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TOTAL DOMATIC NUMBER OF CACTI

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The total domatic number of a graph was introduced by E. J. Cockayne, R. M. Dawes and S. T. Hedetniemi in [1]. By the author of this paper it was investigated in [2].

All considered graphs are undirected, without loops and multiple edges. All of them are finite.

A subset D of the vertex set $V(G)$ of an undirected graph G is called total dominating, if for each vertex $x \in V(G)$ there exists a vertex $y \in D$ adjacent to x in G. A partition of $V(G)$, all of whose classes are total dominating sets in G, is called a total domatic partition of G. The maximum number of classes of a total domatic partition of G is called the total domatic number of G and is denoted by $d_t(G)$.

Instead of a total domatic partition we can speak also about a total domatic colouring of G. Such a colouring has the property that each vertex of G is adjacent to vertices of all colours of this colouring. The maximum number of colours of a total domatic colouring is the total domatic number of G. Evidently both definitions are equivalent.

A cactus is an undirected graph in which each edge belongs to at most one circuit. It is a generalization of a tree; every tree is a cactus, but not vice versa. The domatic number (a certain analogy of the total domatic number) of cacti was studied in [3].

Here we shall study the total domatic number of cacti. First let us begin with some considerations.

A cactus will be called round, if any of its edges is contained in exactly one circuit.

The block tree $T(G)$ of a graph G is a tree whose vertex set is the union of the set B of blocks of G and the set A of articulations of G and in which a vertex $a \in A$ is adjacent to a vertex $b \in B$ if and only if the articulation a of G belongs to the block b of G (no two elements of A and no two elements of B are adjacent in $T(G)$).

Evidently $T(G)$ is a tree. Its terminal vertices are in B, therefore its diameter is always even.

A round cactus G for which the diameter of $T(G)$ is 0 is a circuit. We shall prove a theorem.

Theorem 1. *Let C be a circuit. Then $d_t(C) = 2$ if and only if the length of C is divisible by 4; otherwise $d_t(C) = 1$.*

Proof. Let the length of C be k. Denote the vertices of C by $u_1, \ldots, u_k$ so that the edges of C are u_iu_{i+1} for $i = 1, \ldots, k-1$ and u_ku_1. If k is divisible by 4, we can define the partition $\{D_1, D_2\}$ in such a way that $u_i \in D_1$ if and only if $i \equiv 1 \pmod 4$ or $i \equiv 2 \pmod 4$ and $u_i \in D_2$ otherwise. Evidently $\{D_1, D_2\}$ is a total domatic partition of C and $d_t(C) \geqq 2$. The total domatic number cannot exceed the minimum degree of a vertex of the graph [1], therefore $d_t(C) = 2$.

Now suppose that $d_t(C) = 2$. Then there exists a total domatic partition $\{D_1, D_2\}$ of C. Without loss of generality suppose that $u_1 \in D_1$. Then exactly one of the vertices u_2, u_k is in D_1 and exactly one in D_2. Without loss of generality let $u_2 \in D_1$. Then $u_3 \in D_2$, $u_4 \in D_2$ and by induction we may prove that $u_i \in D_1$ if and only if $i \equiv 1 \pmod 4$ or $i \equiv 2 \pmod 4$ and $u_i \in D_2$ otherwise. On the other hand, $u_k \in D_2$, $u_{k-1} \in D_2$ and thus neither k, nor $k-1$ is congruent to 1 or 2 modulo 4. Hence k must be divisible by 4. Otherwise $d_t(C) = 1$. □

Remark. The assertion that the total domatic number of a circuit of a length divisible by 4 is equal to 2 was stated in [1] without proof.

Now we shall prove another theorem concerning the terminal blocks of cacti.

Theorem 2. *Let G be a non-trivial cactus in which at least one terminal block is a circuit of a length congruent to 2 modulo 4. Then $d_t(G) = 1$.*

Remark. A non-trivial cactus [3] is a cactus having more than one block.

