Ivan Havel
Domination in $n$-cubes with diagonals

*Mathematica Slovaca*, Vol. 48 (1998), No. 2, 105--115

Persistent URL: http://dml.cz/dmlcz/129229

Math. Slovaca, 48 (1998), No. 2, 105–115

Mathematica Slovaca


# DOMINATION IN $n$-CUBES WITH DIAGONALS

IVAN HAVEL

(*Communicated by Martin Škoviera*)

ABSTRACT. The paper studies domination in cubes with diagonals, the graphs that arise from ordinary hypercubes $Q_n$ by adding new edges joining opposite vertices. We establish basic relations between domination and domatic numbers of cubes with diagonals and those of hypercubes, and use them to show that the domatic number of $Q_6$ is 5 (so far it was only known that it equals either 4 or 5).

## 1. Introduction and notation

The graphs of the $n$-cubes with diagonals form an important subclass of the generalized hypercubes and therefore also of the s.c. cube like graphs ([3], [4]). They arise in a natural way: to a well-known hypercube graph $Q_n$, one adds $2^{n-1}$ new edges joining pairs of opposite vertices. It is shown in [10] that for $n$ even $n$-cubes with diagonals are isomorphic to the s.c. *extended odd graphs*, that were defined and first studied in [7]. Also for $n$ odd, the $n$-cubes with diagonals are isomorphic to already known graphs — these are the s.c. *halfcubes* defined again in [7]. The extended odd graphs have already been studied from the point of view of their chromatic properties. It is proved in [10] that the chromatic number of an extended odd graph is 4; a stronger result claiming that the chromatic number of an arbitrary cube-like graph is different from 3 is in [8]. Together with halfcubes that are bipartite, extended odd graphs have also been studied from the point of view of their interval properties ([7], [1]).

In the present paper, we try to find out, whether and in which way the known results on domination in hypercubes ([6], [12], [5]) hold also in the case of $n$-cubes with diagonals. We establish some relations between domination concepts for hypercubes and those for $n$-cubes with diagonals. For certain dimensions, we determine exact values of both the domination and domatic numbers

AMS Subject Classification (1991): Primary 03D30.
Key words: domination, cube with diagonals, hypercube.

Research supported by the grant GA ČR 201/94/1064.

of the $n$-cubes with diagonals. Using a computer we succeeded to show that the domatic number of the 5-cube with diagonals equals 5; this made possible to solve a problem from [5]: we show that the domatic number of the 6-dimensional hypercube $Q_6$ also equals 5.

The notion of the hypercube graph $Q_n$ is used in the usual sense; we define it for $n \geq 1$ as follows: the vertex set $V(Q_n)$ consists of all $2^n$ binary vectors of length $n$, and two vertices $u, v \in V(Q_n)$ are adjacent if and only if they differ in exactly one coordinate. We recall that the usual distance $\mathrm{d}_{Q_n}$ in $Q_n$ is the known Hamming distance. We will denote it in a simplified manner by $\mathrm{d}_{\mathcal{H}}$; thus for arbitrary binary vectors $u$, $v$ of the same length $\nu \geq 1$ and $k \in [0, \dots, \nu]$, $\mathrm{d}_{\mathcal{H}}(u, v) = k$ if and only if $u$ and $v$ differ in exactly $k$ coordinates. For the sake of brevity, the graph of the $n$-cube with diagonals will be throughout the paper denoted by $R_n$ (its full denotation, e.g. according to [3], should be $Q_n(1, n)$). We define it for $n \geq 2$ as follows: $V(R_n) = V(Q_n)$, and for $u, v \in V(R_n)$, $u$ and $v$ are adjacent if and only if $\mathrm{d}_{\mathcal{H}}(u, v) = 1$ or $\mathrm{d}_{\mathcal{H}}(u, v) = n$.

