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DIRECT PRODUCT DECOMPOSITIONS OF g-DIGRAPHS

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The direct, subdirect and weak direct product decompositions of partially ordered sets and the decompositions of their covering graphs were investigated, e.g., in [2], [4], [5], [6], [7], [8]. The direct product decompositions of a covering graph $C(\bar{G})$ of a digraph $\bar{G}$ and the direct product decompositions of $\bar{G}$ were studied in [9].

The relation between the direct product decompositions of a covering graph $C(\bar{G})$ of a g-digraph $\bar{G}$ and the direct product decompositions of $\bar{G}$ will be studied in the present paper. The notion of a g-digraph will be introduced in the Section 2.

1. Preliminaries

We start by recalling some notions concerning graphs, digraphs and direct products (cf. also [7] and [9]). For all further notions concerning digraphs and graphs we refer the reader to [3].

Let $\bar{G} = (V, \bar{E})$ be a digraph. By the *covering graph of* $\bar{G}$ we mean a graph $C(\bar{G}) = (V, E)$ whose edges are those pairs $\{a, b\}$, for which $(a, b) \in \bar{E}$ or $(b, a) \in \bar{E}$.

Let I be a nonempty set and $G_i = (V_i, E_i)$ $(i \in I)$ be graphs. Let V be the cartesian product of the sets $V_i \left(V = \prod_{i \in I} V_i \right)$. The elements of V will be denoted $a = (a_i)$, $i \in I$, where $a_i = a(i) \in V_i$. Let G be a graph whose set of vertices is V and whose set of edges consists of those pairs $\{x, y\}$, $x, y \in V$ which satisfy the following condition: there is $i \in I$ such that $\{x_i, y_i\} \in E_i$ and $x_j = y_j$ for each $j \in I \setminus \{i\}$. Then G is said to be a *direct product of the graphs* G_i $(i \in I)$ and we write $G = \prod_{i \in I} G_i$. We omit the symbol $i \in I$ very often if no misunderstanding can arise.

The *direct product of digraphs* is defined similarly.

If a mapping $f\colon V_1 \to V_2$ is an isomorphism of a graph $G_1 = (V_1, E_1)$ into a graph $G_2 = (V_2, E_2)$, then we shall write $G_1 \overset{f}{\simeq} G_2$. If such an isomorphism does exist, then we write shortly $G_1 \simeq G_2$.

If $G \overset{f}{\simeq} \Pi G_i$, then we shall say that ΠG_i is a *decomposition of* G (*with respect to the map* f).

In the present paper every decomposition ΠG_i, where $G_i = (V_i, E_i)$, is supposed to be nontrivial (i.e. $|V_i| > 1$ for each $i \in I$).

If $G \overset{f}{\simeq} \prod_{i \in I} G_i$ where G is connected, then I must be finite (cf. [11]).

The subgraph of a graph $G = (V, E)$ induced by a set $W \subseteq V$ will be denoted by $G\langle W \rangle$.

An analogous terminology and notions are used for digraphs and partially ordered sets.

Let $G = (V, E)$ be a graph. If there exists a four-element set $W = \{a, b, c, d\} \subseteq V$ such that $G\langle W \rangle = (W, F)$, where $F = \{\{a, b\}, \{b, c\}, \{c, d\}, \{a, d\}\}$, then we say that the graph $G\langle W \rangle$ is a *square* (*in* G) and we denote it by $S(a, b, c, d)$. If $\bar{G}$ is a digraph and $C(\bar{G}\langle W \rangle) = S(a, b, c, d)$, then $\bar{G}\langle W \rangle$ is called a *square* (*in* $\bar{G}$) and denoted by $\bar{S}(a, b, c, d)$.

Let $k \in I$. The edge $\{a, b\}$ of ΠG_i will be called a *k-edge* whenever $a_j = b_j$ for each $j \in I \setminus \{k\}$.

Lemma 1 [9]. *Let* $S(a, b, c, d)$ *be a square in* ΠG_i. *If* $\{a, b\}$ *is an r-edge and* $\{b, c\}$ *is an s-edge,* $r \neq s$, *then* $a_r = d_r$ *and* $c_s = d_s$.

