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ON EDGE DOMATIC NUMBER OF HYPERCUBES

PIER VITTORIO CECCHERINI* — IVAN HAVEL†**

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ABSTRACT. Certain upper bounds of the edge domatic number of a hypercube Q_n are determined. Since they equal for $n < 10$ (with the exception of $n = 8$) the exact values for these small n follow.

1. Introduction and notation

The purpose of this note is to make a contribution to the problem of domination in hypercubes. Both the graphs of hypercubes Q_n, and concepts related to domination are for many reasons important and interesting (cf. [5]). We deal with the so called edge domatic number of Q_n. The edge domatic number of a graph was introduced by Zelinka (cf. [10] and also [5], [6], [12]), and it has been evaluated only in few particular cases (cf. [8], [11], [13]). One reason why the notions connected with edge domination in a graph G (like the edge domination number and the edge domatic number of G) are worth studying is that they correspond to analogous notions of ordinary (i.e. vertex) domination in the line graph $\mathcal{L}(G)$ of G. In particular, the edge domatic number corresponds to the ordinary domatic number, a parameter introduced in [1] by E. J. Cockayne and S. T. Hedetniemi. We also establish a relation between the edge domination in G and maximal matchings of G and use it to obtain an upper bound of the edge domatic number of Q_n.

We consider finite, undirected, and simple graphs G with the vertex set $V(G)$ and the edge set $E(G)$. A common definition of the *hypercube* Q_n, $n \geq 1$, is used: The vertices of Q_n are all 0–1 vectors of the size n, two vertices being adjacent if they differ in exactly one coordinate.

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† Ivan Havel died the 28th of November 1999.

As to the concepts of domination, we recall the definitions of only those we directly need:

Two different edges of G are called *adjacent* if they have a vertex in common.

Let $E' \subseteq E(G)$. We say that E' *dominates* $E(G)$ (cf. [9]) if for any $e \in E(G) \setminus E'$ there exists $e' \in E'$ such that e and e' are adjacent. $\gamma_e(G)$, the *edge domination number of* G, is the size of a smallest $E' \subseteq E(G)$ such that E' dominates $E(G)$. (Observe that $\gamma_e(G) = \gamma(\mathcal{L}(G))$, where $\gamma(\mathcal{L}(G))$ denotes the (vertex) domination number of the line graph of G.)

A partition of $E(G)$ is called an *edge domatic partition* if any class of it dominates $E(G)$. The maximum number of classes in an edge domatic partition of $E(G)$ is called the *edge domatic number of* G and is denoted by $\mathrm{ed}(G)$.

A *matching of* G is any set of edges of G such that no two of them are adjacent. A matching is *maximal* if it is contained in no larger matching. The size of a smallest maximal matching of G will be denoted by $m(G)$.

The concept of an edge domatic partition (and also that of edge domatic number) of a graph G can easily be expressed in terms of edge coloring.

For $d \geq 1$, let us call an *edge d-coloring* of G any mapping of $E(G)$ onto $\{1, \dots, d\}$. (Observe that we do not ask adjacent edges to be colored differently.) Let c be an edge d-coloring of G; we say that c is an *edge domatic d-coloring of* G if the partition of $E(G)$ given by the color classes of c is an edge domatic partition of $E(G)$. It is not difficult to see that c is an edge domatic d-coloring of G if and only if

$$\begin{aligned}&\big(e \in E(G) \ \ \& \ \ i \in \{1, \dots, d\}\big)\\ \Longrightarrow\ & c(e) = i \ \vee \ \big(\exists\, e' \in E(G)\big)\big(c(e') = i \ \ \& \ \ e' \text{ is adjacent to } e\big).\end{aligned}$$

Then $\mathrm{ed}(G)$, the edge domatic number of G, equals the maximum d such that there is an edge domatic d-coloring of G.

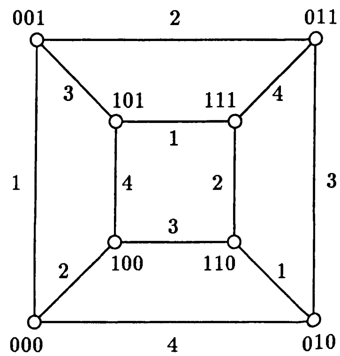

FIGURE 1. The edge domatic 4-coloring γ_1 of Q_3.