Proof. Evidently $d_t(G) \leqq 2$. Suppose that $d_t(G) = 2$ and let $\{D_1, D_2\}$ be a total domatic partition of G. Let C be the mentioned terminal block of G, let k be its length. Let the vertices of C be denoted as in the proof of Theorem 1 and in such a way that a_k is the articulation of G. (A terminal block is a block containing exactly one articulation.) Without loss of generality let $u_1 \in D_1$. If $u_2 \in D_1$, then, similarly as in the proof of Theorem 1, $u_i \in D_1$ if and only if $i \equiv 1 \pmod 4$ or $i \equiv 2 \pmod 4$ and $u_i \in D_2$ otherwise, with a possible exception of u_k. This implies $u_{k-2} \in D_2$, $u_{k-1} \in D_1$. If $u_k \in D_1$, then u_1 is adjacent to no vertex of D_2; if $u_k \in D_2$, then u_{k-1} is adjacent to no vertex of D_1. Now if $u_2 \in D_2$, then $u_i \in D_1$ if and only if $i \equiv 0 \pmod 4$ or $i \equiv 1 \pmod 4$ and $u_i \in D_2$ otherwise, with a possible exception of u_k. This implies $u_{k-2} \in D_1$, $u_{k-1} \in D_1$. If $u_k \in D_1$, then u_{k-1} is adjacent to no vertex of D_2; if $u_k \in D_2$, then u_1 is adjacent to no vertex of D_1. In all the cases we obtain a contradiction with the assumption that $\{D_1, D_2\}$ is a total domatic partition of G. Therefore $d_t(G) = 1$.

Now we prove other theorems concerning terminal blocks of cacti.

Lemma 1. *Let G be a non-trivial cactus, let $d_t(G) = 2$, let $\mathscr{D}$ be a total domatic partition of G. Let C be a terminal block of G of a length $k \equiv 1 \pmod 4$, let a be* *the articulation contained in C. Then a in C is adjacent only to vertices of the class of $\mathscr{D}$ to which it belongs.*

Proof. Let the vertices of C be denoted as in the proof of Theorem 1, let $a = u_k$. Let $\mathscr{D} = \{D_1, D_2\}$ and without loss of generality let $u_k \in D_1$. Suppose that the assertion does not hold; hence at least one of the vertices u_1, u_{k-1} belongs to D_2. Without loss of generality let $u_1 \in D_2$. Thus $u_2 \in D_2$, $u_3 \in D_1$ and, by induction, $u_i \in D_2$ if and only if $i \equiv 1 \pmod 4$ or $i \equiv 2 \pmod 4$ and $u_i \in D_1$ otherwise. Then $u_{k-2} \in D_1$, $u_{k-1} \in D_1$ and u_{k-1} is adjacent to no vertex of D_2, which is a contradiction. □

Lemma 2. *Let G be a non-trivial cactus, let $d_t(G) = 2$, let $\mathscr{D}$ be a total domatic partition of G. Let C be a terminal block of G of a length $k \equiv 3 \pmod 4$, let a be the articulation contained in C. Then a in C is adjacent only to vertices of the class of $\mathscr{D}$ to which it does not belong.*

Proof. Let again the vertices of C be denoted as in the proof of Theorem 1, let $a = u_k$. Let $\mathscr{D} = \{D_1, D_2\}$ and without loss of generality let $u_k \in D_1$. Suppose that the assertion does not hold; hence at least one of the vertices u_1, u_{k-1} belongs to D_1. Without loss of generality let $u_1 \in D_1$. Then $u_2 \in D_2$, $u_3 \in D_2$ and, by induction, $u_i \in D_2$ if and only if $i \equiv 2 \pmod 4$ or $i \equiv 3 \pmod 4$ and $u_i \in D_1$ otherwise. Then $u_{k-2} \in D_1$, $u_{k-1} \in D_2$ and u_{k-1} is adjacent to no vertex of D_2, which is a contradiction. □

Now we prove a theorem.