Lemma 2 [9]. *Let* $S(a, b, c, d)$ *be a square in* ΠG_i. *If* $\{a, b\}$ *is an r-edge and* $\{b, c\}$ *an s-edge, then* $\{c, d\}$ *is an r-edge and* $\{a, d\}$ *an s-edge.*

A square $S(a, b, c, d)$ in ΠG_i $(i \in I)$ will be called an *r-square* whenever all its edges are r-edges for some $r \in I$. If such $r \in I$ does not exist, it will be called a *mixed square.*

Let $\bar{G} = (V, \bar{E})$ be a digraph and $C(\bar{G}) \overset{f}{\simeq} \Pi G_i$, where $G_i = (V_i, E_i)$ $(i \in I)$. We shall say that the *decomposition* ΠG_i (*with respect to the map* f) *of* $C(\bar{G})$ *induces a decomposition of* $\bar{G}$ if there exist such diagraphs $\bar{G}_i = (V_i, \bar{E}_i)$ that $C(\bar{G}_i) = G_i$ for each $i \in I$ and $\bar{G} \overset{f}{\simeq} \Pi \bar{G}_i$.

Let $C(\bar{G}) \overset{f}{\simeq} \Pi G_i$. We shall say that the edge (a, b) of $\bar{G}$ and the edge $\{a, b\}$ of $C(\bar{G})$ are *k-edges* (*with respect to the isomorphism* f) if $\{f(a), f(b)\}$ is a k-edge of ΠG_i. In an analogous way the other notions concerning the direct product ΠG_i can be introduced for $\bar{G}$ and $C(\bar{G})$.

Let $\bar{S}_i$ $(i = 1, 2, 3)$ and $\bar{S}$ be as in the Figure:

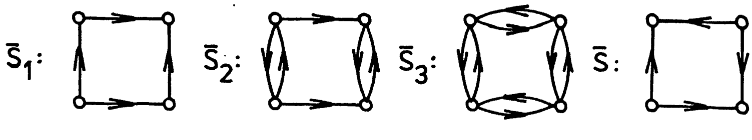

Fig. 1

Theorem 1 [9]. *Let* $C(\bar{G}) \overset{f}{\simeq} \Pi G_i$, *where* $\bar{G} = (V, \bar{E})$ *is a weakly connected digraph. The decomposition* ΠG_i *of* $C(\bar{G})$ *induces a decomposition of* $\bar{G}$ *iff the following condition is fulfilled:*

If $\bar{S}(a, b, c, d)$ *is a mixed square in* $\bar{G}$, *then there exists* $i \in \{1, 2, 3\}$ *with* $\bar{S}(a, b, c, d) \simeq \bar{S}_i$.

2. Decompositions of g-digraphs

A path from a to b in a graph (a digraph) will be said to be an $a - b$ *path.*

Let $\bar{G} = (V, \bar{E})$ be a digraph. An edge $(a, b) \in \bar{E}$ will be called *transitive* if there exists a vertice $c \in V$, $c \neq a$, $c \neq b$ such that there is a (directed) $a - c$ path and also a (directed) $c - b$ path.

The following lemma is easy to verify (cf. also [9], Lemma 8).

Lemma 3. *If* $\bar{S}(a, b, c, d)$ *is a square of an acyclic digraph* $\bar{G}$ *with no transitive edge, then* $\bar{S}(a, b, c, d)$ *is isomorphic either to* $\bar{S}_1$ *or to* $\bar{S}$.

We say that an acyclic digraph $\bar{G}$ is a *g-digraph* iff all (directed) paths between the same vertices have the same length.

For the vertices a, b of $\bar{G}$, $\bar{d}(a, b)$ shall denote the length of a shortest $a - b$ path and $d(a, b)$ shall denote the length of a shortest $a - b$ semipath (i.e. the length of a shortest $a - b$ path in $C(\bar{G})$). Obviously, $\bar{d}(a, b) \geqq d(a, b)$.

A *source* in $\bar{G}$ is a vertex which can reach all the others.

Lemma 4. *Let* $\bar{G} = (V, \bar{E})$ *be a g-digraph and let* $z \in V$ *be a source of* $\bar{G}$. *Then* $\bar{d}(z, a) = d(z, a)$ *for each* $a \in V$.

Proof. If $d(z, a) = k$, then there exist vertices $a_0, a_1, \ldots, a_k \in V$ such that

$$z = a_0, a_1, \ldots, a_k = a \tag{1}$$

is a $z - a$ semipath. Let a_i be the first vertex of the semipath (1) such that $(a_i, a_{i-1}) \in \bar{E}$ (clearly, $i \geqq 3$). Since $\bar{G}$ is a g-digraph, $\bar{d}(z, a_i) + \bar{d}(a_i, a_{i-1}) = \bar{d}(z, a_{i-1}) = i - 1$. So $\bar{d}(z, a_i) = i - 2$. Thus $d(z, a_i) \leqq i - 2$, a contradiction.