EXAMPLE. Fig. 1 shows an edge domatic 4-coloring γ_1 of Q_3, proving that $\mathrm{ed}(Q_3) \geq 4$ (in fact, $\mathrm{ed}(Q_3) = 4$).

Zelinka proved in [12] that, for $m \geq 1$,

$$\mathrm{ed}(Q_{3m}) \geq 4m .$$

Our aim in this paper is:

1) using Forcade's lower bound for the size of a smallest maximal matching of Q_n ([3]) to establish the upper bound

$$\mathrm{ed}(Q_n) \leq n \cdot 2^{n-1} \Big/ \left\lceil \frac{n \cdot 2^n}{3n-1} \right\rceil , \qquad n \geq 1 , \tag{1}$$

2) using Zelinka's construction ([12]) to prove that

$$\mathrm{ed}(Q_n) \geq \left\lfloor \frac{4n}{3} \right\rfloor , \qquad n \geq 1 , \tag{2}$$

3) combining the results 1) and 2) to determine exact values of $\mathrm{ed}(Q_n)$ for several small values of n.

2. Results

We start with establishing the equality between the edge domination number and the size of a smallest maximal matching.

PROPOSITION 1. *For any graph G,*

$$\gamma_e(G) = m(G) .$$

The proof is easy (and the fact itself is mentioned e.g. in [2]): Since every maximal matching in G dominates $E(G)$, we have $\gamma_e(G) \leq m(G)$. To prove $m(G) \leq \gamma_e(G)$ we consider a set of edges E' of G dominating $E(G)$ with $|E'| = \gamma_e(G)$. If E' already is a matching we are done; if not, we turn E' into a maximal matching in G by a finite number of steps, each of them consists of deleting one edge from E' and adding another one. An edge e' being deleted is anyone fulfilling the condition that there is $e \in E'$ adjacent to e', an edge e'' being added is anyone adjacent to no edge of $E' \setminus \{e'\}$ (one easily verifies that such e'' exists).

Consider an edge domatic d-coloring of G with maximum possible d, i.e. such that $d = \mathrm{ed}(G)$. Since every color class is an edge dominating set of G we have

$$|E(G)| \geq \mathrm{ed}(G) \cdot \gamma_e(G) ,$$

hence
$$\operatorname{ed}(G) \le \frac{|E(G)|}{\gamma_e(G)},$$
and using Proposition 1, also
$$\operatorname{ed}(G) \le \frac{|E(G)|}{m(G)}.$$

Now we turn to hypercubes. It is proved in [3] that
$$m(Q_n) \ge \frac{n \cdot 2^n}{3n-1}, \qquad n \ge 1,$$
so we have the upper bound (1) from here.

In order to prove a lower bound for $\operatorname{ed}(Q_n)$, we first introduce the following auxiliary concepts: if c is an edge domatic d-coloring of a graph G, for $v \in V(G)$ we put
$$\operatorname{Col}(c,v) = \Big\{ i \in \{1,\dots,d\} : \ (\exists\, e \in E(G))\big((v \text{ is end vertex of } e) \ \& \ c(e) = i\big) \Big\}.$$

Let c_1, c_2 be edge d-colorings of G. We call them *complementary* if for every $v \in V(G)$
$$\operatorname{Col}(c_1,v) \cup \operatorname{Col}(c_2,v) = \{1,\dots,d\}.$$

Now we are going to present certain construction of an edge coloring of a graph $G \times K_2$, where $\times$ denotes usual "box" Cartesian product of graphs; in our special case (when the second factor is K_2) we have
$$\begin{aligned} V(G \times K_2) &= \{(x,i) : \ x \in V(G) \ \& \ i \in \{0,1\}\}, \\ E(G \times K_2) &= \{(x,i)(y,i) : \ xy \in E(G) \ \& \ i \in \{0,1\}\} \\ &\qquad \cup \{(x,0)(x,1) : \ x \in V(G)\}. \end{aligned}$$

The construction produces an edge $(d+1)$-coloring of $G \times K_2$ from two edge d-colorings (not necessarily distinct) of a graph G.

