$|\Phi(f_1) - \Phi(f_2)| \leqq p(|f_1 - f_2|\;|f_1|^{p-1} + |f_1 - f_2|\;|f_2|^{p-1}),$$

hence by the Hölder inequality

$$\|\Phi(f_1) - \Phi(f_2)\| \leqq p\,\|f_1 - f_2\|_p(\|f_1\|_p^{p/q} + \|f_2\|_p^{p/q}).$$

If $\{f_n\}$ is a fundamental sequence in $\mathscr{L}_p$, it is bounded and is therefore carried by Φ into a fundamental sequence $\{g_n\}$, $g_n = \Phi(f_n)$ in $\mathscr{L}_1$.

Let g be a limit of this sequence in $\mathscr{L}_1$, by the completeness of $\mathscr{L}_1$ this exists. By the continuity of Ψ the function $f = \Psi(g)$ is the limit of $\{f_n\}$ in $\mathscr{L}_p$.

Corollary 21. *The spaces $\mathscr{L}_p$, $p \geqq 1$ are mutually homeomorphic.*

§ 4. On the value of the integral

We turn now our attention to the theorem about the value of the integral. We shall need the following two lemmas.

For the rest of this paper $(X, \mathscr{S})$ will be a measurable space (see [2]) and μ will be a strong submeasure on $\mathscr{S}$.

Lemma 22. *Let $B_1 \subset A_1 \subset B_2 \subset A_2 \subset \ldots \subset B_n \subset A_n$ be a sequence of sets from $\mathscr{S}$ with $\mu(B_n) < \infty$; then*

$$\sum_{i=1}^{n}(\mu(A_i) - \mu(B_i)) \leqq \mu\Big(\bigcup_{i=1}^{n}(A_i - B_i)\Big).$$

Proof. For $n = 1$ the inequality is an easy consequence of the subadditivity of μ.

Let $n = 2$; then, using the subadditivity and the strong subadditivity of μ, we get

$$\mu(B_1)+\mu(B_2)+\mu((A_1-B_1)\cup(A_2-B_2))\geqq\mu(B_2)+\mu(A_1\cup(A_2-B_2))\geqq$$
$$\geqq\mu(B_2\cap(A_1\cup(A_2-B_2)))+\mu(B_2\cup A_1\cup(A_2-B_2))=$$
$$=\mu(A_1)+\mu(A_2)$$

from which the inequality follows. The proof in general proceeds by induction.

Lemma 23. *(see [8]). Let E_0 be the linear subspace of a partially ordered upward filtering linear space E. Let l_0 be a linear monotone functional on E_0 and let p be a seminorm on E. Let $l_0(x)\leqq p(x)$ for $x\in E_0^+$. Then there exists the linear monotone functional l on E with the properties:*

(i) *l is an extension of f_0,*

(ii) *$l(x)\leqq p(x)$ for x in E^+.*

Lemma 24. *Let $\mathscr{R}$ be a subring of $\mathscr{S}$. If φ is a measure on $\mathscr{R}$, with $\varphi\leqq\mu$ then there exists a finitely additive measure ν on $\mathscr{S}$, such that ν is an extension of φ and $\nu\leqq\mu$ on $\mathscr{S}$.*

Proof. Denote $\bar{E}_0=\mathscr{L}_1(\mathscr{R},\varphi)$, $E=\mathscr{L}_1(\mathscr{S},\mu)$, $E_0=\bar{E}_0\cap E$, $l_0=\mathscr{I}_\varphi$ and $p(f)=\mathscr{I}_\mu|f|$. Then l_0 is a linear functional on E_0, p is the seminorm on E and $l_0\leqq p$ on E_0^+. It follows from the last lemma that there exists the linear monotone functional l on E such that l is an extension of l_0 and $l\leqq p$ on E^+. Let us define the set function ν as follows: $\nu(A)=l\chi_A$. By the additivity and monotonicity of l it follows that ν is a finitely additive measure on $\mathscr{S}$. Since $\nu(A)=l\chi_A\leqq p\chi_A=\mathscr{I}_\mu\chi_A=\mu(A)$ for A in $\mathscr{S}$, we have $\nu\leqq\mu$ on $\mathscr{S}$.

Theorem 25. *Let f be a nonnegative integrable function on X; then*

$$\mathscr{I}_\mu f=\max_\nu \mathscr{I}_\nu f\,,$$

where the maximum is taken over all finitely additive measures ν such that $0\leqq\nu\leqq\mu$ on $\mathscr{S}$.

Proof. We put

$$\mathscr{D}=\{\{x\,;f(x)\geqq a\}\,;a>0\}\cup\{\emptyset\}\,.$$

Denote by $\mathscr{R}$ the family of all sets of the form

$\bigcup\limits_{i=1}^{n}(A_i-B_i)$, where $B_1\subset A_1\subset B_2\subset A_2\subset\ldots\subset B_n\subset A_n$ is a sharply increasing sequence of sets from $\mathscr{D}$ and the sets A_i-B_i $(i=1,2,\ldots,n)$ are pairwise disjoint. We put

$$\varphi\Big(\bigcup_{i=1}^{n}(A_i-B_i)\Big)=\sum_{i=1}^{n}(\mu(A_i)-\mu(B_i))\,.$$

It is easy to see that $\mathscr{R}$ is a ring. By Lemma 22 φ is a finitely additive measure on $\mathscr{R}$ with $\varphi\leqq\mu$. By the last lemma there exists a finitely additive measure ν on $\mathscr{S}$, such that ν is an extension of φ and $\nu\leqq\mu$ on $\mathscr{S}$. Since $\mathscr{I}_\mu f$ depends only on the values of μ on $\mathscr{D}$, it is clear that $\mathscr{I}_\mu f=\mathscr{I}_\nu f$. Since for all finitely additive measures τ on $\mathscr{S}$ with $\tau\leqq\mu$ there holds $\mathscr{I}_\tau f\leqq\mathscr{I}_\mu f$, we get

$$\mathscr{I}_\mu f=\max_\nu \mathscr{I}_\nu f\,.$$

Corollary 26. *If μ is a strong submeasure on a σ-ring $\mathscr{S}$, then*

$$\mu = \max_{\nu} \nu$$

where $0 \leqq \nu \leqq \mu$ and ν is a finitely additive measure on $\mathscr{S}$.

Proof. This is a clear conclusion of the last theorem and the fact that $\mathscr{I}_\tau \chi_A = \tau(A)$ for every A and for every pre-measure τ on $\mathscr{S}$.

Let now m be a supermeasure on $(X, \mathscr{S})$. Similarly one can prove as the last theorem the following one:

Theorem 27. *Let m be the, strong super-measure on $\mathscr{S}$. Let f be a nonnegative integrable function; then*

$$\mathscr{I}_m f = \min_{\nu} \mathscr{I}_\nu f$$

and

$$m(A) = \min_{\nu} \nu(A) \quad A \in \mathscr{S},$$

where the minimum is taken over all finitely additive measures ν on $\mathscr{S}$ with $\nu \geqq \mu$.

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НЕЛИНЕАРНЫЕ ИНТЕГРАЛЫ

Ян Шипош

Резюме

Пусть μ-положительная, возрастающая и строго полуаддитивная функция множества определенная на некотороч σ-кольце.

В статье доказывается, что интеграл введеный в [7] является строго полуаддитивным функционалом.

Кроме того доказывается — в случае непрерывной μ — что L_p — пространство Банаха.