Using Lemma 4 from [10] the following lemma is easy to verify.

Lemma 5. *Let* $a = (a_i)$, $b = (b_i)$ *be the vertices of* $\prod_{i \in I} G_i$, $I = \{1, 2, \ldots, n\}$. *Then*

$$d(a, b) = \sum_{i=1}^{n} d(a_i, b_i).$$

Lemma 6. *Let* $\bar{G} = (V, \bar{E})$ *be a g-digraph,* $z \in V$ *be a source of* $\bar{G}$ *and let* $C(\bar{G}) \overset{f}{\simeq} \Pi G_i$ $(i \in I)$, $I = \{1, 2, \ldots, n\}$. *Then every mixed square in* $\bar{G}$ *is isomorphic to* $\bar{S}_1$.

Proof. Suppose that a mixed square $\bar{S}(p, q, x, y)$ in $\bar{G}$ is not isomorphic to $\bar{S}_1$. Then (Lemma 3) $\bar{S}(p, q, x, y) \simeq \bar{S}$. Let $(p, q) \in \bar{E}$. Then $(x, q) \in \bar{E}$, $(x, y) \in$

$\in \bar{E}$, $(p, y) \in \bar{E}$. Since z is a source of $\bar{G}$ we get $\bar{d}(z, p) + 1 = \bar{d}(z, q)$ and $\bar{d}(z, x) + 1 = \bar{d}(z, y)$. From Lemma 4 it follows that $d(z, p) + 1 = d(z, q)$ and $d(z, x) + 1 = d(z, y)$. Further (since the map f is an isomorphism) we get

$$d(f(z), f(p)) + 1 = d(f(z), f(q)) \tag{2}$$

and

$$d(f(z), f(x)) + 1 = d(f(z), f(y)). \tag{3}$$

Since $S(f(p), f(q), f(x), f(y))$ is a mixed square in ΠG_i, then there exist $r, s \in I$, $r \neq s$, such that $\{f(p), f(q)\}$, $\{f(x), f(y)\}$ are r-edges and $\{f(x), f(q)\}$, $\{f(p), f(y)\}$ are s-edges. If we denote $f(z) = (z_i)$, $f(p) = (p_i)$, $f(q) = (q_i)$, $f(x) = (x_i)$, $f(y) = (y_i)$, $i \in I$, then (Lemma 1) we get

$$p_s = q_s,\ q_r = x_r,\ x_s = y_s,\ p_r = y_r \quad \text{and} \quad p_i = q_i = x_i = y_i \tag{4}$$

for each $i \in I \setminus \{r, s\}$.
From (2), (3) and Lemma 5 we have

$$\sum_{i \neq r, s} d(z_i, p_i) + d(z_r, p_r) + d(z_s, p_s) + 1 = \sum_{i \neq r, s} d(z_i, q_i) + d(z_r, q_r) + d(z_s, q_s) \tag{5}$$

and

$$\sum_{i \neq r, s} d(z_i, x_i) + d(z_r, x_r) + d(z_s, x_s) + 1 = \sum_{i \neq r, s} d(z_i, y_i) + d(z_r, y_r) + d(z_s, y_s). \tag{6}$$

Hence (with respect to (4)) it follows immediately that

$$d(z_r, p_r) + 1 = d(z_r, q_r) \tag{7}$$

and

$$d(z_r, q_r) + 1 = d(z_r, p_r), \tag{8}$$

a contradiction. In the case when $(q, p) \in \bar{E}$ we obtain a contradiction in a similar way.

Theorem 1 and Lemma 6 imply the following

Theorem 2. *Let a g-digraph $\bar{G}$ have a source. Then every decomposition ΠG_i of $C(\bar{G})$ induces a decomposition of $\bar{G}$.*

3. Decompositions of graded partially ordered sets

All partially ordered sets dealt with in this paper are assumed to be almost discrete.

Every partially ordered set $(P, \leqq)$ (shortly P) may be represented as a digraph $\bar{G} = (P, \bar{E})$ such that $(a, b) \in \bar{E}$ iff b covers a ($a \prec b$) (cf. also [9] and [12]). Clearly, this digraph is acyclic and has no transitive edge.