Construction:
Let c_1, c_2 be two edge d-coloring of a graph G not necessarily distinct. The construction produces an edge $(d+1)$-coloring c of the graph $G \times K_2$ defined as follows: Let $x, y, z \in V(G)$, let $xy \in E(G)$. We put
$$c\big((x,0)(y,0)\big) = c_1(xy), \quad c\big((x,1)(y,1)\big) = c_2(xy), \quad c\big((z,0)(z,1)\big) = d+1.$$

The resulting coloring c will be denoted $\operatorname{Constr}(G, c_1, c_2)$.

The following statement may be verified directly:

PROPOSITION 2.

1) *If c_1, c_2 are complementary edge d-colorings of a graph G and* $c_1' = \operatorname{Constr}(G, c_1, c_2)$, $c_2' = \operatorname{Constr}(G, c_2, c_1)$, *then c_1', c_2' are complementary edge $(d+1)$-colorings of the graph $G \times K_2$.*

2) *If c_1, c_2 are edge domatic d-colorings of a graph G and* $c = \operatorname{Constr}(G, c_1, c_2)$, *then c is an edge domatic $(d+1)$-coloring of the graph $G \times K_2$.*

Applying our construction and Proposition 2 to hypercubes, we obtain the following statement:

COROLLARY. *Given a pair c_1, c_2 of complementary edge domatic d-colorings of Q_n, it is possible to construct a pair c_1', c_2' of complementary edge domatic $(d+1)$-colorings of Q_{n+1}.*

To prove the following statement we will essentially use Zelinka's construction from [12].

PROPOSITION 3. *For $m \geq 1$ there exist complementary edge domatic $(4m)$-colorings $c_1^{(m)}$, $c_2^{(m)} \in Q_{3m}$.*

Proof. We start with some auxiliary constructions. Let T be the set of all twelve strings of length 3 over the alphabet $\{0, 1, *\}$, containing exactly one asterisk

$$T = \{*00, *01, \ldots, 11*\}.$$

We define two mappings γ_1 and γ_2 of T onto $\{1, 2, 3, 4\}$ by putting

$$\begin{aligned}
\gamma_1(00*) = \gamma_1(1{*}1) = \gamma_1(*10) = \gamma_2(01*) = \gamma_2(1{*}1) = \gamma_2(*00) = 1\,,\\
\gamma_1(0{*}1) = \gamma_1(11*) = \gamma_1(*00) = \gamma_2(0{*}1) = \gamma_2(*10) = \gamma_2(10*) = 2\,,\\
\gamma_1(*01) = \gamma_1(1{*}0) = \gamma_1(01*) = \gamma_2(00*) = \gamma_2(1{*}0) = \gamma_2(*11) = 3\,,\\
\gamma_1(0{*}0) = \gamma_1(10*) = \gamma_1(*11) = \gamma_2(0{*}0) = \gamma_2(*01) = \gamma_2(11*) = 4\,.
\end{aligned}$$

Observe that there is a natural $1{-}1$ correspondence between T and $E(Q_3)$, e.g. the string $*00$ corresponds to the edge joining vertices 000 and 100. Also observe that γ_1 is in fact the edge domatic 4-coloring of Q_3 given in Fig. 1 and γ_2 is symmetric to γ_1 by a vertical axis. Moreover, γ_1 and γ_2 are obviously complementary and may be taken as the needed colorings $c_1^{(1)}$, $c_2^{(1)}$ in case $m = 1$.

