Beloslav Riečan

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# A SIMPLIFIED PROOF OF THE DANIELL INTEGRAL EXTENSION THEOREM IN ORDERED SPACES

BELOSLAV RIEČAN

The aim of the paper is a simplification of the Fremlin proof of the Matthes--Wright extension theorem (see [1]). Especially, we omit the notion of $\sigma$-depressed sets and decompose this proof into a few simple steps.

We shall work with $\sigma$-complete Riesz spaces, i.e. linear spaces being boundedly $\sigma$-complete lattices and fulfilling the identity $a + (b \vee c) = (a + b) \vee (a + c)$. A Riesz space is called weakly $\sigma$-distributive if for every bounded double sequence $(a_{ij})_{i,j}$ such that $a_{ij} \searrow 0$ $(j \to \infty,\ i = 1, 2, \ldots)$ it is $\bigwedge_{\varphi \in N^N} \bigvee_i a_{i\varphi(i)} = 0$.

**Theorem.** *Let $X$, $G$ be $\sigma$-complete Riesz spaces, $G$ be weakly $\sigma$-distributive, $A$ be a Riesz subspace of $X$ such that every element of $X$ is dominated by some element of $A$. Let $J_0\colon A \to G$ be a linear, positive and continuous (i.e. $x_n \searrow 0 \Rightarrow J_0(x_n) \searrow 0$) map. Then $J_0$ can be extended to a linear, positive, continuous map, defined on the $\sigma$-complete subspace generated by $A$.*

The proof will consist of a few propositions. First we define

$$A^+ = \{b \in X;\ \exists a_n \in A,\ a_n \nearrow b\},$$
$$J^+\colon A^+ \to G,\ J^+(b) = \lim_{n \to \infty} J_0(a_n),\ a_n \nearrow b\,.$$

(It is easy to prove that $J^+(b)$ does not depend on the choice of the sequence $(a_n)_n$ converging to $b$.) Dually we define $A^-$ and $J^-$.

**Proposition 1.** *If $b \in A^+$, $c \in A^-$, $c \leqq b$, then $J^-(c) \leqq J^+(b)$.*

Proof. Choose $b_n \in A$, $c_n \in A$ $(n = 1, 2, \ldots)$ such that $b_n \nearrow b$, $c_n \searrow c$. Then $b_n - c_n \nearrow b - c$, hence

$$J^+(b) = \lim_{n \to \infty} J_0(b_n) = \lim_{n \to \infty} J_0(b_n - c_n) + \lim_{n \to \infty} J_0(c_n) =$$
$$= J^+(b - c) + J^-(c) \geqq J^-(c),$$

since $b \geqq c$ and so $b - c \geqq 0$.

**Definition 1.** *By $L$ we denote the set of all $x \in L$ with the following properties: There is $\alpha \in G$ and there are $a_{ij}, b_{ij} \in G$ such that $a_{ij} \searrow 0$, $b_{ij} \searrow 0$ $(j \to \infty, i = 1, 2, \ldots)$ and such that to every $\varphi \in N^N$ there are $x_1^\varphi \in A^-$, $x_2^\varphi \in A^+$ satisfying the relations $x_1^\varphi \leqq x \leqq x_2^\varphi$ and*

$$J^+(x_2^\varphi) - \bigvee_i a_{i\varphi(i)} \leqq \alpha \leqq J^-(x_1^\varphi) + \bigvee_i b_{i\varphi(i)}.$$

(The preceding definition substitutes the classical definition $\inf\{J^+(y);\ y \geqq x, y \in A^+\} = \sup\{J^-(z);\ z \leqq x, z \in A^-\}$ and enables to use something similar to the "epsilon" technic in the abstract position.)

**Proposition 2.** *If $x \in L$ and $\alpha$ is the corresponding element of $G$, then*

$$\alpha = \bigwedge\{J^+(x_2);\ x_2 \geqq x, x_2 \in A^+\} = \bigvee\{J^-(x_1);\ x_1 \leqq x, x_1 \in A^-\}.$$

Proof.*) Let $x_1 \in A^-$, $x_1 \leqq x$. Then $x_1 \leqq x_2^\varphi$, hence $J^-(x_1) \leqq J^+(x_2^\varphi)$ by Prop. 1. Therefore

$$J^-(x_1) - \alpha \leqq J^+(x_2^\varphi) - \alpha \leqq \bigvee_i a_{i\varphi(i)}$$

for all $\varphi \in N^N$, hence $J^-(x_1) - \alpha \leqq 0$ by the $\sigma$-distributivity of $G$. We have proved that $\alpha$ is an upper bound of the set $\{J^-(x_1);\ x_1 \leqq x, x_1 \in A^-\}$.

