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THE PRINCIPAL JOIN PROPERTY IN DEMI-P-LATTICES

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ABSTRACT. We characterize, both algebraically and via their dual space, those semi-De Morgan algebras in the subclass of demi-p-lattices having the principal join property.

We show that, for demi-p-lattices having the principal join property, the principal intersection property holds.

1. Introduction

The equational class of demi-p-lattices was introduced by H. P. Sankappanavar in [15]. It is a subclass of semi-De Morgan algebras and it contains the equational subclass of almost-p-lattices. These classes generalize distributive pseudocomplemented lattices (called here p-lattices).

Principal congruences play an important role in universal algebra and their properties on p-lattices have been studied by numerous authors so it is natural to ask wether some of these properties can be extended to demi-p-lattices.

In [4], Beazer characterized those p-lattices having the principal join property, i.e. those algebras such that the join of any two principal congruences is a principal congruence. These algebras are called *congruence principal* by I. Chajda in [7] and they are defined there as being such that every compact congruence is principal.

In this paper we characterize demi-p-lattices having the principal join property extending the corresponding results obtained for p-lattices by Beazer [4] and show that almost-p-lattices with the principal join property can be described in exactly the same way as Beazer described p-lattices having this property.

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I. Chajda [8] studies algebras whose principal congruences form a sublattice of their congruence lattice. We provide an answer to this problem in what concerns demi-p-lattices since we prove, generalizing a result of Beazer [4], that in this equational class those algebras having the principal join property have the principal intersection property.

2. Preliminaries

We start by recalling some definitions and essential results as well as some notation.

Definition 2.1. An algebra $(L; \vee, \wedge, ', 0, 1)$ is a *semi-De Morgan algebra* if the following conditions hold:

(S1) $(L; \vee, \wedge, 0, 1)$ is a distributive lattice with 0 and 1.
(S2) $0' \approx 1$ and $1' \approx 0$.
(S3) $(a \vee b)' \approx a' \wedge b'$.
(S4) $(a \wedge b)'' \approx a'' \wedge b''$.
(S5) $a''' \approx a'$.

We will denote by SDMA this equational class of algebras.

The following rules hold in SDMA:

(S6) $(a \wedge b)' \approx (a'' \wedge b'')' \approx (a' \vee b')''$.
(S7) $(a \vee b)' \approx (a'' \vee b'')' \approx (a' \wedge b')''$.
(S8) $a \leq b \implies b' \leq a'$.

A semi-De Morgan algebra is a De Morgan algebra if and only if it satisfies the identity $a'' \approx a$.

If L is a SDMA and the equation $a' \wedge a'' \approx 0$ holds in L, then L is called a *demi-p-lattice*. A semi-De Morgan algebra L is called an *almost-p-lattice* if it satisfies $a' \wedge a \approx 0$.

An almost-p-lattice is a distributive pseudocomplemented lattice (p-lattice, for short) iff $a \leq a''$.

If in any $L \in$ SDMA, we write $\mathrm{DM}(L) = \{a \in L : a = a''\}$, then $\big(\mathrm{DM}(L); \dot\vee, \wedge, ', 0, 1\big)$ is a De Morgan algebra where $a \dot\vee b$ is defined to be $(a' \wedge b')'$.

The map $'' \colon L \to L$ defined by $a \mapsto a''$ is a homomorphism onto $\mathrm{DM}(L)$ and its kernel is $\phi = \big\{(a, b) \in L \times L : a' = b'\big\}$. Therefore $L/\phi \cong \mathrm{DM}(L)$.

A semi-De Morgan algebra L is a demi-p-lattice iff $\big(\mathrm{DM}(L); \dot\vee, \wedge, ', 0, 1\big)$ is a Boolean algebra. For a demi-p-lattice L we let $B(L) = \mathrm{DM}(L)$.

These results were proved by Sankappanavar in [15].

The congruence lattice of the semi-De Morgan algebra L will be denoted by $\mathrm{Con}(L)$ and the corresponding congruence lattice on the lattice reduct of L will be denoted by $\mathrm{Con}_{\mathrm{lat}\,L}(L)$.

If L is a semi-De Morgan algebra, then $\theta(a,b)$ will denote the *principal congruence of* $\mathrm{Con}(L)$ *collapsing the pair* $a, b \in L$. If S is any sublattice of the lattice reduct of L, then $\theta_{\mathrm{lat}\,S}(a,b)$ will denote the *principal lattice congruence of* S *collapsing a pair* $a, b \in S$. The *smallest congruence of* $\mathrm{Con}_{\mathrm{lat}\,L}(L)$ *collapsing* S will be denoted by $\theta_{\mathrm{lat}\,L}(S)$.

For any congruence $\theta \in \mathrm{Con}\,L$, we write $[a]_\theta$ to denote the class of θ containing $a \in L$. The class $[a]_\theta$ is a convex sublattice of the lattice reduct of L. The restriction of $\theta_{\mathrm{lat}\,L}(b,c)$ to $[a]_\theta$ will be denoted by $\theta_{\mathrm{lat}\,L}(b,c)\big|_{[a]_\theta}$.

We write $D_0 = [0]_\phi$ and $D_1 = [1]_\phi$ as in [16]. It is clear that D_0 is an ideal and D_1 is a filter.