Every acyclic digraph $\bar{G} = (P, \bar{E})$ with no transitive edge represents a partially ordered set $(P, \leqq)$ such that $a \prec b$ iff $(a, b) \in \bar{E}$ (the ordering on P is determined by this covering relation).

If $\bar{G} = (P, \bar{E})$ represents a partially ordered set $(P, \leqq)$, we shall say that $\bar{G}$ *is a digraph of* $(P, \leqq)$. From the above mentioned facts it follows that $a \leqq b$ in $(P, \leqq)$ iff there exists an $a - b$ path in $\bar{G}$.

The *direct product of partially ordered sets* is defined in the usual way. For all further notions concerning the partially ordered sets we refer the reader to [1].

From definitions of the direct product of the digraphs and of the partially ordered sets we obtain immediately.

Lemma 7. *Let* $(P_i, \leqq)$ *be a partially ordered set and* $\bar{G}_i = (P_i, \bar{E}_i)$ *be their digraph for each* $i \in I$. *Then* $\Pi \bar{G}_i = (P, \bar{E})$ *is a digraph of* $(\Pi P_i, \leqq) = (P, \leqq)$.

Lemma 8. *Let* $(P_1, \leqq_1)$, $(P_2, \leqq_2)$ *be partially ordered sets and* $\bar{G}_1 = (P_1, \bar{E}_1)$, $\bar{G}_2 = (P_2, \bar{E}_2)$ *be their digraphs. If a map* $f\colon P_1 \to P_2$ *is an isomorphism of* $\bar{G}_1$ *into* $\bar{G}_2$, *then* f *is an isomorphism of* $(P_1, \leqq_1)$ *into* $(P_2, \leqq_2)$.

Proof. Let $a, b \in P_1$, $a \leqq_1 b$. Then there exists an $a - b$ path in $\bar{G}_1$. Since $\bar{G}_1 \overset{f}{\simeq} \bar{G}_2$ there exists an $f(a) - f(b)$ path in $\bar{G}_2$. Hence $f(a) \leqq_2 f(b)$ in $(P_2, \leqq_2)$. Similarly we obtain that if $a, b \in P_2$, $a \leqq_2 b$, then $f^{-1}(a) \leqq_1 f^{-1}(b)$.

The *covering graph* $C(P)$ *of a partially ordered set* P is the graph whose vertices are the elements of P and whose edges are those pairs $\{a, b\}$, $a, b \in P$, for which $a \prec b$ or $b \prec a$.

If a partially ordered set P has a least element and all maximal chains between the same endpoinds have the same length, then we say that P is a *graded partially ordered set*.

Obviously, $C(P) = C(\bar{G})$, where $\bar{G}$ is the digraph of P.

Let P be a graded partially ordered set and $\bar{G}$ be a digraph of P. Then $\bar{G}$ is a g-digraph and it obviously has a source. By Theorem 2 and Lemma 7 this implies the following corollary, which is a generalization of a result from [2].

Corollary. *Let* P *be a graded partially ordered set and let* $C(P) \overset{f}{\simeq} \prod_{i \in I} G_i$, $I = \{1, 2, \ldots, n\}$, *where* $G_i = (P_i, E_i)$ $(i \in I)$ *are graphs. Then the decomposition* ΠG_i *of* $C(P)$ *induces a decomposition of* P.

Proof. Let $\bar{G} = (P, \bar{E})$ be a digraph of P. Then (Theorem 2) $\bar{G} \overset{f}{\simeq} \Pi \bar{G}_i$ and $C(\bar{G}_i) = G_i$ for each $i \in I$, where $\bar{G}_i = (P_i, \bar{E}_i)$ $(i \in I)$ are acyclic digraphs with no transitive edges. Hence $\bar{G}_i$ is a digraph of a partially ordered set P_i for each $i \in I$. From Lemma 7 it follows that $\Pi \bar{G}_i$ is a digraph of a partially ordered set ΠP_i. Then, by Lemma 8, $P \overset{f}{\simeq} \Pi P_i$ and (since $C(\bar{G}_i) = C(P_i)$) $C(P_i) = G_i$ for each $i \in I$.

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РАЗЛОЖЕНИЯ *g*-ОРГРАФОВ НА ПРЯМЫЕ ПРОИЗВЕДЕНИЯ

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Резюме

В статье рассматриваются некоторые отношения между разложениями *g*-орграфов и градуированных частично упорядоченных множеств на прямые произведения и разложениями их покрывающих графов.