Theorem 3. *Let G be a round cactus with exactly one articulation. The total domatic number of G is 2 if and only if no block of G is a circuit of a length congruent with 2 modulo 4 and either there exists at least one block of G being a circuit of a length divisible by 4, or there exists at least one block of G being a circuit of a length congruent with 1 modulo 4 and at least one block of G being a circuit of a length congruent with 3 modulo 4.*

Proof. In G all blocks are terminal, therefore by Theorem 2 none of them can be a circuit of a length congruent with 2 modulo 4. Suppose that there exists a block C of G being a circuit of a length divisible by 4. Then there exists a total domatic partition of C with two classes and thus a total domatic colouring of C by the colours 1 and 2. It may be chosen so that the unique articulation a of G has the colour 1. In such a way we may take the total domatic colourings of all blocks of G being circuits of lengths divisible by 4. If there exists a block of G being a circuit of a length congruent to 1 modulo 4, then let its vertices be denoted as in the proof of Theorem 1 and so that $u_1 = a$. Colour all vertices u_i for $i \equiv 1 \pmod 4$ or $i \equiv 2 \pmod 4$ by 1 and all the others by 2. If there exists a block of G being a circuit of a length congruent with 3 modulo 4, we denote its vertices again as in the proof of Theorem 1 and so that $u_1 = a$. We colour all vertices u_i for $i \equiv 0 \pmod 4$ or $i \equiv 1 \pmod 4$ by 1 and all the others by 2. The colouring obtained in this way is a total domatic colouring of G by two colours and thus $d_t(G) = 2$.

If there is no block of G being a circuit of a length divisible by 4, but there exists at least one of a length congruent with 1 modulo 4 and at least one of a length congruent with 3 modulo 4, then we proceed in the same way. The vertex a is adjacent to vertices coloured by 1 (or by 2) in the circuits of the lengths congruent with 1 (or with 3 respectively) modulo 4. Also other vertices are adjacent to vertices of both the colours, therefore we have a total domatic colouring by two colours and $d_t(G) = 2$.

The remaining cases are those when all blocks are circuits of lengths congruent with 1 modulo 4 and when all blocks are circuits of lengths congruent with 3 modulo 4. Let the first case occur and let $\{D_1, D_2\}$ be a total domatic partition of G. Without loss of generality let $a \in D_1$. According to Lemma 1 the vertex a is adjacent only to vertices of D_1 and $\{D_1, D_2\}$ is not a total domatic partition, which is a contradiction. Similarly in the second case according to Lemma 2 the vertex a would be adjacent only to vertices of D_2, which is again a contradiction. □

A graph G is said to be uniquely totally domatic if in G there exists exactly one maximal total domatic partition of G, i.e., a domatic partition of G with $d_t(G)$ classes.

Theorem 4. *Let G be a round cactus with exactly one articulation, let $d_t(G) = 2$. The graph G is uniquely totally domatic if and only if it contains no circuit of a length divisible by* 4.

Proof. If G contains no circuit of a length divisible by 4, then all of its blocks are circuits of lengths congruent with 1 or with 3 modulo 4. Then for each vertex it is uniquely determined, whether it belongs to the same class of a maximal total domatic partition of G as the articulation, or not. This follows from Lemma 1 and Lemma 2 and from the fact that any vertex of degree 2 must be adjacent to exactly one vertex of each class of a maximal total domatic partition. Hence G is uniquely totally domatic.

Now let G contain a circuit C of a length k divisible by 4. Let its vertices be denoted as in the proof of Theorem 1 and so that u_k is the articulation of G. Suppose that there is a total domatic partition $\{D_1, D_2\}$ of G. Now we construct another partition $\{D_1', D_2'\}$ in the following way. If a vertex x does not belong to C, then $x \in D_1'$ if and only if $x \in D_1$; otherwise it is in D_2'. For each $i = 1, \ldots, k - 1$ let the vertex u_i be in D_1' if and only if $u_{k-i} \in D_1$; otherwise let it be in D_2'. If $u_k \in D_1$, then let $u_k \in D_1'$; otherwise let $u_k \in D_2'$. Evidently $\{D_1', D_2'\}$ is also a total domatic partition of G. Exactly one of the vertices u_1, u_{k-1} is in the same class of $\{D_1, D_2\}$ as u_k; this vertex is in the other class of $\{D_1', D_2'\}$, then u_k and thus $\{D_1', D_2'\} \neq \{D_1, D_2\}$ and G is not uniquely totally domatic. □

Before formulating a theorem concerning other cacti than those with one articulation, we shall proceed with some considerations on the tree $T(G)$.