Let us recall here two basic operations on binary vectors: if $u = (u_1, \dots, u_n)$ is a binary vector of length $n \geq 1$ (and therefore $u_i \in \{0, 1\}$, $i = 1, \dots, n$), we denote by $\overline{u}$ the complementary vector of $u$, defined as follows:

$$\overline{u} = (\overline{u_1}, \dots, \overline{u_n}),$$

where $\overline{0} = 1$ and $\overline{1} = 0$. If $u$ and $v$ are two binary vectors of the same length $n \geq 1$, $u = (u_1, \dots, u_n)$, $v = (v_1, \dots, v_n)$, we put

$$u \oplus v = (u_1 \oplus v_1, \dots, u_n \oplus v_n),$$

where $\oplus$ is the addition mod 2 ($0 \oplus 0 = 1 \oplus 1 = 0$, $0 \oplus 1 = 1 \oplus 0 = 1$).

An important role in our constructions will be played by the *Hamming code*; for its existence and properties cf., e.g., [9]. We summarize all we need from this area of coding theory in the following statement:

Let $k > 1$ and $n = 2^k - 1$. Then in $Q_n$, there exists the Hamming code, i.e., the set $H_n \subseteq V(Q_n)$ such that

$\mathcal{H}_0$: $H_n$ contains the zero vector $(v_1, \dots, v_n)$ for which $v_i = 0$, $i = 1, \dots, n$.

$\mathcal{H}_1$: $|H_n| = 2^{n-k}$, the Hamming distance of any two different elements of $H_n$ is at least 3, and for every $u \in V(Q_n)$, there is exactly one $h(u) \in H_n$ fulfilling $\mathrm{d}_{\mathcal{H}}\big(u, h(u)\big) \leq 1$.

$\mathcal{H}_2$: $v \in H_n \implies \overline{v} \in H_n$.

$\mathcal{H}_3$: If $1 \leq i \leq n$, $\iota \in \{0, 1\}$, and $H^{\iota}_{n,i} = \big\{(v_1, \dots, v_n) \in H_n ;\ v_i = \iota\big\}$, then $|H^0_{n,i}| = |H^1_{n,i}| = 2^{n-k-1}$.

Further, let us choose arbitrarily $w \in V(Q_n)$ and put

$$H_n(w) = \{v \oplus w ;\ v \in H_n\}.$$

Then also $\mathcal{H}_1'$–$\mathcal{H}_3'$ hold, where $\mathcal{H}_j'$ is obtained from the corresponding $\mathcal{H}_j$ by substituting $H_n(w)$ for $H_n$.

From the area of domination in graphs, we shall need the following definitions and facts:

A set of vertices $D \subseteq V(G)$ is a *dominating set* of $G$ if every vertex not in $D$ is adjacent to at least one vertex in $D$. The *domination number* $\gamma(G)$ is the minimum cardinality taken over all minimal dominating sets. A partition of $V(G)$ into dominating sets of $G$ is called a *domatic partition.* The maximum order of a domatic partition of $G$ is called the *domatic number* of $G$ and is denoted by $d(G)$.

$d(G)$ can also be defined in an equivalent way, using the s.c. domatic coloring of $G$. Let $c$ be a coloring of vertices of $G$, let $c(V(G)) = \{1,\dots,k\}$. We say that $c$ is a *domatic coloring* of $G$ if to every $v \in V(G)$ and every $i \in \{1,\dots,k\}$, $i \neq c(v)$, there is $u \in V(G)$ with $c(u) = i$, $u$ adjacent to $v$. (Observe that in a domatic coloring, vertices of the same color may be adjacent.) Then the maximum number of colors of a domatic coloring of $G$ is $d(G)$.

One verifies directly the following facts:

$$\gamma(G) \geq \frac{|V(G)|}{\Delta(G)+1},$$

where $\Delta(G)$ is the maximum degree of vertices in $G$.

If $H$ is a spanning subgraph of $G$, then $\gamma(G) \leq \gamma(H)$.

Further,

$$d(G) \leq \frac{|V(G)|}{\gamma(G)}$$

and also $d(G) \leq \delta(G)+1$, where $\delta(G)$ is the minimum degree of vertices in $G$ ([2]). For a regular graph $G$, $d(G) = \delta(G)+1$ only if $\delta(G)+1$ divides $|V(G)|$ ([11]).

We are closing this introductory paragraph by presenting several results concerning the domination in hypercubes that are known from the previous work and are related to our subject.