In [11], D. Hobby presented a duality for semi-De Morgan algebras based on the Priestley duality for distributive lattices. If L is a SDMA, the dual $d(L)$ is a structure $(P, \tau, \leq, \lhd)$ where $(P, \tau, \leq)$ is the Priestley dual of the distributive lattice reduct of L (i.e. P is $d(L)$, the set of prime ideals of the distributive lattice L, $\leq$ is the set inclusion and τ has as subbase the collection of all sets $X_a = \{x \in P : a \notin x\}$ and $\complement X_a = \{x \in P : a \in x\}$ with $a \in L$) and $\lhd$ is a binary relation defined on P. For some purposes the relation $\lhd$ is inconvenient, so a new binary relation is defined in P by $x \leftarrow y$ iff x is a minimal element of $\{z \in P : z \supseteq \{a \in L : a' \notin y\}\}$.

Since Hobby's duality is built on Priestley duality, $d(L)$ will denote both the Priestley space dual of the distributive lattice reduct of L and the dual space of the SDMA itself.

From [9; p. 209] we know that, in Priestley duality, $\mathrm{Con}_{\mathrm{lat}\,L}(L)$ is order isomorphic to the lattice of open subsets of $d(L)$. In this order-isomorphism, any congruence $\alpha \in \mathrm{Con}_{\mathrm{lat}\,L}(L)$ corresponds to an open subset A such that

$$(X,Y) \in \alpha \iff X \cap \complement A = Y \cap \complement A$$

where X, Y are elements of $e(d(L))$, the set of clopen decreasing subsets of $d(L)$, and $\complement A$ denotes the complementary set of A.

In Hobby's duality, it follows from [11; Corollary 3.20] that if $L \in \mathrm{SDMA}$, $\mathrm{Con}(L)$ is order isomorphic to the lattice of open subsets of $d(L)$ that are closed under $(\leftarrow)^{-1}$. In this isomorphism the open subset of $d(L)$ corresponding to the congruence relation ϕ is the complement of the closed set D, considered in [11; p. 158], such that

$$\complement D = \big\{x \in d(L) : \ (\forall y \in d(L))(x \not\leftarrow y)\big\}.$$

Thus, for any $X, Y \in e(d(L))$,

$$(X,Y) \in \phi \iff X \cap D = Y \cap D.$$

For a congruence $\theta \leq \phi$ the corresponding open set is an open subset of $\complement D$. Note that any subset of $\complement D$ is closed under $(\leftarrow)^{-1}$.

In what follows, congruences below congruence ϕ play a fundamental role since every lattice congruence $\theta \leq \phi$ is a congruence of the semi-De Morgan algebra L. Therefore the intervals $[\triangle, \phi] \subseteq \mathrm{Con}(L)$ and $[\triangle, \phi] \subseteq \mathrm{Con}_{\mathrm{lat}\, L}(L)$ coincide. So, as far as these congruences are concerned, we can focus our attention on $\mathrm{Con}_{\mathrm{lat}\, L}(L)$ since both Priestley's and Hobby's dualities lead to the same results.

For more details about Hobby's duality we refer the reader to [11].

If P is a partially ordered set, then, for $x \in P$, $\downarrow x = \{y \in P : \ y \leq x\}$ and $\uparrow x = \{y \in P : \ y \geq x\}$.

We will say that an algebra L has the *principal join property*, abbreviated PJP, if the join of principal congruences on L is again a principal congruence. An algebra L has the *principal intersection property*, PIP, if the intersection of principal congruences is a principal congruence.

For all other unexplained notation and terminology we refer the reader to [16], [4] and [13].

3. The principal join property

We begin with a useful lemma, proved in [13], that extends to SDMAs the corresponding lemma stated for p-lattices by Beazer [5; Lemma 3.1].

Lemma 3.1. ([13; Lemma 3.3]) *Let L be an* SDMA, *$a, x, y \in L$, $x \leq y$ and $x' = y'$. Then*

(i) $\theta(x,y) = \theta_{\mathrm{lat}\, L}(x,y)$,

(ii) (a) $\theta(x,y) = \theta(x \vee x', y \vee x') \vee \theta(x \wedge x', y \wedge x')$,

(b) $\theta_{\mathrm{lat}\, L}(d,e)\big|_{[a]_\phi} = \theta_{\mathrm{lat}[a]_\phi}(d,e)$ *for any $d, e \in [a]_\phi$.*

Following arguments similar to those used by Beazer in the proof of [5; Lemma 3.4] we have the following corollary.

Corollary 3.2. *Let L be an* SDMA. *Let $(b_i, c_i)_{i \in I}$ be elements of $L \times L$ such that $b_i, c_i \in [a]_\phi$ for any $i \in I$. Then*

$$\left(\bigvee_{i \in I} \{\theta_{\mathrm{lat}\, L}(b_i, c_i)\} \right)\Bigg|_{[a]_\phi} = \bigvee_{i \in I} \{\theta_{\mathrm{lat}[a]_\phi}(b_i, c_i)\}.$$

With the previous results in hand we can show some conditions held in SDMAs having the PJP.

LEMMA 3.3. *Let L be an* SDMA *having the* PJP*. If $a \in L$, then the sublattice $[a]_\phi$ of the lattice reduct of L has the* PJP.

