Jacek Hejduk; Eliza Wajch

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COMPACTNESS IN THE SENSE OF THE CONVERGENCE WITH RESPECT TO A SMALL SYSTEM

JACEK HEJDUK—ELIZA WAJCH

The purpose of the paper is to generalize Fréchet's theorem characterizing the compactness of families of measurable real functions in the sense of the convergence with respect to a finite measure (cf. [1, 3, 4]). Some necessary and sufficient conditions (analogous to those from [1, 3, 4]) for a family of measurable real functions to be compact in the sense of the convergence with respect to a small system will be proved.

Before proceeding to the body of the article, let us introduce some notation and establish some useful facts.

Let X be a nonempty abstract set and $\mathscr{S}$- a σ-field of subsets of X. Suppose that we are given a sequence $(\mathscr{E}_n)$ of subfamilies of $\mathscr{S}$ which satisfies the following conditions:

(I) $\emptyset \in \mathscr{E}_n$ for each $n \in N$;

(II) for any $n \in N$, there exists a sequence (k_i) of positive integers such that if $A_i \in \mathscr{E}_{k_i}$ for $i \in N$, then $\bigcup_{i=1}^{\infty} A_i \in \mathscr{E}_n$;

(III) for any $n \in N$, $A \in \mathscr{E}_n$ and $B \in \mathscr{S}$ such that $B \subset A$, we have $B \in \mathscr{E}_n$;

(IV) for any $n \in N$, $A \in \mathscr{E}_n$ and $B \in \bigcap_{m=1}^{\infty} \mathscr{E}_m$, we have $A \cup B \in \mathscr{E}_n$;

(V) $\mathscr{E}_n \supset \mathscr{E}_{n+1}$ for each $n \in N$.

The sequence $(\mathscr{E}_n)$ is said to be *a small system on* $\mathscr{S}$ (cf. [2, 6, 7]). If, in addition, $(\mathscr{E}_n)$ has the following property:

(VI) if (A_n) is a nonincreasing sequence of $\mathscr{S}$-measurable sets for which there exists $m \in N$ such that $A_n \notin \mathscr{E}_m$ for any $n \in N$, then $\bigcap_{n=1}^{\infty} A_n \notin \bigcap_{n=1}^{\infty} \mathscr{E}_n$,

then it is called *an upper semicontinuous small system* (cf. [6, Definition 2]). In the sequel, we shall assume that $(\mathscr{E}_n)$ fulfils (I)—(V). If it proves necessary, we shall in addition insist that $(\mathscr{E}_n)$ is upper semicontinuous.

Let us observe that the family $\mathscr{J} = \bigcap_{n=1}^{\infty} \mathscr{E}_n$ forms a σ-ideal on $\mathscr{S}$ (cf. [6]). Of course, for any σ-ideal $\mathcal{J}^*$ on $\mathcal{S}$, there exists a small system $(\mathcal{E}_n^*)$ such that $\mathcal{J}^* = \bigcap_{n=1}^{\infty} \mathcal{E}_n^*$; however, there are σ-ideals which are not the intersections of any upper semicontinuous small systems (cf. [6, Corollary 5]).

One says that a property holds $\mathcal{J}$-almost everywhere (abbr. $\mathcal{J}$-a.e.) on X if the set of points not having this property belongs to $\mathcal{J}$. The family of all $\mathcal{J}$-a.e. finite $\mathcal{S}$-measurable real functions defined on X will be denoted by $\mathbf{M}[\mathcal{S}, \mathcal{J}]$.

In [8] E. Wagner introduced the definition of the convergence with respect to a σ-ideal. We will recall the notion of the convergence with respect to a small system, which was investigated in [6].

Definition 1. *A sequence $(f_n) \subset \mathbf{M}[\mathcal{S}, \mathcal{J}]$ converges with respect to the small system $(\mathcal{E}_n)$ to a function $f \in \mathbf{M}[\mathcal{S}, \mathcal{J}]$ if for any $\delta > 0$ and any $m \in N$, there exists $n_0 \in N$ such that $\{x \in X: |f_n(x) - f(x)| > \delta\} \in \mathcal{E}_m$ whenever $n \geqslant n_0$.*

Definition 2. *A family $\Phi \subset \mathbf{M}[\mathcal{S}, \mathcal{J}]$ is called*:

(a) *compact in the sense of the convergence with respect to the small system $(\mathcal{E}_n)$ (abbr. $(\mathcal{E}_n)$-compact) if each sequence of functions from Φ contains a subsequence converging with respect to $(\mathcal{E}_n)$ to some function from $\mathbf{M}[\mathcal{S}, \mathcal{J}]$*;

(b) *compact in the sense of the convergencse with respect to the σ-ideal $\mathcal{J}$ (abbr. $\mathcal{J}$-compact) if each sequence of functions from Φ contains a subsequence converging $\mathcal{J}$-a.e. to some function from $\mathbf{M}[\mathcal{S}, \mathcal{J}]$.*

It follows from [6, Theorem 1] that $(\mathcal{E}_n)$-compactness implies $\mathcal{J}$-compactness; however, the converse holds if and only if the small system $(\mathcal{E}_n)$ is upper semicontinuous (cf. [6, Corollary 1 and Remark 2]). Some characterization of $\mathcal{J}$-compactness was given in [5]. Here, we shall be primarily concerned with the $(\mathcal{E}_n)$-compactness.

Definition 3. *An $\mathcal{S}$-measurable real function f defined on X is $(\mathcal{E}_n)$-bounded if, for each $n \in N$, there exists a positive integer t_n such that $\{x \in X: |f(x)| > t_n\} \in \mathcal{E}_n$. Denote by $\mathbf{M}[\mathcal{S}, (\mathcal{E}_n)]$ the family of all $(\mathcal{E}_n)$-bounded functions.*

Proposition 1. (a) *The inclusion $\mathbf{M}[\mathcal{S}, (\mathcal{E}_n)] \subset \mathbf{M}[\mathcal{S}, \mathcal{J}]$ always holds.*

(b) *The equality $\mathbf{M}[\mathcal{S}, (\mathcal{E}_n)] = \mathbf{M}[\mathcal{S}, \mathcal{J}]$ holds if and only if $(\mathcal{E}_n)$ is upper semicontinuous.*

Proof. (a) Consider any $f \in \mathbf{M}[\mathcal{S}, (\mathcal{E}_n)]$. Let (t_n) be a sequence of positive integers such that the sets $A_n = \{x \in X: |f(x)| > t_n\}$ belong to $\mathcal{E}_n$. It follows from (III) that $\bigcap_{n=1}^{\infty} A_n \in \mathcal{J}$. Since $\{x \in X: |f(x)| = +\infty\} \subset \bigcap_{n=1}^{\infty} A_n$, we obtain that $f \in \mathbf{M}[\mathcal{S}, \mathcal{J}]$.

(b) Suppose that $(\mathcal{E}_n)$ is upper semicontinuous and let $g \in \mathbf{M}[\mathcal{S}, \mathcal{J}]$. If $g \notin \mathbf{M}[\mathcal{S}, (\mathcal{E}_n)]$, then there exists $k \in N$ such that $\{x \in X: |g(x)| > n\} \notin \mathcal{E}_k$ for each

$n \in N$. If follows from (VI) that $\bigcap_{n=1}^{\infty} \{x \in X: |g(x)| > n\} \notin \mathcal{J}$, which is impossible; hence $g \in \mathbf{M}[\mathcal{S}, (\mathcal{E}_n)]$ and, consequently, $\mathbf{M}[\mathcal{S}, (\mathcal{E}_n)] = \mathbf{M}[\mathcal{S}, \mathcal{J}]$.

Conversely, suppose that $(\mathcal{E}_n)$ is not upper semicontinuous. There exist a positive integer m and a strictly nonincreasing sequence (B_n) of members of $\mathcal{S}$ such that $\bigcap_{n=1}^{\infty} B_n \in \mathcal{J}$ and $B_n \notin \mathcal{E}_m$ for each $n \in N$. Let us define

$$h(x) = \begin{cases} 1 & \text{for } x \in X \setminus B_1, \\ n & \text{for } x \in B_n \setminus B_{n+1}, \\ +\infty & \text{for } x \in \bigcap_{n=1}^{\infty} B_n. \end{cases}$$

The function h is $\mathcal{J}$-a.e. finite but not $(\mathcal{E}_n)$-bounded.