The following table yields the known values (for $1 \leq n \leq 7$) of $\gamma(Q_n)$ (cf. [6]):

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $\gamma(Q_n)$ | 1 | 2 | 2 | 4 | 7 | 12 | 16 |

$\gamma(Q_n)$ can also be determined for the dimensions of the form $n = 2^k - 1$; as a minimum dominating set in $Q_{2^k-1}$ $(k > 1)$, the Hamming code can be taken. Hence we get

$$\gamma(Q_{2^k-1}) = 2^{2^k-k-1}, \qquad k > 1.$$

As to the domatic number $d(Q_n)$, it is shown in [12] that

$$d(Q_n) \leq d(Q_{n+1}), \qquad n \geq 1,$$

and

$$d(Q_{2^k-1}) = d(Q_{2^k}) = 2^k, \qquad k \geq 1.$$

L a b o r d e ([5]), using Hamming code, gave an alternative proof of the result of [12]. He also showed, disproving a conjecture from [12], that $d(Q_5) = 4$ and further, using $\gamma(Q_6) = 12$, also

$$4 \leq d(Q_6) \leq 5. \tag{1}$$

Thus for the known values of $d(Q_n)$ we have the following table:

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | ... | 15 | 16 | ... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $d(Q_n)$ | 2 | 2 | 4 | 4 | 4 | 4–5 | 8 | 8 | | 16 | 16 | |

We are going to solve the problem of $d(Q_6)$ below: constructing a domatic coloring of $R_5$ by 5 colors and using the general statement claiming that $d(R_n) \leq d(Q_{n+1})$, we show that

$$d(Q_6) = 5.$$

## 2. $\gamma(R_n)$

Trying to determine exact values and bounds of the domination number $\gamma(R_n)$, we start with the following obvious remark:

Since $Q_n$ is a spanning subgraph of $R_n$, we have

$$\gamma(R_n) \leq \gamma(Q_n), \qquad n \geq 2. \tag{2}$$

**PROPOSITION 1.**

$$\gamma(Q_{n+1}) \leq 2 \cdot \gamma(R_n), \qquad n \geq 2. \tag{3}$$

P r o o f . Let $A \subseteq V(R_n)$ be a dominating set in $R_n$. We put

$$\begin{aligned} A' &= \{(v_1, \dots, v_n, 0);\ (v_1, \dots, v_n) \in A\}, \\ B &= \{(\overline{v_1}, \dots, \overline{v_n}, 1);\ (v_1, \dots, v_n) \in A\}, \\ C &= A' \cup B, \end{aligned}$$

and show that $C$ dominates $Q_{n+1}$. Let $u \in V(Q_{n+1})$, $u = (u_1, \dots, u_n, u_{n+1})$. We have to show either that $u \in C$, or that $u$ is adjacent (in $Q_{n+1}$) to a vertex of $C$. To this end, put $w = (u_1, \dots, u_n)$.

I. Assume first $u_{n+1} = 0$. If $w \in A$, then $u \in A'$. If $w \notin A$, then there is $v \in A$, $v = (v_1, \dots, v_n)$ such that either $d_{\mathcal{H}}(w, v) = 1$ or $d_{\mathcal{H}}(w, v) = n$. Let $d_{\mathcal{H}}(w, v) = 1$. Then also $d_{\mathcal{H}}((u_1, \dots, u_n, 0), (v_1, \dots, v_n, 0)) = 1$; we have, however, $(u_1, \dots, u_n, 0) = u$ and $(v_1, \dots, v_n, 0) \in A'$, hence we are done. Let $d_{\mathcal{H}}(w, v) = n$; then $w = \overline{v}$. Since $(\overline{v_1}, \dots, \overline{v_n}, 1) \in B$, $d_{\mathcal{H}}((u_1, \dots, u_n, 0), (\overline{v_1}, \dots \dots, \overline{v_n}, 1)) = 1$, and $(u_1, \dots, u_n, 0) = u$, we are done again.