Proof. Let L be an SDMA having the PJP. Let d, e, f, g be elements of $[a]_\phi$ such that $d \le e$ and $f \le g$.

We want to show that $\theta_{\mathrm{lat}[a]_\phi}(d,e) \vee \theta_{\mathrm{lat}[a]_\phi}(f,g)$ is a principal congruence of the sublattice $[a]_\phi$.

Clearly $\theta(d,e) \le \phi$ and $\theta(f,g) \le \phi$. Thus $\theta(d,e) = \theta_{\mathrm{lat}\,L}(d,e)$ and $\theta(f,g) = \theta_{\mathrm{lat}\,L}(f,g)$.

Since L has the PJP, there exist $p,q \in L$ with $p \le q$ such that $\theta(d,e) \vee \theta(f,g) = \theta(p,q)$. But $\theta(p,q) \le \phi$, so $p' = q'$ and $\theta(p,q) = \theta_{\mathrm{lat}\,L}(p,q)$.

Observe that $\theta(d,e) \vee \theta(f,g) \le \theta_{\mathrm{lat}\,L}([a]_\phi)$. So $\theta(p,q) \le \theta_{\mathrm{lat}\,L}([a]_\phi)$.

By [13; Lemma 3.4] we know that there exist $m,n \in [a]_\phi$ with $m \le n$ such that $p \wedge m = q \wedge m$ and $p \vee n = q \vee n$.

It is clear that $\theta_{\mathrm{lat}\,L}\big((p\wedge n)\vee m, (q\wedge n)\vee m\big) \le \theta_{\mathrm{lat}\,L}(p,q)$.

Since
$$p \wedge \big((p\wedge n)\vee m\big) = p \wedge n = q \wedge \big((p\wedge n)\vee m\big)$$
and
$$p \vee \big((q\wedge n)\vee m\big) = \big(q \wedge (p\vee n)\big)\vee m = q \vee m = q \vee \big((q\wedge n)\vee m\big),$$
we conclude that $\theta_{\mathrm{lat}\,L}(p,q) \le \theta_{\mathrm{lat}\,L}\big((p\wedge n)\vee m, (q\wedge n)\vee m\big)$.

Thus $\theta_{\mathrm{lat}\,L}(p,q) = \theta_{\mathrm{lat}\,L}\big((p\wedge n)\vee m, (q\wedge n)\vee m\big) = \theta_{\mathrm{lat}\,L}(h,j)$ where $h = (p\wedge n)\vee m$ and $j = (q\wedge n)\vee m$. Since $m' = n' = a'$, we have, by (S7) and (S5),
$$\big((p\wedge n)\vee m\big)' = \big((p\wedge n'')\vee m''\big)' = \big((p\wedge a'')\vee a''\big)' = a''' = a'.$$
Analogously, $\big((q\wedge n)\vee m\big)' = a'$. Therefore $h,j \in [a]_\phi$.

Using Lemma 3.1(ii)(b) and Corollary 3.2 we have:
$$\begin{aligned}\theta_{\mathrm{lat}[a]_\phi}(d,e) \vee \theta_{\mathrm{lat}[a]_\phi}(f,g) &= \theta_{\mathrm{lat}\,L}(d,e)\big|_{[a]_\phi} \vee \theta_{\mathrm{lat}\,L}(f,g)\big|_{[a]_\phi}\\ &= \big(\theta_{\mathrm{lat}\,L}(d,e) \vee \theta_{\mathrm{lat}\,L}(f,g)\big)\big|_{[a]_\phi}\\ &= \theta_{\mathrm{lat}\,L}(h,j)\big|_{[a]_\phi}\\ &= \theta_{\mathrm{lat}[a]_\phi}(h,j).\end{aligned}$$
Thus $[a]_\phi$ has the PJP. □

A consequence of the previous lemma and [1; Theorem 4.5] is the following:

COROLLARY 3.4. *Let L be an* SDMA *having the* PJP *and $a \in L$. Then there is no 3-element chain in the poset of prime ideals of the lattice $[a]_\phi$.*

In what follows we shall see that it is possible to characterize those algebras with the PJP in the variety of demi-p-lattices. With this in mind we extend to demi-p-lattices a lemma proved for p-lattices by Beazer [4; Lemma 3.2].

LEMMA 3.5. *A congruence relation of a demi-p-lattice L is principal iff it is of the form $\theta(0,a)\vee\theta(d,e)$ for some $a\in B(L)$, $d,e\in L$ with $d\le e$ and $d'=e'$.*

Proof. First we will show that, for any $x,y\in L$ with $x\le y$,

$$\theta(x,y)=\theta(0,x'\wedge y'')\vee\theta\big(x\wedge(y\wedge x')',y\wedge(y\wedge x')'\big).$$

Let ρ denote $\theta(0,x'\wedge y'')\vee\theta\big(x\wedge(y\wedge x')',y\wedge(y\wedge x')'\big)$.

Note that $(x'',y'')\in\theta(x,y)$, so $(x'\wedge x'',x'\wedge y'')\in\theta(x,y)$ and, since L is a demi-p-lattice, $(0,x'\wedge y'')\in\theta(x,y)$. Thus $\theta(0,x'\wedge y'')\le\theta(x,y)$.