Let G be a round cactus such that the diameter of $T(G)$ is greater than 2. Let P be a diametral path in $T(G)$. The terminal vertices of P are evidently blocks of G. Let a be a vertex of P adjacent to one of the terminal vertices of P; this is an articulation of G. The articulation a has the property that it is contained in exactly one non-terminal block of G (otherwise there would exist a path longer that P in $T(G)$). We introduce some notation. By $G'(a)$ we denote the subgraph of G consisting of all terminal blocks containing a, by $G''(a)$ the graph obtained from G by deleting all vertices of $G'(a)$ except a. In $G''(a)$ the vertex a has evidently the degree 2; let v_1, v_2 be the vertices adjacent to a in $G''(a)$. By $G^-(a)$ we denote the graph obtained from $G''(a)$ by deleting the vertex a and adding the edge $e^-(a)$ joining v_1 and v_2. By $G^+(a)$ we denote the graph obtained from $G''(a)$ by replacing a by two adjacent vertices a_1, a_2 and joining a_1 with v_1 and a_2 with v_2 by an edge. Obviously all the mentioned graphs are cacti.

Consider the following three assertions:

A. The graph $G''(a)$ has the total domatic number equal to 2.

A$^-$. The graph $G^-(a)$ has a total domatic partition with two classes such that the end vertices of $e^-(a)$ belong to the same class.

A$^+$. The graph $G^+(a)$ has a total domatic partition with two classes such that the end vertices of $e^+(a)$ belong to the same class.

The next theorem will enable us to determine the total domatic number of a round cactus recurrently by means of that of a round cactus for which the diameter of the block tree is smaller.

Theorem 5. *Let G be a round cactus such that the diameter of $T(G)$ is at least 4. The cactus G has the total domatic number equal to 2 if and only if $G'(a)$ contains no circuit of a length congruent with 2 modulo 4 and at least one of the following cases occurs:*

(a) *$G'(a)$ contains a circuit of a length divisible by 4 and either* **A**, *or* **A**$^-$, *or* **A**$^+$ *holds.*

(b) *$G'(a)$ contains a circuit of a length congruent with 1 modulo 4 and a circuit of a length congruent with 3 modulo 4 and either* **A**, *or* **A**$^-$, *or* **A**$^+$ *holds.*

(c) *$G'(a)$ consists of circuits of lengths congruent with 1 modulo 4 and either* **A**, *or* **A**$^+$ *holds.*

(d) *$G'(a)$ consists of circuits of lengths congruent with 3 modulo 4 and either* **A**, *or* **A**$^-$ *holds.*