Definition 4. *A family* $\Phi \subset \mathbf{M}[\mathcal{S}, \mathcal{J}]$ *is called*:

(a) $(\mathcal{E}_n)$*-equibounded if, for any* $n \in N$, *there exists a positive integer* t_n *such that* $\{x \in X: |f(x)| > t_n\} \in \mathcal{E}_n$ *whenever* $f \in \Phi$;

(b) $\mathcal{J}$*-equibounded if there exists a sequence* (t_n) *of positive integers such that* $\bigcap_{n=m}^{\infty} \bigcup_{m=1}^{\infty} \{x \in X: |f_n(x)| > t_n\} \in \mathcal{J}$ *for every sequence* (f_n) *of functions from* Φ.

Proposition 2. (a) $(\mathcal{E}_n)$*-equiboundedness implies* $\mathcal{J}$*-equiboundedness.*

(b) $\mathcal{J}$*-equiboundedness implies* $(\mathcal{E}_n)$*-equiboundedness if and only if the small system* $(\mathcal{E}_n)$ *is upper semicontinuous.*

Proof. (a) Lemma 1 of [6] implies the existence of a sequence (k_i) of positive integers such that if $A_i \in \mathcal{E}_{k_i}$, then $\bigcup_{i=n}^{\infty} A_i \in \mathcal{E}_n$ for each $n \in N$. Suppose that $\Phi \subset \mathbf{M}[\mathcal{S}, \mathcal{J}]$ is $(\mathcal{E}_n)$-equibounded. There exists a sequence (t_i) of positive integers such that $\{x \in X: |f(x)| > t_i\} \in \mathcal{E}_{k_i}$ for any $f \in \Phi$ and $i \in N$. If Φ is not $\mathcal{J}$-equibounded, then there is a sequence $(f_i) \subset \Phi$ such that $\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \{x \in X: |f_i(x)| > t_i\} \notin \mathcal{J}$, which contradicts (III).

(b) Assume that $(\mathcal{E}_n)$ is upper semicontinuous, and $\Phi \subset \mathbf{M}[\mathcal{S}, \mathcal{J}]$ is $\mathcal{J}$-equibounded. There is a sequence (r_i) of positive integers such that, for any sequence $(g_i) \subset \Phi$, $\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} \{x \in X: |g_i(x)| > r_i\} \in \mathcal{J}$. If Φ is not $(\mathcal{E}_n)$-equibounded, then we can find $m \in N$ such that, for each $i \in N$, there exists $g_i \in \Phi$ for which $\{x \in X: |g_i(x)| > r_i\} \notin \mathcal{E}_m$. By virtue of (III), the sets $A_n = \bigcup_{i=n}^{\infty} \{x \in X: |g_i(x)| > r_i\}$ form a nonincreasing sequence such that $A_n \notin \mathcal{E}_m$ for any $n \in N$. According to (VI), $\bigcap_{n=1}^{\infty} A_n \notin \mathcal{J}$ — a contradiction; hence Φ is $(\mathcal{E}_n)$ — equibounded.

If $(\mathscr{E}_n)$ is not upper semicontinuous, then the family $\{h\}$, where h is the function constructed in the proof of Proposition 1 (b), is $\mathscr{J}$-equibounded and not $(\mathscr{E}_n)$-equibounded.

For a function $f \in \mathbf{M}[\mathscr{S}, \mathscr{J}]$, finite on a set $A \subset X$, let us denote $\operatorname{osc}(f, A) = \sup\{|f(x) - f(y)|: x, y \in A\}$; of course, if $A = \emptyset$, then $\operatorname{osc}(f, A) = -\infty$.

By *a partition of X* we shall mean a finite subfamily $\mathscr{P}$ of $\mathscr{S}$ such that $\cup\{P: P \in \mathscr{P}\} = X$.

Definition 5. *A family* $\Phi \subset \mathbf{M}[\mathscr{S}, \mathscr{J}]$ *is called*:

(a) $(\mathscr{E}_n)$*-equimeasurable if, for any* $\delta > 0$ *and* $n \in N$, *there exist a partition* $\mathscr{P}$ *of* X *and a collection* $\{A_f : f \in \Phi\}$ *of members of* $\mathscr{E}_n$ *such that* $\operatorname{osc}(f, P \backslash A_f) \leqslant \delta$ *whenever* $f \in \Phi$ *and* $P \in \mathscr{P}$;

(b) $\mathscr{J}$*-equimeasurable if, for any* $\delta > 0$, *there exist a sequence* $(\mathscr{P}_n)$ *of partitions of X and a collection* $\{A_f^n : f \in \Phi, n \in N\}$ *of* $\mathscr{S}$*-measurable sets such that, for any sequence* (f_n) *of functions from* Φ, *we have* $\bigcap_{i=1}^{\infty} \bigcup_{n=i}^{\infty} A_{f_n}^n \in \mathscr{J}$ *and, moreover,* $\operatorname{osc}(f, P \backslash A_f^n) \leqslant \delta$ *for any* $f \in \Phi$, $n \in N$ *and* $P \in \mathscr{P}_n$.