It is obvious that $\theta\big(x\wedge(y\wedge x')',y\wedge(y\wedge x')'\big)\le\theta(x,y)$. Therefore $\rho\le\theta(x,y)$.

Since $(0,x'\wedge y'')\in\rho$, we have $\big(1,(x'\wedge y'')'\big)=\big(1,(x'\wedge y)'\big)\in\rho$ and consequently $\big(x,x\wedge(y\wedge x')'\big)\in\rho$ and $\big(y,y\wedge(y\wedge x')'\big)\in\rho$. It follows from $\big(x\wedge(x'\wedge y)',y\wedge(x'\wedge y)'\big)\in\rho$ that $(x,y)\in\rho$ and so $\theta(x,y)\le\rho$.

Thus we have $\theta(x,y)=\rho$.

It is clear that $x'\wedge y''\in B(L)$ and $x\wedge(y\wedge x')'\le y\wedge(y\wedge x')'$.

To show that $\big(x\wedge(y\wedge x')'\big)'=\big(y\wedge(y\wedge x')'\big)'$, observe that, by (S6), (S3) and (S5),

$$\big(x\wedge(y\wedge x')'\big)'=\big(x''\wedge(y''\wedge x')'\big)'=\big(x'\vee(y''\wedge x')\big)''=x'''=x'$$

and that by distributivity, by (S6), (S3), (S4) and (S5) and by the fact that in a demi-p-lattice $(y'\vee y'')''=1$,

$$\begin{aligned}\big(y\wedge(y\wedge x')'\big)'&=\big(y''\wedge(y''\wedge x')'\big)'\\&=\big(y'\vee(y''\wedge x')\big)''\\&=\big((y'\vee y'')\wedge(y'\vee x')\big)''\\&=(y'\vee y'')''\wedge(y'\vee x')''\\&=(y''\wedge x'')'=(y\wedge x)'=x'.\end{aligned}$$

For the converse, suppose that $a\in B(L)$, $d,e\in L$, $d\le e$ and $d'=e'$. We will show that $\theta(0,a)\vee\theta(d,e)=\theta(a'\wedge d,a\vee e)$.

Let θ denote $\theta(0,a)\vee\theta(d,e)$. Since $(d,e)\in\theta$, we have $(a'\wedge d,a'\wedge e)\in\theta$. From $(0,a)\in\theta$ it follows that $(a',1)\in\theta$ and $(a'\wedge e,e)\in\theta$. From $(0,a)\in\theta$ we have also $(e,a\vee e)\in\theta$.

Thus, by transitivity, $(a'\wedge d,a\vee e)\in\theta$, so $\theta(a'\wedge d,a\vee e)\le\theta$.

For the reverse inclusion, observe that $\big(d\vee(a'\wedge d),d\vee a\vee e\big)=(d,a\vee e)\in\theta(a'\wedge d,a\vee e)$ and therefore $\big(d\wedge e,(a\vee e)\wedge e\big)=(d,e)\in\theta(a'\wedge d,a\vee e)$.

Thus $\theta(d,e)\le\theta(a'\wedge d,a\vee e)$.

It follows from $a\in B(L)$ that $a\wedge a'=0$ and, therefore, $(0,a)\in\theta(a'\wedge d,a\vee e)$.

Thus $\theta(0,a)\le\theta(a'\wedge d,a\vee e)$.

It is now clear that $\theta\le\theta(a'\wedge d,a\vee e)$. □

Now we can prove the following theorem.

THEOREM 3.6. *Let L be a demi-p-lattice. Then L has the* PJP *if and only if the join of any two principal congruences less than or equal to ϕ is a principal congruence.*

Proof.

($\Longrightarrow$) is obvious.

($\Longleftarrow$): Let $\theta(p,q)$ and $\theta(r,s)$ be two principal congruences of L such that $p \le q$ and $r \le s$. We wish to prove that $\theta(p,q) \vee \theta(r,s)$ is a principal congruence of L.

By Lemma 3.5 there exist $a, b \in B(L)$ and $d, e, f, g \in L$ such that $d \le e$, $f \le g$, $d' = e'$ and $f' = g'$ satisfying $\theta(p,q) = \theta(0,a) \vee \theta(d,e)$ and $\theta(r,s) = \theta(0,b) \vee \theta(f,g)$.

Then $\theta(p,q) \vee \theta(r,s) = \theta(0,a) \vee \theta(0,b) \vee \theta(d,e) \vee \theta(f,g)$.

First we will show that $\theta(0,a) \vee \theta(0,b) = \theta\bigl(0, (a \vee b)''\bigr)$. In fact, since $a, b \in B(L)$, $a = a'' \le (a \vee b)''$ and $b = b'' \le (a \vee b)''$, it is clear that

$$\theta(0,a) \vee \theta(0,b) \le \theta\bigl(0, (a \vee b)''\bigr). \tag{3.1}$$

By [16; Corollary 3.6], we know that $\theta(0, b'') = \theta_{\mathrm{lat}\, L}(b', 1)$.