Let $\beta$ be another upper bound of the set $\{J^-(x_1);\ x_1 \leqq x, x_1 \in A^-\}$. Then $\beta \geqq J^-(x_1^\varphi)$, hence $\alpha - \beta \leqq \alpha - J^-(x_1^\varphi) \leqq \bigvee_i b_{i\varphi(i)}$ for all $\varphi \in N^N$. Therefore $\alpha - \beta \leqq 0$, hence $\alpha$ is the least upper bound of the set $\{J^-(x_1);\ x_1 \leqq x, x_1 \in A^-\}$.

The second assertion can be proved dually.

**Definition 2.** *If $x \in L$, then the common value $\bigvee\{J^-(x_1);\ x_1 \leqq x, x_1 \in A^-\} = \bigwedge\{J^+(x_2);\ x_2 \geqq x, x_2 \in A^+\}$ will be denoted by $J(x)$.*

**Proposition 3.** *To every bounded triple sequence $(a_{n,i,j})_{n,i,j}$ such that $a_{n,i,j} \searrow 0$ $(j \to \infty, n, i = 1, 2, \ldots)$ and every $b > 0$ there exists $(a_{i,j})_{i,j}$ bounded, such that $a_{i,j} \searrow 0$ $(j \to \infty\ i = 1, 2, \ldots)$ and*

$$b \wedge \left(\sum_{n=1}^{\infty} \bigvee_{i=1}^{\infty} a_{n,i,\varphi(i+n)}\right) \leqq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$$

*for every $\varphi \in N^N$.*

Proof. Put $b_{i,j} = \bigvee_{k=1}^{i-1} 2^k a_{k,i-k,j}$, $a_{i,j} = b \wedge b_{i,j}$. Then $b_{i+k,j} \geqq 2^k a_{k,i,j}$, hence

*) Only in the proof of this proposition we use the $\sigma$-distributivity of $G$.

$$\leqq b \wedge \left( \left( \sum_{k=1}^{n} 2^{-k} \right) \bigvee_{i=1}^{n+p} b_{i,\varphi(i)} \right) \leqq \bigvee_{r=1}^{\infty} a_{r,\varphi(r)} .$$

**Proposition 4.** *$L$ is a linear subspace of $X$ and $J: L \to G$ is a linear map.*

Proof. The assertion is a straightforward application of the definitions, the linearity of $J_0$ and Proposition 3.

**Proposition 5.** *If $b_n \nearrow b$, $b_n \in A^+$ $(n=1, 2, \ldots)$, then $b \in A^+$ and $J^+(b) = \lim_{n\to\infty} J^+(b_n)$.*

Proof. If $a_{n,m} \nearrow b_n$ $(m \to \infty,\ n=1, 2, \ldots)$, $a_{n,m} \in A$, then it suffices to put $a_n = \bigvee_{k=1}^{n} a_{k,n}$ for obtaining $a_n \nearrow b$. Further

$$J^+(b) = \lim_{n\to\infty} J_0(a_n) \leqq \lim_{n\to\infty} J^+(b_n) \leqq J^+(b).$$

**Proposition 6.** *If $x_n \in L$ $(n=1,2,\ldots)$, $x_n \nearrow x$, then $x \in L$ and $J(x) = \lim_{n\to\infty} J(x_n)$.*

Proof. Let $a \in A$ be such an element that $x \leqq a$. Put $x_0 = 0$. Then $x_n - x_{n+1} \in L$ $(n=1, 2, \ldots)$, hence to every $\varphi \in N^N$ there are $a_{n,i,j} \searrow 0$ $(j \to \infty)$, $y_n \in A^+$, $y_n \geqq x_n - x_{n+1}$ such that

$$J(x_n - x_{n-1}) \geqq J^+(y_n) - \bigvee_{i=1}^{\infty} a_{n,i,\varphi(i+n)} .$$