Proposition 3. (a) *If* $f \in \mathbf{M}[\mathscr{S}, (\mathscr{E}_n)]$, *then the family* $\{f\}$ *is* $(\mathscr{E}_n)$*-equimeasurable.*

(b) *If* $f \in \mathbf{M}[\mathscr{S}, \mathscr{J}]$, *then the family* $\{f\}$ *is* $\mathscr{J}$*-equimeasurable.*

(c) *The small system* $(\mathscr{E}_n)$ *is upper semicontinuous if and only if, for any* $f \in \mathbf{M}[\mathscr{S}, \mathscr{J}]$, *the family* $\{f\}$ *is* $(\mathscr{E}_n)$*-equimeasurable.*

Proof. (a) Let us fix $\delta > 0$ and $n_0 \in N$. If $f \in \mathbf{M}[\mathscr{S}, (\mathscr{E}_n)]$, then there exists $t > 0$ such that the set $A = \{x \in X: |f(x)| > t\}$ is a member of $\mathscr{E}_{n_0}$. Let $\mathscr{P}^*$ be a partition of $[-t, t]$ which consists of intervals of diameter less than δ. If $\mathscr{P} = \{f^{-1}(P^*): P^* \in \mathscr{P}^*\} \cup \{A\}$, then $\operatorname{osc}(f, P \backslash A) \leqslant \delta$ for each $P \in \mathscr{P}$.

(b) If $f \in \mathbf{M}[\mathscr{S}, \mathscr{J}]$ then $\bigcap_{n=1}^{\infty} \{x \in X: |f(x)| > n\} \in \mathscr{J}$. Putting $A_n = \{x \in X: |f(x)| > n\}$ for $n \in N$ and arguing as in the proof of (a), we obtain a sequence $(\mathscr{P}_n)$ of partitions of X such that $\operatorname{osc}(f, P \backslash A_n) \leqslant \delta$ for a fixed $\delta > 0$, any $n \in N$ and $P \in \mathscr{P}_n$.

(c) If $(\mathscr{E}_n)$ is upper semicontinuous, then $\mathbf{M}[\mathscr{S}, \mathscr{J}] = \mathbf{M}[\mathscr{S}, (\mathscr{E}_n)]$ by Proposition 1 (b), thus, to complete the proof, it suffices to suppose that $(\mathscr{E}_n)$ is not upper semicontinuous and to show that $\{h\}$ is not $(\mathscr{E}_n)$-equimeasurable, where h is the function constructed in the proof of Proposition 1 (b)

Let $\mathscr{P}$ be an arbitrary partition of X and let $C \in \mathscr{E}_m$. We may assume that $\bigcap_{n=1}^{\infty} B_n \subset C$ (cf. proof of Proposition 1 (b)). It is easily seen that $B_n \backslash C \neq \emptyset$ for each $n \in N$ (otherwise, B_n would belong to $\mathscr{E}_m$ for some n). As the family $\mathscr{P}$ is finite, there exists $P \in \mathscr{P}$ such that $(P \backslash C) \cap (B_n \backslash B_{n+1}) \neq \emptyset$ for infinitely many n. This implies that $\operatorname{osc}(h, P \backslash C) = +\infty$, which concludes the proof.

Proposition 4. (a) *$(\mathscr{E}_n)$-equimeasurability implies $\mathscr{J}$-equimeasurability.*

(b) *$\mathscr{J}$-equimeasurability implies $(\mathscr{E}_n)$-equimeasurability if and only if $(\mathscr{E}_n)$ is upper semicontinuous.*

Proof. (a) First of all, let us take a sequence (k_i) of positive integers such that if $A_i \in \mathscr{E}_{k_i}$, then $\bigcup_{i=n}^{\infty} A_i \in \mathscr{E}_n$ for any $n \in N$ (cf. [6, Lemma 1]). Suppose that $\Phi \subset \mathbf{M}[\mathscr{S}, \mathscr{J}]$ is $(\mathscr{E}_n)$-equimeasurable. For a fixed $\delta > 0$, there exist a sequence $(\mathscr{P}_i)$ of partitions of X and a collection $\{A_f^i : f \in \Phi,\ i \in N\}$ of $\mathscr{S}$-measurable sets such that $A_f^i \in \mathscr{E}_{k_i}$ and $\operatorname{osc}(f, P \backslash A_f^i) \leqslant \delta$ for any $f \in \Phi$, $P \in \mathscr{P}_i$ and $i \in N$. By virtue of (III), $\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_{f_i}^i \in \mathscr{J}$ for any sequence $(f_i) \subset \Phi$, so Φ is $\mathscr{J}$-equimeasurable.