Since $b = b''$, $(x,y) \in \theta(0,b)$ if and only if $x \wedge b' = y \wedge b'$. Therefore $\bigl(a, (a \vee b)''\bigr) \in \theta(0,b)$ because, by (S4) and (S5), and by distributivity,

$$(a \vee b)'' \wedge b' = \bigl((a \vee b) \wedge b'\bigr)'' = \bigl((a \wedge b') \vee (b \wedge b')\bigr)'' = a'' \wedge b' = a \wedge b'.$$

Since $(0,a) \in \theta(0,a)$ and $\bigl(a, (a \vee b)''\bigr) \in \theta(0,b)$ we have $\bigl(0, (a \vee b)''\bigr) \in \theta(0,a) \vee \theta(0,b)$. Thus

$$\theta\bigl(0, (a \vee b)''\bigr) \le \theta(0,a) \vee \theta(0,b). \tag{3.2}$$

From (3.1) and (3.2) it follows that $\theta(0,a) \vee \theta(0,b) = \theta\bigl(0, (a \vee b)''\bigr)$.

Now observe that $\theta(d,e) \le \phi$ and $\theta(f,g) \le \phi$, so, by the hypothesis, there exist $m, n \in L$ such that $\theta(d,e) \vee \theta(f,g) = \theta(m,n)$ with $m \le n$. Since $\theta(m,n) \le \phi$ we have $m' = n'$.

Then $\theta(p,q) \vee \theta(r,s) = \theta\bigl(0, (a \vee b)''\bigr) \vee \theta(m,n)$ where $(a \vee b)'' \in B(L)$, $m' = n'$ and $m \le n$.

Applying Lemma 3.5 we conclude that $\theta(p,q) \vee \theta(r,s)$ is a principal congruence of L. □

Our objective is to characterize demi-p-lattices having the PJP in terms of the height of their poset of prime ideals, so we will need some results concerning the Priestley dual of a distributive lattice.

The characterization of the open subsets of the Priestley dual of a distributive lattice L corresponding to a congruence $\theta_{\mathrm{lat}\, L}(F)$ where F is a filter of L was obtained in [14; Proposition 12]. From [6; p. 325] we know that the open subsets of the Priestley dual corresponding to a congruence $\theta_{\mathrm{lat}\, L}(I)$ where I is an ideal of L are obtained dually. These are important results in what follows, so we introduce them here:

Lemma 3.7. *Let L be a distributive lattice, F and I be a filter and an ideal in L, respectively. Let A_F and A_I be the open subsets of $d(L)$ such that:*

$$\begin{aligned}(X,Y) \in \theta_{\mathrm{lat}\,L}(F) &\iff X \cap \complement A_F = Y \cap \complement A_F\,,\\ (X,Y) \in \theta_{\mathrm{lat}\,L}(I) &\iff X \cap \complement A_I = Y \cap \complement A_I\end{aligned}$$

where X, Y are elements of $e\big(d(L)\big)$.

Then:

(i) *A_F is the increasing subset $\{x \in d(L) : \ x \cap F \neq \emptyset\}$.*

(ii) *A_I is the decreasing subset $\{x \in d(L) : \ I \not\subseteq x\}$.*

Proof. By [14; Proposition 12], we know that A_F is the open increasing subset of $d(L)$ such that $\complement A_F = \bigcap\limits_{f \in F} \{x \in d(L) : \ f \notin x\}$. Thus

$$A_F = \bigcup_{f \in F} \{x \in d(L) : \ f \in x\} = \{x \in d(L) : \ x \cap F \neq \emptyset\}.$$

Dually

$$\complement A_I = \bigcap_{i \in I} \{x \in d(L) : \ i \in x\} = \{x \in d(L) : \ I \subseteq x\}.$$

□

Lemma 3.8. *Let L be a distributive lattice, and I and F be an ideal and a filter in L, respectively.*

Let $A_I = \{x \in d(L) : \ I \not\subseteq x\}$ and $A_F = \{x \in d(L) : \ x \cap F \neq \emptyset\}$. Then the maps $\alpha \colon A_I \to d(I)$ such that $\alpha(x) = x \cap I$ and $\beta \colon A_F \to d(F)$ such that $\beta(x) = x \cap F$ are order isomorphisms.

Proof. We will only show that α is an order isomorphism because the proof for β is similar.

Since I is a sublattice of the lattice L, applying [3; p. 74, Theorem 5], we have $x \cap I \in d(I) \cup \{\emptyset, I\}$ for each $x \in A_I$. Since $I \not\subseteq x$, we have $x \cap I \neq I$ and it is obvious that $x \cap I \neq \emptyset$. Thus for each $x \in A_I$, $x \cap I \in d(I)$.

To see that the map is onto, let us consider $y \in d(I)$.

By [3; Theorem 5], there exists $x \in d(L)$ such that $x \cap I = y$. Let us suppose that $x \notin A_I$. Then $I \subseteq x$, and hence $x \cap I = I$, contradicting the hypothesis $x \cap I = y$.

To show that α is an order isomorphism, suppose that $x_1, x_2 \in A_I$ and $x_1 \subseteq x_2$. Then $x_1 \cap I \subseteq x_2 \cap I$. On the other hand, if $x_1 \cap I \subseteq x_2 \cap I$, applying the proof of [1; Lemma 3.13] we can conclude that $x_1 \subseteq x_2$. □

When L is a demi-p-lattice, congruences $\theta(D_0)$ and $\theta(D_1)$ are below congruence ϕ since $D_0 = [0]_\phi$ and $D_1 = [1]_\phi$. Therefore $\theta(D_0) = \theta_{\mathrm{lat}\,L}(D_0)$ and

$\theta(D_1) = \theta_{\mathrm{lat}\,L}(D_1)$. According to Lemma 3.7 the open subsets of $d(L)$ corresponding to these congruences are respectively a decreasing subset and an increasing subset. We will denote these subsets by A_0 and A_1, respectively.