II. Let now $u_{n+1} = 1$. Consider $\overline{w} = (\overline{u_1}, \dots, \overline{u_n})$. If $\overline{w} \in A$, then $(u_1, \dots, u_n, 1) \in B$, i.e., $u \in B \subseteq C$. If $\overline{w} \notin A$, then there is $v \in A$, $v = (v_1, \dots, v_n)$ such that either $d_{\mathcal{H}}(v, \overline{w}) = 1$ or $d_{\mathcal{H}}(v, \overline{w}) = n$. Let first $d_{\mathcal{H}}(v, \overline{w}) = 1$. Then $d_{\mathcal{H}}(\overline{v}, w) = 1$ and also $d_{\mathcal{H}}((\overline{v_1}, \dots, \overline{v_n}, 1), (u_1, \dots, u_n, 1)) = 1$. Since $(\overline{v_1}, \dots, \overline{v_n}, 1) \in B$ and $(u_1, \dots, u_n, 1) = u$, we are done. If $d_{\mathcal{H}}(v, \overline{w}) = n$, then $w = \overline{v}$. Since $(\overline{v_1}, \dots, \overline{v_n}, 1) \in B$ and $u = (u_1, \dots, u_n, 1) = (\overline{v_1}, \dots, \overline{v_n}, 1)$, we have $u \in B$, which accomplishes the proof. □

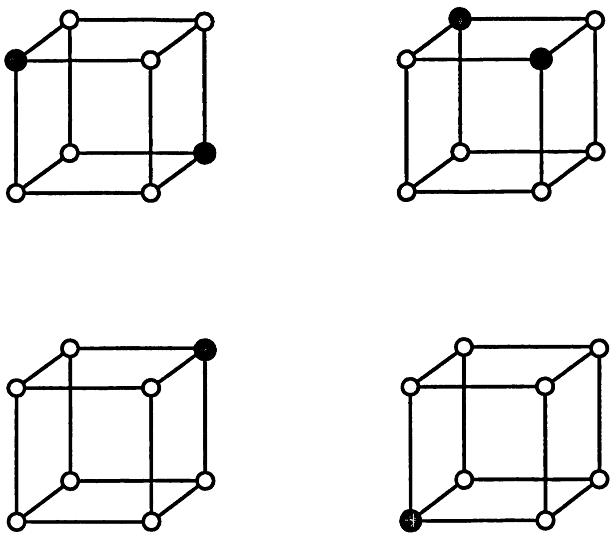

FIGURE 1.

Using (2) and (3), we obtain as a corollary $\gamma(R_3) = 2$, $\gamma(R_4) = 4$ and $\gamma(R_5) \geq 6$. Fig. 1 shows, on the other hand, a 6-element dominating set in $R_5$, hence

$$\gamma(R_5) = 6 .$$

In the figure, all the vertices of $R_5$ but only a half of its edges are drawn. Three more edges (horizontal, vertical and diagonal) are incident in $R_5$ with each vertex: the horizontal edge joins it with the corresponding neighbour to the right or to the left, the vertical edge joins it with the neighbour situated up- or downwards, and the diagonal edge leads to the opposite vertex.

**PROPOSITION 2.** *Let $m \ge 2$, $n = 2^m - 2$. Then*

$$\gamma(R_n) = 2^{n-m}.$$

P r o o f. Let $m \ge 2$, $n = 2^m - 2$. $R_n$ is regular of degree $n + 1 = 2^m - 1$, hence $\gamma(R_n) \ge 2^{n-m}$. In order to prove $\gamma(R_n) \le 2^{n-m}$, we will construct a dominating set $D_n$ of $R_n$ with $|D_n| = 2^{n-m}$. Our construction starts with the Hamming code $H_{n+1}$ in $Q_{n+1}$. From the properties of Hamming code, we derive the following facts: $H_{n+1} \subseteq V(Q_{n+1})$, $|H_{n+1}| = 2^{n-m+1}$, and for every $u \in V(Q_{n+1})$ there is exactly one $h(u) \in H_{n+1}$ fulfilling $\mathrm{d}_{\mathcal{H}}(u, h(u)) \le 1$. Also, if $(v_1, \dots, v_{n+1}) \in H_{n+1}$, then $(\overline{v_1}, \dots, \overline{v_{n+1}}) \in H_{n+1}$.