(b) Assume that $(\mathscr{E}_n)$ is upper semicontinuous and $\Phi \subset \mathbf{M}[\mathscr{S}, \mathscr{J}]$ is $\mathscr{J}$-equimeasurable. Suppose that, for some $\delta > 0$ and $m \in N$, the condition of the $(\mathscr{E}_n)$-equimeasurability of Φ is not satisfied. For this δ, take a sequence $(\mathscr{P}_n)$ of partitions of X and a collection $\{A_f^n : f \in \Phi,\ n \in N\}$ of $\mathscr{S}$-measurable sets, fulfilling the conditions of Definition 5 (b). For each $n \in N$, we can find $f_n \in \Phi$ such that if $A \in \mathscr{E}_m$, then there exists $P \in \mathscr{P}_n$ for which $\operatorname{osc}(f_n, P \backslash A) > \delta$. Since $\bigcap_{j=1}^{\infty} \bigcup_{n=i}^{\infty} A_{f_n}^n \in \mathscr{J}$, it follows from (VI) that $A_i = \bigcup_{n=i}^{\infty} A_{f_n}^n \in \mathscr{E}_m$ for some $i \in N$; moreover, $\operatorname{osc}(f_i, P \backslash A_i) \leqslant \delta$ whenever $P \in \mathscr{P}_i$ — a contradiction. This, together with Proposition 3 (c), completes the proof.

To prove the main theorems of the paper, we need two more lemmas.

Lemma 1. *Let (f_n) be a sequence of functions from $\mathbf{M}[\mathscr{S}, (\mathscr{E}_n)]$. If (f_n) converges with respect to $(\mathscr{E}_n)$ to a function f, then $f \in \mathbf{M}[\mathscr{S}, (\mathscr{E}_n)]$.*

Proof. Suppose that $f \notin \mathbf{M}[\mathscr{S}, (\mathscr{E}_n)]$. There exists $m \in N$ such that $\{x \in X: |f(x)| > n\} \notin \mathscr{E}_m$ for all $n \in N$. Properties (II) and (V) of $(\mathscr{E}_n)$ imply the existence of $k \in N$ such that $A \cup B \in \mathscr{E}_m$ for any $A, B \in \mathscr{E}_k$. Let us fix $\delta > 0$. There exists $n_0 \in N$ such that $\{x \in X: |f(x) - f_n(x)| > \delta\} \in \mathscr{E}_k$ whenever $n \leqslant n_0$. Moreover, we can find $t > 0$ such that $\{x \in X: |f_{n_0}(x)| > t\} \in \mathscr{E}_k$. Let us take a positive integer $n > t + \delta$. Then $C = \{x \in X: |f(x)| > n\} \backslash \{x \in X: |f_{n_0}(x)| > t\} \notin \mathscr{E}_k$ (otherwise, the set $\{x \in X: |f(x)| > n\}$ would belong to $\mathscr{E}_m$). On the other hand, $C \subset \{x \in X: |f(x) - f_{n_0}(x)| > \delta\} \in \mathscr{E}_k$; hence, by (III), $C \in \mathscr{E}_k$ — a contradiction.

Lemma 2. *A sequence (f_n) of functions from $\mathbf{M}[\mathscr{S}, \mathscr{J}]$ converges with respect to $(\mathscr{E}_n)$ to some function $f \in \mathbf{M}[\mathscr{S}, \mathscr{J}]$ if and only if, for each $i \in N$ and any $\delta > 0$, there exists $n_0 \in N$ such that $\{x \in X: |f_n(x) - f_m(x)| > \delta\} \in \mathscr{E}_i$ whenever $n, m \geqslant n_0$.*

Proof. Necessity is obvious.