The open subset of $d(L)$ corresponding to congruence ϕ is denoted by $\complement D$.

In [16; Theorem 2.5], Sankappanavar shows that, if L is a demi-p-lattice, $\phi = \theta_{\mathrm{lat}\,L}(D_0) \vee \theta_{\mathrm{lat}\,L}(D_1)$. Therefore we have by Lemma 3.7:

COROLLARY 3.9. *Let L be a demi-p-lattice. Then $\complement D = A_0 \cup A_1$.*

From Corollary 3.4 and from Lemma 3.8 we infer the following.

COROLLARY 3.10. *Let L be a demi-p-lattice having the* PJP. *Then there is a 3-element chain neither in $d(D_0)$ nor in $d(D_1)$. Furthermore, there is no 3-element chain in any of the subsets A_0 and A_1 of $d(L)$.*

Observe that, in a distributive lattice L, if a filter F is principal, i.e. if there exists $a \in L$ such that $F = \uparrow a$, then $\theta_{\mathrm{lat}\,L}(F) = \theta_{\mathrm{lat}\,L}(a,1)$ is a principal congruence and, by Lemma 3.7, the corresponding subset of $d(L)$ is the clopen increasing subset

$$\{x \in d(L) :\ x \cap \uparrow a \neq \emptyset\} = \{x \in d(L) :\ a \in x\} = \complement X_a\,.$$

Dually when there exists $b \in L$ such that the ideal $I = \downarrow b$, then $\theta_{\mathrm{lat}\,L}(I) = \theta_{\mathrm{lat}\,L}(0,b)$ is a principal congruence and the corresponding subset of $d(L)$ is the clopen decreasing subset

$$\{x \in d(L) :\ \downarrow b \nsubseteq x\} = \{x \in d(L) :\ b \notin x\} = X_b\,.$$

Since in a distributive lattice L a principal congruence $\theta_{\mathrm{lat}\,L}(a,b)$ with $a \leq b$ is such that $\theta_{\mathrm{lat}\,L}(a,b) = \theta_{\mathrm{lat}\,L}(a,1) \cap \theta_{\mathrm{lat}\,L}(0,b)$, we conclude that the open subset of the dual space $d(L)$ corresponding to this principal congruence is the clopen convex subset of $d(L)$ $X_b \setminus X_a$.

The converse is also true. The congruences corresponding to clopen convex subsets of $d(L)$ are principal congruences of the distributive lattice L.

In fact, if $A \subseteq d(L)$ is a clopen convex subset of $d(L)$, then there exist clopen decreasing subsets X_a and X_b of $d(L)$ such that $A = X_b \setminus X_a$ and the corresponding congruence is the principal congruence $\theta_{\mathrm{lat}\,L}(a,b)$ (see the proof of [2; Lemma 3]).

It is now possible to characterize those open sets that correspond, under duality, to principal congruences below ϕ.

LEMMA 3.11. *Let L be a demi-p-lattice and θ a congruence less than or equal to ϕ. Then θ is a principal congruence of the demi-p-lattice L if and only if the corresponding open subset of the dual space $d(L)$ is a clopen convex subset of $d(L)$ contained in $\complement D$.*

As a consequence of the above we have:

LEMMA 3.12. *Let L be a demi-p-lattice. If L has the* PJP*, then there exists no 3-element chain $x < y < z$ in $d(L)$ such that $x \in A_0$ and $z \in A_1$.*

Proof. Suppose that $x < y < z$ is a chain of prime ideals of L such that $x \in A_0$ and $z \in A_1$. Since $d(L)$ is a Priestley space and $x \not\geq y$ there is a clopen decreasing set X_a with $a \in L$ such that $x \in X_a$ and $y \notin X_a$.

We know that $\downarrow x$ and $\complement A_0$ are closed subsets of $d(L)$ and they are disjoint since $\downarrow x$ is decreasing, $\complement A_0$ is increasing and $x \notin \complement A_0$, so, by compactness and total disconnectedness, there is a clopen decreasing subset of $d(L)$ X_b with $b \in L$ such that $\downarrow x \subseteq X_b$ and $\complement A_0 \subseteq \complement X_b$. Therefore $X_b \subseteq A_0$ and $x \in X_b$. Consequently $x \in X_a \cap X_b = X_{a\wedge b} \subseteq A_0$ and $y \notin X_{a\wedge b}$ because $y \notin X_a$.

Now, $X_{a\wedge b}$ is the clopen decreasing (convex) subset corresponding to the principal congruence $\theta_{\mathrm{lat}\,L}(0, a\wedge b)$. Since $X_{a\wedge b} \subseteq A_0$, we know that $\theta_{\mathrm{lat}\,L}(0, a\wedge b) = \theta(0, a \wedge b)$.