Then

$$J(x_n) = \sum_{k=1}^{n} J(x_k - x_{k-1}) \geqq J^+\left( \left( \sum_{k=1}^{n} y_k \right) \wedge a \right) - J(a) \wedge \left( \sum_{k=1}^{n} \bigvee_{i=1}^{\infty} a_{n,i,\varphi(i+n)} \right) \geqq$$

$$\geqq J^+\left( \left( \sum_{k=1}^{n} y_k \right) \wedge a \right) - \bigvee_{i=1}^{\infty} a_{i,\varphi(i)},$$

where $(a_{i,j})_{i,j}$ is the sequence mentioned in Proposition 3. Since $y_n \geqq x_n - x_{n-1} \geqq 0$, the sequence $\left( \sum_{k=1}^{n} y_k \right)_{n=1}^{\infty}$ is non-decreasing. Since $\left( \sum_{k=1}^{n} y_k \right) \wedge a \leqq a$ for every $n$, there is $y = \bigwedge_{n=1}^{\infty} \left( \left( \sum_{k=1}^{n} y_k \right) \wedge a \right) = \left( \bigvee_{n=1}^{\infty} \sum_{k=1}^{n} y_k \right) \wedge a \geqq \left( \bigvee_{n=1}^{\infty} x_n \right) \wedge a = x \wedge a = x$ and, of course, $y \in A^+$,

$$\alpha = \lim_{n\to\infty} J(x_n) \geqq \lim_{n\to\infty} J^+\left( \left( \sum_{k=1}^{n} y_k \right) \wedge a \right) - \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} = J^+(y) - \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} .$$

We have examined the left-hand side of the definition of the relation $x \in L$: there is

$y \in A^+$, $y \geqq x$ and there is $a_{i,j} \searrow 0$ $(j \to \infty)$ such that $J^+(y) - \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \leqq \alpha$. We shall consider the right-hand side now.

We have also elements $b_{n,i,j} \searrow 0$ $(j \to \infty)$ and $z_n \leqq x_n$, $z_n \in A^-$ such that

$$J(x_n) \leqq J^-(z_n) + \bigvee_i b_{n,i,\varphi(n+i)}.$$

Put $b_{0,i,j} = \alpha - J(x_j)$ $(i, j = 1, 2, \ldots)$. Since $J(x_n) \nearrow \bigvee_n J(x_n) = \alpha$, we have $b_{0,i,j} \searrow 0$ $(j \to \infty)$. By Prop. 3 there are $b_{i,j} \searrow 0$ $(j \to \infty)$ such that

$$J(a) \wedge \left( \sum_n \bigvee_i b_{n,i,\varphi(i+n)} \right) \leqq \bigvee_i b_{i,\varphi(i)}.$$

For every $n$ there holds

$$\alpha = b_n + J(x_n) \leqq J^-(z_n) + J(a) \wedge \left( \sum_n \bigvee_i b_{n,i,\varphi(i+n)} \right).$$

Choose $n \geqq \varphi(1)$ and put $z = z_n$. Then evidently $z \in A^-$, $z \leqq x_n \leqq x$ and

$$\alpha \leqq J^-(z) + \bigvee_i b_{i,\varphi(i)}.$$

Now the right-hand side has been examined, too. By the definition, $J(x) = \alpha = \lim_{n \to \infty} J(x_n)$.

## REFERENCES


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*Katedra teórie pravdepodobnosti*
*a matematickej štatistiky*
*Matematicko-fyzikálnej fakulty UK*
*Mlynská dolina*
*842 15 Bratislava*

# УПРОЩЕННОЕ ДОКАЗАТЕЛЬСТВО ТЕОРЕМЫ О ПРОДОЛЖЕНИИ ИНТЕГРАЛА ДАНИЭЛЯ В УПОРЯДОЧЕННЫХ ПРОСТРАНСТВАХ

Белослав Риечан

## Тезюме

В рабoта приводится новое доказательство следующей теоремы: Пусть $X$ и $G$ $\sigma$-полные пространства Риса (т. е. действительные линейные пространства являющиеся одновременно относительно $\sigma$-полными структурами и выполняющими тождество $y+(b\vee c) = (a+b) \vee (a+c)$). Пусть $G$ является $\sigma$-дистрибутивным

$$\left(\text{т. е. из } a_{ij}\searrow 0 \ (j\to\infty,\ i=1, 2, \dots) \text{ вытекает } \bigwedge_{\varphi\in N^N} \bigvee_i a_{i\varphi(i)}=0\right).$$

Пусть $A$ подпространство Риса пространства $X$ такое, что для всякого $x \in X$ существует $a \in A$ так, что $x \leqq a$. Пусть $J_0 : A \to G$ линейное, положительное и непрерывное (т. е. $x_n\searrow 0 \Rightarrow J_0(x_n)\searrow 0$) отображение. Тогда $J_0$ возможно продолжить до линейного, положительного и непрерывного отображения, определенного на $\sigma$-полном подпространстве порожденном множеством $A$.

Теорема принадлежит Маттэсу и Райту и приведенное доказательство является упрощением ранее публикованного доказательства принадлежащего Фремлину. В часности здес не используется понятие $\sigma$-сжатого множества и доказательство разложено в несколько простых шагов.