Consider the partition $V(Q_{n+1}) = V^0_{n+1} \cup V^1_{n+1}$, where

$$V^\iota_{n+1} = \{u = (u_1, \dots, u_n, u_{n+1}) \in V(Q_{n+1});\ u_{n+1} = \iota\}, \quad \iota = 0, 1,$$

and put

$$H^\iota_{n+1} = H_{n+1} \cap V^\iota_{n+1}, \qquad \iota = 0, 1.$$

Also from the properties of Hamming code, we have $|H^0_{n+1}| = |H^1_{n+1}| = 2^{n-m}$. Further, if $u \in V^0_{n+1}$, $u = (u_1, \dots, u_n, 0)$ and $h(u) = (v_1, \dots, v_n, v_{n+1})$, then $h(u) \in V^1_{n+1}$ if and only if $u_i = v_i$, $i = 1, \dots, n$. Hence, if $u \in V^0_{n+1}$, $u = (u_1, \dots, u_n, 0)$ and $(u_1, \dots, u_n, 1) \notin H_{n+1}$, then $h(u) \in H^0_{n+1}$.

We are ready to define

$$D_n = \{(u_1, \dots, u_n);\ (u_1, \dots, u_n, 0) \in H^0_{n+1}\}.$$

Let us verify that $D_n$ dominates $R_n$. Given $(u_1, \dots, u_n) \in V(R_n) \setminus D_n$, consider $w = (u_1, \dots, u_n, 0)$ and find $h(w) \in H_{n+1}$ such that $\mathrm{d}_{\mathcal{H}}(w, h(w)) \le 1$. Let $h(w) = (v_1, \dots, v_n, v_{n+1})$. If $v_{n+1} = 0$ and therefore $h(w) \in H^0_{n+1}$, then $\mathrm{d}_{\mathcal{H}}((u_1, \dots, u_n), (v_1, \dots, v_n)) = 1$, and we are done. If, on the other hand, $v_{n+1} = 1$ and $h(w) \in H^1_{n+1}$, then $u_i = v_i$, $i = 1, \dots, n$. But, in this case also $(\overline{v_1}, \dots, \overline{v_n}, 0) \in H_{n+1}$, hence $(\overline{v_1}, \dots, \overline{v_n}) \in D_n$, and this vertex is a neighbour (in $R_n$) of $(u_1, \dots, u_n)$. □

One verifies directly (or using the Proposition 2 just proved) that $\gamma(R_2) = 1$; hence the known values of $\gamma(R_n)$ may be put in the following table:

| $n$ | 2 | 3 | 4 | 5 | 6 | ... | 14 | ... |
|---|---|---|---|---|---|---|---|---|
| $\gamma(R_n)$ | 1 | 2 | 4 | 6 | 8 | | $2^{10}$ | |

## 3. $d(R_n)$

**PROPOSITION 3.**

$$d(R_n) \le d(Q_{n+1}), \qquad n \ge 2. \tag{4}$$

Proof. Let $c$ be a domatic coloring of the graph $R_n$ by $k$ colors,

$$c\colon V(R_n) \to \{1, \dots, k\}.$$

Define the coloring $c'$ of vertices of $Q_{n+1}$ (also by $k$ colors),

$$c'\colon V(Q_{n+1}) \to \{1, \dots, k\},$$

putting

$$c'\big((v_1, \dots, v_n, v_{n+1})\big) = \begin{cases} c\big((v_1, \dots, v_n)\big) & \text{if } v_{n+1} = 0, \\ c\big((\overline{v_1}, \dots, \overline{v_n})\big) & \text{if } v_{n+1} = 1. \end{cases}$$

We are going to verify that $c'$ is a domatic coloring of $Q_{n+1}$. Let $v = (v_1, \dots, v_n, v_{n+1}) \in V(Q_{n+1})$, let $i \in \{1, \dots, k\}$, $i \ne c'(v)$.