In a similar way, since $y \not\geq z$, there is an X_c with $c \in L$ such that $y \in X_c$ and $z \in \complement X_c$, and from the fact that $\complement A_1$ is a closed decreasing subset such that $z \notin \complement A_1$ it follows that there is an X_d with $d \in L$ such that $\complement A_1 \subseteq X_d$ and $z \in \complement X_d$. Then $z \in \complement X_c \cap \complement X_d = \complement X_{c\vee d} \subseteq A_1$ and $y \notin \complement X_{c\vee d}$ because $y \in X_c$.

Thus we obtained $\complement X_{c\vee d}$, the clopen increasing subset corresponding to the principal congruence $\theta_{\mathrm{lat}\,L}(c \vee d, 1) = \theta(c \vee d, 1)$.

It is clear that $X_{a\wedge b} \cup \complement X_{c\vee d}$ is an open subset of $\complement D$, so it corresponds to a congruence of the demi-p-lattice below ϕ. It is not a convex subset because $x \in X_{a\wedge b}$, $z \in \complement X_{c\vee d}$ and $y \notin X_{a\wedge b} \cup \complement X_{c\vee d}$. Therefore the corresponding congruence $\theta(0, a \wedge d) \vee \theta(c \vee d, 1)$ is not a principal congruence, which is a contradiction. □

Now we can characterize demi-p-lattices with the PJP.

THEOREM 3.13. *Let L be a demi-p-lattice and $\complement D$ be the open subset of $d(L)$ corresponding to congruence ϕ. Then the following are equivalent:*

(i) *L has the* PJP.
(ii) *There is no 3-element chain in $\complement D$, and $\complement D$ is a convex subset of $d(L)$.*
(iii) *There exists a 3-element chain neither in $d(D_0)$ nor in $d(D_1)$, and there exists no 3-element chain $x < y < z$ in $d(L)$ such that $D_0 \not\subseteq x$ and $z \cap D_1 \neq \emptyset$.*

Proof.

(i) $\Longrightarrow$ (ii): By Corollary 3.9 we know that $\complement D = A_0 \cup A_1$.

Now observe that, if there is a chain $x < y < z$ with $x \in \complement D$ such that $x \in A_1$, then $y, z \in A_1$ because, by Lemma 3.7, A_1 is an increasing subset. This contradicts Corollary 3.10. Therefore, if $x \in \complement D$, then $x \in A_0$ and $x \notin A_1$.

But, by Lemma 3.7, A_0 is a decreasing subset, therefore, if $z \in A_0$, the 3-element chain would be in A_0 which is absurd by Corollary 3.10.

By Lemma 3.12, $z \notin A_1$, thus it is impossible to have a 3-element chain $x < y < z$ with $x, z \in \complement D$.

So we conclude that there is no 3-element chain in $\complement D$ and that $\complement D$ is a convex subset of $d(L)$.

(ii) $\implies$ (i): Now suppose that (ii) holds. In order to show that L has the PJP, it is enough, by Theorem 3.6, to show that for any two principal congruences $\theta_1 \leq \phi$ and $\theta_2 \leq \phi$, $\theta_1 \vee \theta_2$ is principal.

Let A_{θ_1} and A_{θ_2} be the open subsets of $d(L)$ corresponding to θ_1 and θ_2, respectively. By Lemma 3.11 we know that A_{θ_1} and A_{θ_2} are clopen convex subsets of $d(L)$ contained in $\complement D$, so it is clear that $A_{\theta_1} \cup A_{\theta_2}$ is also a clopen subset of $d(L)$ and that it is a convex subset since it is contained in the convex subset $\complement D$ and there is no 3-element chain in $\complement D$. Therefore, by Lemma 3.11, the congruence $\theta_1 \vee \theta_2$ corresponding to $A_{\theta_1} \cup A_{\theta_2}$ is a principal congruence of the demi-p-lattice L.

(ii) $\iff$ (iii): by Lemma 3.8 and Corollary 3.9. □

In [13; Lemma 4.3], we proved that, if L is a demi-p-lattice, the subset $D \subseteq d(L)$ is an antichain, so we conclude:

COROLLARY 3.14. *If L is a demi-p-lattice with the* PJP*, then there is no 4-element chain in $d(L)$.*

In [16; Theorem 2.5] Sankappanavar shows that in almost-p-lattices $\phi = \theta(D_1)$. Thus, in this variety, we have always $\complement D = A_1$ by Lemma 3.7.

It follows that we can extend to almost p-lattices the corresponding theorem stated for p-lattices with the PJP by Beazer [4; Theorem 3.8].

THEOREM 3.15. *Let L be an almost-p-lattice. Then the following are equivalent:*

(i) *L has the* PJP.
(ii) *There is no 4-element chain in $d(L)$.*
(iii) *D_1 has the* PJP.
(iv) *There is no 3-element chain in $d(D_1)$.*
(v) *For any $d, x, y \in D_1$ with $d \leq x \leq y$ there exists $u, v \in D_1$ such that $d = x \wedge u$, $x \vee u = y \wedge v$ and $y \vee v = 1$.*

Proof.
(i) $\implies$ (ii): by Corollary 3.14.
(ii) $\implies$ (i): When L is an almost p-lattice,

$$D = \{x \in d(L) : \ x \text{ is a minimal element of } d(L)\}$$

by [13; Theorem 4.9]. So, for almost p-lattices, the existence of a 4-element chain in $d(L)$ is equivalent to the existence of a 3-element chain in $\complement D$.