Sufficiency. Let (k_i) be a sequence of positive integers such that if $A_i \in \mathscr{E}_{k_i}$, then $\bigcup_{i=n}^{\infty} A_i \in \mathscr{E}_n$ for each $n \in N$ (cf. [6, Lemma 1]). Consider any subsequence (h_n) on (f_n). For each $i \in N$, there exists $n_i \in N$ such that $\left\{x \in X\colon |h_n(x) - h_m(x)| > \frac{1}{2^i}\right\} \in \mathscr{E}_{k_i}$ whenever $n, m \geqslant n_i$. We may assume that $n_{i+1} > n_i$ for each $i \in N$. Denote $g_i = h_{n_i}$, $A_i = \left\{x \in X\colon |g_{i+1}(x) - g_i(x)| > \frac{1}{2^i}\right\}$ and $A = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i$. Of course, $A \in \mathscr{J}$. If $x \in X \setminus A$, then there exists $i_0 \in N$ such that $x \notin A_i$ whenever $i \geqslant i_0$; hence, for each $i > j \geqslant i_0$, we have that $|g_i(x) - g_j(x)| \leqslant \sum_{r=j}^{i-1} |g_{r+1}(x) - g_r(x)| \leqslant \leqslant \frac{1}{2^{j-1}}$; this implies that $(g_i(x))$ is a Cauchy sequence. Define $g(x) = \lim_{i \to \infty} g_i(x)$ for $x \in X \setminus A$ and $g(x) = 0$ for $x \in A$. We shall show that (g_i) converges with respect to $(\mathscr{E}_n)$ to the function g.

Let us fix $\delta > 0$ and $n \in N$. Take a positive integer $m > n$ such that $\frac{1}{2^m} < \delta$, and suppose that $\{x \in X\colon |g_j(x) - g(x)| > \delta\} \notin \mathscr{E}_n$ for some $j \geqslant m + 2$. Consider any $x \in X \setminus A$ such that $|g_j(x) - g(x)| > \delta$. There exists $i > j$ such that $|g_i(x) - - g(x)| < \frac{1}{2^{m+1}}$. Let us observe that $\frac{1}{2^m} < |g_j(x) - g(x)| \leqslant |g_i(x) - g(x)| + + \sum_{r=i}^{\infty} |g_{r+1}(x) - g_r(x)|$; hence $\frac{1}{2^{m+1}} < \sum_{r=j}^{\infty} |g_{r+1}(x) - g_r(x)|$. This yields that $x \in \in \bigcup_{r=j}^{\infty} A_r$. Consequently, by virtue of (IV) and (V), $\{x \in X\colon |g_j(x) - g(x)| > \delta\} \in \in \mathscr{E}_n$ — a contradiction. Therefore (g_i) converges with respect to $(\mathscr{E}_n)$ to g. To conclude the proof, it suffices to apply Lemma 2 of [6].

Theorem 1. (cf. [1, 3, 4]). *A family $\Phi \subset \mathbf{M}[\mathscr{S}, (\mathscr{E}_n)]$ is $(\mathscr{E}_n)$-compact if and only if it is $(\mathscr{E}_n)$-equibounded and $(\mathscr{E}_n)$-equimeasurable.*

Proof. To begin with, let us fix $\delta > 0$ and $n_0 \in N$. It follows from (II) and (V) that there is $k_0 \in N$ such that $\bigcup_{i=1}^{4} A_i \in \mathscr{E}_{n_0}$ whenever $A_i \in \mathscr{E}_{k_0}$ for $i = 1, 2, 3, 4$.

Necessity. Suppose that, for each $n \in N$, we can find a function $f_n \in \Phi$ such that $\{x \in X\colon |f_n(x)| > n\} \notin \mathscr{E}_{n_0}$. The sequence (f_n) contains a subsequence converging with respect to $(\mathscr{E}_n)$ to some $f \in \mathbf{M}[\mathscr{S}, \mathscr{J}]$. By Lemma 1, $f \in \in \mathbf{M}[\mathscr{S}, (\mathscr{E}_n)]$, so there exists $t > 0$ such that $\{x \in X\colon |f(x)| > t\} \in \mathscr{E}_{k_0}$. Moreover, there exists $n > t + \delta$ such that $\{x \in X\colon |f_n(x) - f(x)| > \delta\} \in \mathscr{E}_{k_0}$. Then the set $A = \{x \in X\colon |f_n(x)| > n$ and $|f(x)| \leqslant t\}$ does not belong to $\mathscr{E}_{k_0}$. On the other hand, arguing similarly as in the proof of Lemma 1, we can show that $A \in \mathscr{E}_{k_0}$. The contradiction obtained proves that Φ is $(\mathscr{E}_n)$ — equibounded.