I. Assume first $v_{n+1} = 0$. Then $c'(v) = c\big((v_1, \dots, v_n)\big)$ and there is $u = (u_1, \dots, u_n)$ such that $c(u) = i$, $u$ being a neighbour of $(v_1, \dots, v_n)$ in $R_n$. If $\mathrm{d}_{\mathcal{H}}\big(u, (v_1, \dots, v_n)\big) = 1$, then also $\mathrm{d}_{\mathcal{H}}\big((u_1, \dots, u_n, 0), (v_1, \dots, v_n, 0)\big) = 1$. Since $c'\big((u_1, \dots, u_n, 0)\big) = c(u) = i$, we are done. If $\mathrm{d}_{\mathcal{H}}\big(u, (v_1, \dots, v_n)\big) = n$, then $u_i = \overline{v_i}$ and $c'\big((v_1, \dots, v_n, 1)\big) = c\big((\overline{v_1}, \dots, \overline{v_n})\big) = c\big((u_1, \dots, u_n)\big) = i$. Since $(v_1, \dots, v_n, 1)$ is a neighbour (in $Q_{n+1}$) of $(v_1, \dots, v_n, 0)$, we are done again.

II. Assume now $v_{n+1} = 1$. Then $c'(v) = c\big((\overline{v_1}, \dots, \overline{v_n})\big)$, and again there is $u = (u_1, \dots, u_n)$ such that $c(u) = i$, $u$ being adjacent in $R_n$ with $(\overline{v_1}, \dots, \overline{v_n})$. If $\mathrm{d}_{\mathcal{H}}\big(u, (\overline{v_1}, \dots, \overline{v_n})\big) = 1$, then $\mathrm{d}_{\mathcal{H}}\big((u_1, \dots, u_n, 0), (\overline{v_1}, \dots, \overline{v_n}, 0)\big) = 1$, $\mathrm{d}_{\mathcal{H}}\big((\overline{u_1}, \dots, \overline{u_n}, 1), (v_1, \dots, v_n, 1)\big) = 1$ and $c'\big((\overline{u_1}, \dots, \overline{u_n}, 1)\big) = c\big((u_1, \dots, u_n)\big) = i$. If $\mathrm{d}_{\mathcal{H}}\big(u, (\overline{v_1}, \dots, \overline{v_n})\big) = n$, then $u_i = v_i$, $i = 1, \dots, n$. Further, $c'\big((v_1, \dots, v_n, 0)\big) = c\big((v_1, \dots, v_n)\big) = c\big((u_1, \dots, u_n)\big) = i$, $(v_1, \dots, v_n, 0)$ being a neighbour (in $Q_{n+1}$) of $(v_1, \dots, v_n, 1)$. □

Similarly to the hypercubes $Q_n$, we have a monotony of the domatic number also for cubes with diagonals:

$$d(R_n) \le d(R_{n+1}), \qquad n \ge 2.$$

We prove it analogously to the case of hypercubes (cf. [12]): Given a domatic coloring $c$ of the graph $R_n$, define the vertex coloring $c'$ of $R_{n+1}$ as follows:

for $(v_1, \dots, v_n, v_{n+1}) \in V(R_{n+1})$ put $c'\big((v_1, \dots, v_n, v_{n+1})\big) = c\big((v_1, \dots, v_n)\big)$.

One verifies directly that $c'$ is a domatic coloring of $R_{n+1}$.

**PROPOSITION 4.**

$$d(R_{2^k-2}) = d(R_{2^k-1}) = 2^k\,, \qquad k \geq 2\,.$$

Proof. Let $k \geq 2$, put $m = 2^k$.

First we construct a domatic coloring $c$ of $R_{m-2}$ by $m$ colors. We start with the known fact that $d(Q_{m-1}) = m$, and that the domatic coloring $c'$ of $Q_{m-1}$ by $m$ colors may be constructed from the Hamming code $H_{m-1} \subseteq V(Q_{m-1})$ as follows (cf. [5]): according to $\mathcal{H}_1$, for each $v' \in V(Q_{m-1})$ there is exactly one $h(v') \in H_{m-1}$ fulfilling $\mathrm{d}_{\mathcal{H}}(v', h(v')) \leq 1$. If $\mathrm{d}_{\mathcal{H}}(v', h(v')) = 0$ (i.e., $v' = h(v')$), we put $c'(v') = 0$. If $\mathrm{d}_{\mathcal{H}}(v', h(v')) = 1$, then $v'$ and $h(v')$ differ in exactly one coordinate; let it be the $j$-th one ($1 \leq j \leq m-1$). In this case, we put $c'(v') = j$. Now, we are ready to define $c$. For $v \in V(R_{m-2})$, $v = (v_1, \dots, v_{m-2})$, we put $v' = (v_1, \dots, v_{m-2}, 0)$ and $c(v) = c'(v')$. Let now $i \in [0, m-1]$, $M = \{v \in V(R_{m-2})\,;\ c(v) = i\}$, let $u \in V(R_{m-2}) \setminus M$, $u = (u_1, \dots, u_{m-2})$. Consider $u' = (u_1, \dots, u_{m-2}, 0)$; we have $c'(u') \neq i$. Also,