But, from [16; Theorem 2.5], $\phi = \theta(D_1)$. Then, in an almost-p-lattice L, $\complement D$ is always a convex subset of $d(L)$ since $\complement D = A_1$.

Thus by Theorem 3.13, L has the PJP.

(i) $\iff$ (iv): by Theorem 3.13 since $\phi = \theta(D_1)$, and consequently $\complement D = A_1$.

The equivalence of statements (iii), (iv) and (v) is given by Beazer [4; Lemma 3.5]. □

4. The principal intersection property

For p-lattices it is known ([4; Theorem 3.10]) that any algebra having PJP has the PIP. The same is true for demi-p-lattices.

THEOREM 4.1. *Let L be a demi-p-lattice. If L has the* PJP, *then L has the* PIP.

Proof. Let L be a demi-p-lattice with the PJP. Let $\theta(a,b)$ and $\theta(c,d)$ be elements of $\mathrm{Con}(L)$. By Lemma 3.5, there exist $e,f \in B(L)$ and $g,h,i,j \in L$ with $g \le h$, $i \le j$, $g' = h'$ and $i' = j'$ such that $\theta(a,b) = \theta(0,e) \vee \theta(g,h)$ and $\theta(c,d) = \theta(0,f) \vee \theta(i,j)$.

Then

$$\begin{aligned}&\theta(a,b) \wedge \theta(c,d)\\ &= \big(\theta(0,e) \wedge \theta(0,f)\big) \vee \big(\theta(0,e) \wedge \theta(i,j)\big) \vee \big(\theta(0,f) \wedge \theta(g,h)\big) \vee \big(\theta(g,h) \wedge \theta(i,j)\big).\end{aligned}$$

We have, by [16; Corollary 3.6], $\theta(0,e) \wedge \theta(0,f) = \theta_{\mathrm{lat}\,L}(e',1) \wedge \theta_{\mathrm{lat}\,L}(f',1)$. It is known that $\theta_{\mathrm{lat}\,L}(e',1) \wedge \theta_{\mathrm{lat}\,L}(f',1) = \theta_{\mathrm{lat}\,L}(e' \vee f',1)$.

We claim that $\theta_{\mathrm{lat}\,L}(e' \vee f',1) = \theta(e' \vee f',1)$.

In fact, again by [16; Corollary 3.6], $(x,y) \in \theta(e' \vee f',1)$ if and only if $x \wedge (e' \vee f') \wedge (e' \vee f')'' = y \wedge (e' \vee f') \wedge (e' \vee f')''$.

By distributivity, (S5) and (S8) we have

$$(e' \vee f') \wedge (e' \vee f')'' = \big(e''' \wedge (e' \vee f')''\big) \vee \big(f''' \wedge (e' \vee f')''\big) = e' \vee f'.$$

Thus $(x,y) \in \theta(e' \vee f',1)$ if and only if $x \wedge (e' \vee f') = y \wedge (e' \vee f')$, which is equivalent to $(x,y) \in \theta_{\mathrm{lat}\,L}(e' \vee f',1)$. Hence $\theta_{\mathrm{lat}\,L}(e' \vee f',1) = \theta(e' \vee f',1)$ and consequently $\theta(0,e) \wedge \theta(0,f) = \theta(e' \vee f',1)$.

From [16; Corollary 3.6] and from $i' = j'$ it is clear that $\theta(0,e) \wedge \theta(i,j) = \theta_{\mathrm{lat}\,L}(e',1) \wedge \theta_{\mathrm{lat}\,L}(i,j)$.

Since in a distributive lattice L the PIP holds, there is $p,q \in L$ such that $\theta_{\mathrm{lat}\,L}(e',1) \wedge \theta_{\mathrm{lat}\,L}(i,j) = \theta_{\mathrm{lat}\,L}(p,q)$ and $\theta_{\mathrm{lat}\,L}(p,q) = \theta(p,q)$ because $\theta_{\mathrm{lat}\,L}(p,q) \le \theta_{\mathrm{lat}\,L}(i,j) \le \phi$.

Therefore $\theta(0,e) \wedge \theta(i,j) = \theta(p,q)$.

Using the same arguments we can show that there exists $m, n \in L$ such that $\theta(0,f) \wedge \theta(g,h) = \theta(m,n)$.

In a similar way, since $g' = h'$ and $i' = j'$, we have $\theta(g,h) \wedge \theta(i,j) = \theta_{\mathrm{lat}\,L}(g,h) \wedge \theta_{\mathrm{lat}\,L}(i,j)$. But distributive lattices have the PIP, therefore there exist $r, s \in L$ such that $\theta_{\mathrm{lat}\,L}(g,h) \wedge \theta_{\mathrm{lat}\,L}(i,j) = \theta_{\mathrm{lat}\,L}(r,s)$ and $\theta_{\mathrm{lat}\,L}(r,s) = \theta(r,s)$ since $\theta_{\mathrm{lat}\,L}(r,s) \leq \theta_{\mathrm{lat}\,L}(i,j) \leq \phi$.

Hence all these meets are principal congruences of the demi-p-lattice L. It is now clear, since the PJP holds in L, that $\theta(a,b) \wedge \theta(c,d)$ is a principal congruence of L. So L has the PIP. □

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