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NOTE ON THE MAPPINGS OF COMPLEX PROJECTIVE SPACES

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ABSTRACT. Here we show that all homotopy classes of maps $f\colon \mathbb{C}P^n \to \mathbb{C}P^m$ except the class of the standard embedding and of that with reversed orientation contain only rather complicated singular maps if n is even and $m < (3/2)n$.

This paper gives some immediate easy applications of the results of [3] and [4]. Let x_i denote the generator of the cohomology ring $H^*(\mathbb{C}P^i;\mathbb{Z})$ of the complex i-dimensional projective space $\mathbb{C}P^i$.

Recall that the set of homotopy classes $[\mathbb{C}P^n, \mathbb{C}P^{n+k-1}]$ can be identified with $H^2(\mathbb{C}P^n;\mathbb{Z}) \approx \mathbb{Z}$. The homotopy class of a map f corresponds to $f^*(x_{n+k-1})$.

Let us also recall that a smooth map $g\colon M \to N$ of positive codimension (i.e. $\dim(M) < \dim(N)$) has a singular point at $x \in M$ if the rank of the differential $\mathrm{d}g$ at x is less than $\dim(M)$. Such a point has type $\Sigma^{1,0}$ if $\operatorname{rank}\bigl(\mathrm{d}g(x)\bigr) = \dim(M) - 1$, and the restriction of g to the set of singular points has maximal rank at x. The $\Sigma^{1,0}$ singular points are the simplest singularities; for example, for a generic map they form the highest dimensional stratum of the set of singular points. Locally at these points the image of the map is a generalized Whitney umbrella (see [1]).

THEOREM 1.

1) *Let $f\colon \mathbb{C}P^n \to \mathbb{C}P^{n+k-1}$ be a smooth map such that $f^*(x_{n+k-1}) \neq \pm x_n$, where n is even and $2k \le n$. Then f has singular points which are not of the type $\Sigma^{1,0}$.*

2) *An arbitrary map $g\colon V^{2n} \to \mathbb{C}P^{n+k-1}$ bordant to a nonzero multiple of f has a singular point which is not of the type $\Sigma^{1,0}$.*

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Remarks.

1) S. Feder has shown that such a map f must have singular points. The theorem says that f must have singularities which are more complicated than the simplest (Whitney umbrella) singularities and that this also holds for maps rationally bordant to f.

2) If $2k > n$ then generic maps in the dimensions given above have no singular points different from the $\Sigma^{1,0}$ points for dimensional reasons.

Proof of Theorem 1. Let ν_f be the stable normal bundle of f (i.e. ν_f equals to the virtual bundle $f^*(\tau\mathbb{C}P^{n+k-1}) - \tau(\mathbb{C}P^n)$. Its kth Pontrjagin class is nonzero by [3; Theorem 3.1]. Therefore

$$\langle p_k(\nu_f) \cup p_1(\tau\mathbb{C}P^n)^{(n-2k)/2}, [\mathbb{C}P^n]\rangle \neq 0 .$$

Indeed, $p_1(\mathbb{C}P^n) = (n+1)x_n^2$, where $x_n \in H^2(\mathbb{C}P^n;\mathbb{Z})$ is the generator, and $p_k(\nu_f)$ (as any nonzero element in $H^{2k}(\mathbb{C}P^n;\mathbb{Z})$) is a non-zero multiple of x_n^{2k}. Now the product $p_k(\nu_f) \cup p_1(\tau\mathbb{C}P^n)^{(n-2k)/2}$ is a nonzero multiple of x_n^n and so it is non-vanishing, when evaluated on the fundamental class of $\mathbb{C}P^n$.

Note that this number $\langle p_k(\nu_f) \cup p_1(\tau\mathbb{C}P^n)^{(n-2k)/2}, [\mathbb{C}P^n]\rangle$ is a linear combination of Pontrjagin numbers in the sense of Conner-Floyd. (Indeed, if we denote by $p(M)$ the total Pontrjagin class of the tangent bundle of a manifold M and by $\bar{p}(M)$ the total Pontrjagin class of the (stable) normal bundle of the manifold M, then we have $p(\nu_f) = f^*p(\mathbb{C}P^{n+k-1})/p(\mathbb{C}P^n) = f^*p(\mathbb{C}P^{n+k-1}) \cup \bar{p}(\mathbb{C}P^n)$.)

Now suppose that f has only $\Sigma^{1,0}$ singular points. Then, by [4; Corollary], a nonzero multiple of f is bordant to an immersion. This means that there exist

a) a nonzero integer N,

b) a $2n$-dimensional manifold V^{2n} cobordant to $N \cdot \mathbb{C}P^n$,

c) an immersion i of V^{2n} into $\mathbb{C}P^{n+k-1}$

such that the bordism class $[i] \in \Omega_{2n}(\mathbb{C}P^{n+k-1})$ of i is N times the class $[f]$. Let us denote by ν_i the normal bundle of i. Since the Pontrjagin numbers are invariants of bordism classes (see [2; Theorem 17.4]) we obtain that

$$\langle p_k(\nu_i) \cup p_1(V^{2n})^{(n-2k)/2}, [V^{2n}]\rangle \neq 0 .$$

In particular, $p_k(\nu_i)$ is nonzero. But this is impossible since $\dim(\nu_i) = 2(k-1)$. The contradiction obtained proves part 1) of the theorem. The proof of part 2) is similar. □

In order to formulate a more general theorem we introduce some notation.

Let Π be a product of complex projective spaces

$$\Pi = \mathbb{C}P^{m_1} \times \mathbb{C}P^{m_2} \times \ldots \times \mathbb{C}P^{m_r}$$

and let π_j denote the projection of Π onto its jth factor. We say that a map

$$f \colon \mathbb{C}P^n \to \Pi$$

has degree $(d_1, d_2, \ldots, d_r)$ if $\pi_j^* \circ f^*(x_{m_j}) = d_j x_n$ for $j = 1, 2, \ldots, r$.

THEOREM 2. *Let f be a map $\mathbb{C}P^n \to \Pi$ of degree $(d_1, d_2, \dots, d_r)$, where n is even.*

Let us consider the formal power series

$$(1+x^2)^{-(n+1)} \prod_{j=1}^{j=r} (1+d_j x^2)^{m_j+1}.$$

If we put $x^{n+1} = 0$ it becomes a polynomial. If this polynomial has degree greater than $k = (m_1 + m_2 + \cdots + m_r) - n$, then f has a singular point, which is not of the type $\Sigma^{1,0}$.

Proof. The proof is essentially the same as that for the previous theorem. The above polynomial gives the total Pontrjagin class of the virtual normal bundle ν_f of the map f if we substitute the generator $x_n \in H^*(\mathbb{C}P^n)$ for x. If its degree is s, and $s > k$, then $p_s(\nu_f) \neq 0$. Now multiplying this by the appropriate power of the class $p_1(\mathbb{C}P^n)$ we obtain a nonzero element of the group $H^{2n}(\mathbb{C}P^n)$, and integrating it on the fundamental class we get a nonzero "Pontrjagin number". Therefore this "Pontrjagin number" does not vanish for any map bordant to any nonzero multiple of f. But a nonzero multiple of f is bordant to an immersion if f has only $\Sigma^{1,0}$ singular points (by [4]) and for an immersion this "Pontrjagin number" must vanish, since the class p_s of the normal bundle of the immersion vanishes, because $4s$ is greater than twice the dimension of this bundle. □

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