$$\{v' \in V(Q_{m-1})\,;\ c'(v') = i\} = H_{m-1}(e^{(i)})\,,$$

where $e^{(i)}$ is the unit vector of the $i$-th direction (all its coordinates equal 0 with the exception of the $i$-th one, which equals 1). Hence, there exists $h(u') \in H_{m-1}(e^{(i)})$ such that $\mathrm{d}_{\mathcal{H}}(u', h(u')) = 1$. Let $h(u') = (w_1, \dots, w_{m-2}, w_{m-1})$. If $w_{m-1} = 0$, then $(w_1, \dots, w_{m-2}) \in M$, $\mathrm{d}_{\mathcal{H}}((w_1, \dots, w_{m-2}), u) = 1$, and we are done. If $w_{m-1} = 1$, then $w_i = u_i$, $i = 1, \dots, m-2$; however, according to $\mathcal{H}_2$, we have $(\overline{w_1}, \dots, \overline{w_{m-2}}, 0) \in H_{m-1}(e^{(i)})$, hence $(\overline{w_1}, \dots, \overline{w_{m-2}}) \in M$. On the other hand, $(\overline{w_1}, \dots, \overline{w_{m-2}})$ is a neighbour (in $R_{m-2}$) of the vertex $u$.

Thus, we showed so far that $d(R_{m-2}) \geq m$. Recall that if $G$ is an arbitrary graph, then for its domatic number $d(G)$ we have $d(G) \leq \delta(G) + 1$, and further, if $G$ is regular, then $d(G) = \delta(G) + 1$ only if $\delta(G) + 1$ divides $|V(G)|$. Since $\delta(R_{m-2}) = m - 1$, we get $d(R_{m-2}) \leq m$, the latter proving

$$d(R_{m-2}) = m\,.$$

From $\delta(R_{m-1}) = m$, we get $m = d(R_{m-2}) \leq d(R_{m-1}) \leq m + 1$. Since $R_{m-1}$ is regular, $d(R_{m-1}) = m+1$ would imply that $m+1$ divides $|V(R_{m-1})|$. However, this is impossible, and therefore

$$d(R_{m-1}) = m\,.$$

□

As a corollary of Proposition 4, we get $d(R_2) = d(R_3) = 4$; further,

$$d(R_3) \leq d(R_4) \leq d(Q_5) = 4\,,$$

and therefore also

$$d(R_4) = 4\,.$$

**PROPOSITION 5.**

$$d(R_5) = 5\,.$$

Proof. From the inequality $d(G) \leq |V(G)|/\gamma(G)$ that holds for any graph $G$ and from $\gamma(R_5) = 6$, we are getting $d(R_5) \leq 5$. The proof is accomplished by Fig. 2, depicting a domatic coloring of $R_5$ by five colors. We found this coloring easily by a simple computer backtracking procedure. (To draw $R_5$, in Fig. 2, we used the same convention as before in Fig. 1 — all the vertices but only a half of edges are drawn.) □

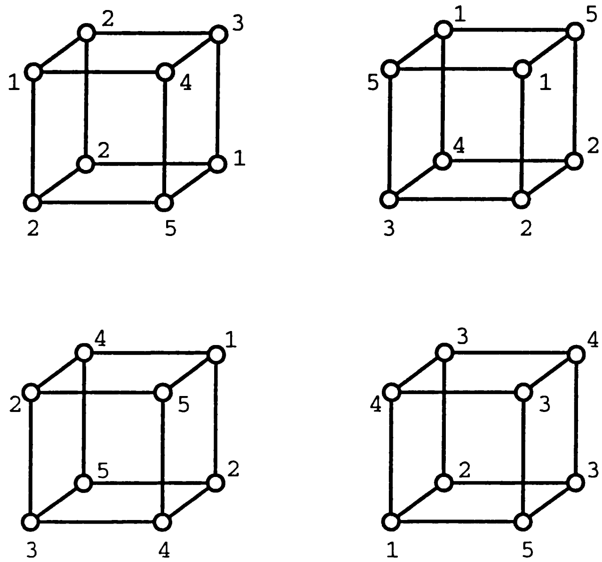


FIGURE 2.