Now suppose that Φ is not $(\mathscr{E}_n)$-equimeasurable. Let δ and n_0 be such that the condition of Definition 5 (a) is not satisfied. Consider any function $g_1 \in \Phi$. By virtue of Proposition 3 (a), there exist a partition $\mathscr{P}_1$ of X and a set $C_1 \in \mathscr{E}_{k_0}$ such that $\operatorname{osc}(g_1, P \setminus C_1) \leqslant \frac{\delta}{3}$ whenever $P \in \mathscr{P}_1$. Assume that, for each $i \in \{1, \ldots, n\}$, we have already defined functions $g_i \in \Phi$, sets $C_i \in \mathscr{E}_{k_0}$ and partitions $\mathscr{P}_i$ of X, such that $\left\{x \in X \colon |g_i(x) - g_j(x)| > \frac{\delta}{3}\right\} \notin \mathscr{E}_{k_0}$ whenever $1 \leqslant i < j \leqslant n$ and, moreover, $\operatorname{osc}(g_i, P \setminus C_i) \leqslant \frac{\delta}{3}$ for any $P \in \mathscr{P}_j$ and $1 \leqslant i \leqslant j \leqslant n$. The choice of δ and n_0 implies the existence of $g_{n+1} \in \Phi$ such that, for any $C \in \mathscr{E}_{n_0}$, there is $P \in \mathscr{P}_n$ for which $\operatorname{osc}(g_{n+1}, P \setminus C) > \delta$. Let us observe that $D_i = \left\{x \in X \colon |g_i(x) - g_{n+1}(x)| > \frac{\delta}{3}\right\} \notin \mathscr{E}_{k_0}$ for each $i \in \{1, \ldots, n\}$. Indeed, if $i \in \{1, \ldots, n\}$, $P \in \mathscr{P}_n$ and $x, y \in P \setminus (C_i \cup D_i)$, then $|g_{n+1}(x) - g_{n+1}(y)| \leqslant |g_{n+1}(x) - g_i(x)| + |g_i(x) - g_i(y)| + |g_i(y) - g_{n+1}(y)|) \leqslant \delta$; thus $\operatorname{osc}(g_{n+1}, P \setminus (C_i \cup D_i)) \leqslant \delta$ and, consequently, $C_i \cup D_i \notin \mathscr{E}_{n_0}$; this implies that $D_i \notin \mathscr{E}_{k_0}$.

By Proposition 3 (a), there exist a partition $\mathscr{P}^*_{n+1}$ of X and a set $C_{n+1} \in \mathscr{E}_{k_0}$ such that $\operatorname{osc}(g_{n+1}, P \setminus C_{n+1}) \leqslant \frac{\delta}{3}$ whenever $P \in \mathscr{P}^*_{n+1}$. Denote $\mathscr{P}_{n+1} = \{P \cap T \colon P \in \mathscr{P}^*_{n+1} \text{ and } T \in \mathscr{P}_n\}$. In this way, we have inductively defined a sequence (g_n) of functions from Φ such that $\left\{x \in X \colon |g_i(x) - g_j(x)| > \frac{\delta}{3}\right\} \notin \mathscr{E}_{k_0}$ whenever $i < j$ $(i, j \in N)$. This, together with Lemma 2, implies that no subsequence of (g_n) is convergent with respect to $(\mathscr{E}_n)$, which is impossible.

Sufficiency. Let us consider any sequence (h_n) of functions from Φ. First of all, we shall prove that

(*) (h_n) contains a subsequence (h_{n_i}) such that there exists $i_0 \in N$ for which $\{x \in X \colon |h_{n_i}(x) - h_{n_j}(x)| > \delta\} \in \mathscr{E}_{n_0}$ whenever $i, j \geqslant i_0$.