Let $m > 1$, we are ready to define the needed colorings $c_1^{(m)}$, $c_2^{(m)}$: given $e \in E(Q_{3m})$, assume the end vertices of e are $(v_1, \ldots, v_{j-1}, 0, v_{j+1}, \ldots, v_{3m})$ and $(v_1, \ldots, v_{j-1}, 1, v_{j+1}, \ldots, v_{3m})$, where $1 \leq j \leq 3m$. Put $k = 3 \cdot \lfloor (j-1)/3 \rfloor$ and call e to be *odd* (resp. *even*) if

$$\sum_{i=1}^{k} v_i + \sum_{i=k+4}^{3m} v_i$$

is odd (resp. even). We determine $e' \in T$ by putting

$$e' = \begin{cases} *v_{k+2}v_{k+3} & \text{if } j = k+1\,, \\ v_{k+1}*v_{k+3} & \text{if } j = k+2\,, \\ v_{k+1}v_{k+2}* & \text{if } j = k+3\,, \end{cases}$$

and define

$$c_1^{(m)}(e) = \frac{4k}{3} + \gamma_1(e')\,, \quad c_2^{(m)}(e) = \frac{4k}{3} + \gamma_2(e') \qquad \text{if } e \text{ is odd}\,,$$

$$c_1^{(m)}(e) = \frac{4k}{3} + \gamma_2(e')\,, \quad c_2^{(m)}(e) = \frac{4k}{3} + \gamma_1(e') \qquad \text{if } e \text{ is even}\,.$$

We have to show that $c_1^{(m)}$ and $c_2^{(m)}$ just constructed possess the needed properties. Let us consider an arbitrary vertex $v = (v_1, \dots, v_{3m}) \in V(Q_{3m})$ and let us take an arbitrary j, $1 \le j \le m$. Determine the parity of the sum $\sum_{i=1}^{3j} v_i + \sum_{i=3j+4}^{3m} v_i$ and consider the following three edges of Q_{3m} incident with v:

$$\begin{aligned} e_{v,j,1} = &(v_1, \dots, v_{3j}, v_{3j+1}, v_{3j+2}, v_{3j+3}, v_{3j+4}, \dots, v_{3m}) \\ &(v_1, \dots, v_{3j}, \overline{v_{3j+1}}, v_{3j+2}, v_{3j+3}, v_{3j+4}, \dots, v_{3m})\,, \\ e_{v,j,2} = &(v_1, \dots, v_{3j}, v_{3j+1}, v_{3j+2}, v_{3j+3}, v_{3j+4}, \dots, v_{3m}) \\ &(v_1, \dots, v_{3j}, v_{3j+1}, \overline{v_{3j+2}}, v_{3j+3}, v_{3j+4}, \dots, v_{3m})\,, \\ e_{v,j,3} = &(v_1, \dots, v_{3j}, v_{3j+1}, v_{3j+2}, v_{3j+3}, v_{3j+4}, \dots, v_{3m}) \\ &(v_1, \dots, v_{3j}, v_{3j+1}, v_{3j+2}, \overline{v_{3j+3}}, v_{3j+4}, \dots, v_{3m})\,, \end{aligned}$$

where, as usual, $\overline{0}$ denotes 1 and $\overline{1}$ denotes 0.

The values of colorings $c_1^{(m)}$ and $c_2^{(m)}$ for these edges are defined using the mappings γ_1 and γ_2. We verify directly that

$$\{c_1^{(m)}(e_{v,j,s}) :\ 1 \le s \le 3\} \cup \{c_2^{(m)}(e_{v,j,s}) :\ 1 \le s \le 3\} = \{4j-3, 4j-2, 4j-1, 4j\}$$

and also that both $c_1^{(m)}$ and $c_2^{(m)}$ are surjections, which accomplishes the proof. □

Propositions 2 and 3 already prove the lower bound (2).

Combining the lower (lb) and upper (ub) bounds for $\mathrm{ed}(Q_n)$, we obtain the following table:

n	3	4	5	6	7	8	9	10	11	12
lb	4	5	6	8	9	10	12	13	14	16
ub	4	5	6	8	9	11	12	14	16	17
$\mathrm{ed}(Q_n)$	4	5	6	8	9	10 or 11	12			

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* *Dipartimento di Matematica "Guido Castelnuovo"*
Università degli Studi di Roma "La Sapienza"
P. le Aldo Moro, 2
I-00185 Rome
ITALY

E-mail: pvcecch@mat.uniroma1.it

** *Mathematical Institute*
Academy of Sciences of the Czech Republic
Žitná 25
CZ-11 567 Prague 1
CZECH REPUBLIC