**COROLLARY.**

$$d(Q_6) = 5\,.$$

Proof. Use (1) and Propositions 3 and 5. □

Now we can summarize the known values of $d(R_n)$ in the following table:

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | ... | 14 | 15 | ... |
|---|---|---|---|---|---|---|---|---|---|---|
| $d(R_n)$ | 4 | 4 | 4 | 5 | 8 | 8 | | 16 | 16 | |

## 4. Concluding remarks

One can formulate various interesting problems related to domination in hypercubes and $n$-cubes with diagonals. Let us mention here only those that follow naturally from our results above and concern therefore the domination and domatic numbers $\gamma(R_n)$ and $d(R_n)$.

1. In (3) of Proposition 1, we have the inequality $\gamma(Q_{n+1}) < 2 \cdot \gamma(R_n)$ for $n = 4$: $\gamma(R_4) = 4$ and $\gamma(Q_5) = 7$. This is the only value of $n$ known with this property. Of course, in addition to $n = 4$, we do know exact values of $\gamma(R_n)$ only for $n = 3, 5$ and for $n = 2^k - 2$, $k \geq 2$. In all these cases, the equality $\gamma(Q_{n+1}) = 2 \cdot \gamma(R_n)$ holds. It would be interesting to look for other values of $n$ for which in (3) the equality does not hold.

2. We pose a similar question also for (4) of Proposition 3. However, here we have the equality in all the cases we know exact values. Hence, in this case, we can ask whether $d(R_n) = d(Q_{n+1})$ for all $n \geq 2$.

REFERENCES


[1] BERRACHEDI, A.—HAVEL, I.—MULDER, H. M.: *Spherical and clockwise spherical graphs.* Report 9456/B, Erasmus University Rotterdam, 1994.

[2] COCKAYNE, E. J.—HEDETNIEMI, S. T.: *Towards a theory of domination in graphs*, Networks **7** (1977), 247–271.

[3] DVOŘÁK, T.—HAVEL, I.—LABORDE, J-M.—LIEBL, P.: *Generalized hypercubes and graph embedding with dilation*, Rostocker Mathem. Kolloquium **39** (1990), 13–20.

[4] HARARY, F.: *Four difficult unsolved problems in graph theory.* In: Recent Advances in Graph Theory (M. Fiedler, ed.), Academia, Praha, 1974, pp. 249–256.

[5] LABORDE, J-M.: *Sur le Nombre Domatique du n-Cube et Une Conjecture de Zelinka*, European J. Combin. **8** (1987), 175–177.

[6] MOLLARD, M.: *Les invariants du n-cube.* These de 3e cycle, IMAG, Grenoble, 1981.

[7] MULDER, H. M.: *The interval function of a graph.* Mathematical Centre Tracts 132, Mathematisch Centrum, Amsterdam, 1980.

[8] PAYAN, CH.: *On the chromatic number of cube-like graphs*, Discrete Math. **103** (1992), 271–277.

[9] SLOANE, N. J. A.—Mac-WILLIAMS, F. J.: *The Theory of Error-Correcting Codes*, North-Holland, Amsterdam, 1978.

[10] SOKOLOVÁ, M.: *The chromatic number of extended odd graphs is four*, Časopis Pěst. Mat. **112** (1987), 308–311.

[11] ZELINKA, B.: *Domatically critical graphs*, Czechoslovak Math. J. **30** (1980), 468 489.

[12] ZELINKA, B.: *Domatic numbers of cube graphs*, Math. Slovaca **32** (1982), 117–119.

Received August 28, 1995

*Mathematical Institute*
*Academy of Sciences*
*of the Czech Republic*
*Žitná 25*
*CZ–115 67 Praha*
*CZECH REPUBLIC*
*E-mail*: haveli@math.cas.cz