By the assumptions we can find $t > 0$, a partition $\mathscr{P}$ of X and the sets $A_n \in \mathscr{E}_{k_0}$, such that $B_n = \{x \in X \colon |h_n(x)| > t\} \in \mathscr{E}_{k_0}$ and $\operatorname{osc}(h_n, P \setminus A_n) \leqslant \frac{\delta}{3}$ for any $n \in N$ and $P \in \mathscr{P}$. Let us fix $P \in \mathscr{P}$ and, for $n \in N$, define

$$a_n = \begin{cases} \sup\{h_n(x)\colon x\in P\backslash(A_n\cup B_n)\} & \text{if } P\backslash(A_n\cup B_n)\neq\emptyset, \\ 0 & \text{if } P\backslash(A_n\cup B_n)=\emptyset. \end{cases}$$

Without any difficulties one can check that $\left\{x\in P\colon |h_n(x)-a_n|>\frac{\delta}{3}\right\}\subset A_n\cup B_n$ for every $n\in N$. As $|a_n|\leqslant t$ for $n\in N$, the sequence (a_n) contains a Cauchy subsequence (a_{n_i}). There exists $i_0\in N$ such that $|a_{n_i}-a_{n_j}|\leqslant\frac{\delta}{3}$ whenever $i, j\geqslant i_0$. Let us observe that $\{x\in P\colon |h_{n_i}(x)-h_{n_j}(x)|>\delta\}\subset\left(\left\{x\in P\colon |h_{n_i}(x)-a_{n_i}|>\frac{\delta}{3}\right\}\cup\right.$ $\left.\cup\left\{x\in P\colon\ |a_{n_j}-h_{n_j}(x)|>\frac{\delta}{3}\right\}\right)\subset(A_{n_i}\cup B_{n_i}\cup A_{n_j}\cup B_{n_j})\in\mathscr{E}_{n_0}$ whenever $i,\ j\geqslant i_0$. Since the family $\mathscr{P}$ is finite, the proof of $(*)$ has been completed.

According to $(*)$, we can inductively define subsequences (h_n^k) of (h_n) such that, for each $k\in N$, (h_n^{k+1}) is a subsequence of (h_n^k) and, moreover, there exists $n_k\in N$ such that $\left\{x\in X\colon |h_n^k(x)-h_m^k(x)|>\frac{1}{k}\right\}\in\mathscr{E}_k$ whenever $n,\ m\geqslant n_k$. It is easily seen that the diagonal sequence (h_n^n) satisfies all assumptions of Lemma 2, which concludes the proof.

Propositions 1(b) and 3(c) point out that the assumption that Φ consists of $(\mathscr{E}_n)$-bounded functions cannot be omitted in the above theorem.

An immediate consequence of Propositions 2, 4 and Theorem 1 is the following.

Theorem 2. *Suppose that a σ-ideal $\mathscr{J}$ is the intersection of an upper semicontinuous small system. A family $\Phi\subset\mathbf{M}[\mathscr{S},\mathscr{J}]$ is $\mathscr{J}$-compact if and only if it is $\mathscr{J}$-equibounded ad $\mathscr{J}$-equimeasurable.*

In connection with the last theorem, the following question can be posed: Does Theorem 2 remain true for an arbitrary σ-ideal?. We do not know the answer to this question.

REFERENCES

[1] CAFIERO, F.: Misura e Integrazione. Edizioni Cremonese Roma — 1959, 308—315 (in Italian).

[2] CAPEK, P.: On small systems. Acta F.R.N. Univ. Comen. Math. XXXIV, 1979, 93—101.

[3] FRÉCHET, M.: Sur les ensembles compacts de fonctions mesurables. Fund. Math. 9, 1927, 25—32.

[4] HANSON, E. H.: A note on compactness. Bull. Amer. Math. Soc. 39, 1933, 397—400.

[5] HEJDUK, J. and WAJCH, E.: A characterization of compactness in the sense of convergence with respect to a σ-ideal. (to appear).

[6] NIEWIAROWSKI, J.: Convergence of sequences of real functions with respect to small systems. Math. Slovaca 38, 1988, 333—340.
[7] RIEČAN, B.: Abstract formulation of some theorems of measure theory. Mat. Fyz. Časopis SAV 16, 1966, 268—273.
[8] WAGNER, E.: Sequences of measurable functions. Fund. Math. 112, 1981, 89—102.

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Institute of Mathematics
University of Łódź
S. Banacha 22, 90-238 Łódź,
POLAND.

КОМПАКТНОСТЬ ПО СПОДИМОСТИ ПО МАЛОМ СИСТЕМЕ

Jacek Hejduk—Eliza Wajch

Резюме

Главной целью этой работы является обобщение теоремы Фреше, характеризующей компактность множеств измеримых функций по сходимости по конечной мере. В статье рассматривается сходимость по малым системам измеримых множеств. Доказаны самые необходимые и достаточные условия для того, чтобы множество измеримых функций было компактно по сходимости по малой системе.