Proof. The assertion concerning the circuit of a length congruent with 2 modulo 4 follows from Theorem 2. Suppose that $d_t(G) = 2$. In the case (a) there exists a total domatic partition $\{D_1, D_2\}$ of G with the property that a is adjacent to vertices of both the classes in a circuit of $G'(a)$ whose length is divisible by 4. If in $G''(a)$ it is also adjacent to vertices of both classes, then evidently the restriction of $\{D_1, D_2\}$ onto $G''(a)$ is a total domatic partition of $G''(a)$ and $\boldsymbol{A}$ holds. If in $G''(a)$ the vertex a is adjacent only to vertices of its own class, then we may take the restriction of $\{D_1, D_2\}$ onto the vertex set of $G^-(a)$; this is evidently a total domatic partition of $G^-(a)$ in which the end vertices of $e^-(a)$ belong to the same class and $\boldsymbol{A}^-$ holds. If in $G''(a)$ the vertex a is adjacent only to vertices not belonging to its own class, then we may take the partition of the vertex set of $G^+(a)$ obtained from the restriction of $\{D_1, D_2\}$ onto $G''(a)$ by replacing a by a_1, a_2 and putting a_1, a_2 into the class in which a was; this is evidently a total domatic partition of $G^+(a)$ in which the end vertices of $e^+(a)$ belong to the same class and $\boldsymbol{A}^+$ holds. In the case (b) there also exists a total domatic partition $\{D_1, D_2\}$ of G with the property that a is adjacent to a vertex of its own class in a circuit of a length congruent with 1 modulo 4 and to a vertex of the other class in a circuit of a length congruent with 3 modulo 4 in $G'(a)$. The rest of the proof is now the same as in the case (a). In the case (c) the vertex a is adjacent in $G'(a)$ only to vertices of its own class. Then in $G''(a)$ the vertex a must be adjacent either to vertices of both the classes, or only of the class other than its own. Analogously as above we prove that $\boldsymbol{A}$ or $\boldsymbol{A}^+$ holds. In the case (d) the vertex a is adjacent in $G'(a)$ only to vertices of the class other than its own. Then in $G''(a)$ the vertex a must be adjacent either to vertices of both the classes, or only of its own class. Analogously as above we prove that $\boldsymbol{A}$ or $\boldsymbol{A}^-$ holds.

Now suppose that the conditions are satisfied. Let (a) occur. Then we take a total domatic colouring of $G'(a)$ described in the proof of Theorem 3. If $\boldsymbol{A}$ holds, we take a domatic colouring of $G''(a)$ by two colours such that a has the same colour in both the colourings. Both the colourings together give a total domatic colouring of G by two colours. If $\boldsymbol{A}^-$ holds, we take a total domatic polouring of G^- by two colours such that the end vertices of $e^-(a)$ have the same colour and this colour is the same as that of a in the colouring of $G'(a)$. Then we colour $G''(a)$ so as $G^-(a)$; we obtain a total domatic colouring of G by two colours. If $\boldsymbol{A}^+$ holds, we take a total domatic colouring of $G^+(a)$ by two colours such that the end vertices of $e^+(a)$ have the same colour and this colour is the same as that of a in the colouring of $G'(a)$. We colour $G''(a)$ so as $G^+(a)$; to a we assign the colour of a_1 and a_2. We obtain again a total domatic colouring of G by two colours. In the case (b) we proceed quite analogously. In the case (c) we colour $G'(a)$ in such a way that a is adjacent only to vertices of its own class and all other vertices are adjacent to vertices of both the classes (as in the proof of Theorem 4). Then in the cases $\boldsymbol{A}$ and $\boldsymbol{A}^+$ we proceed analogously as above. In the case (d) we colour $G'(a)$ in such a way that a is adjacent only to vertices not belonging to its own class and all other vertices are adjacent to vertices of both the classes. Then in the case $\boldsymbol{A}$ and $\boldsymbol{A}^-$ we proceed analogously as above. □

Remark. If the block of $G^-(a)$ containing $e^-(a)$ has the length divisible by 4 and $d_t(G^-(a)) = 2$, then always there exists the required colouring. If its length is congruent with 1 or 3 modulo 4, it is not always so (see Lemma 1, Lemma 2, Theorem 4). Analogously for $G^+(a)$ and $e^+(a)$.

From the cacti which are not round we shall study only the simplest, namely those consisting of two circuits and a path connecting a vertex of one of them with a vertex of the other.

Theorem 6. *Let G be a cactus consisting of two circuits C_1, C_2 of the lengths c_1, c_2 respectively and a path P of the length p connecting a vertex of C_1 with a vertex of C_2. Then $d_t(G) = 2$ if and only if one of the following cases occurs*:

(a) $c_1 \equiv 0 \pmod 4$.

(a′) $c_2 \equiv 0 \pmod 4$.

(b) $c_1 \equiv c_2 \equiv 1 \pmod 4$, $p \equiv 1 \pmod 2$.

(c) $c_1 \equiv c_2 \equiv 3 \pmod 4$, $p \equiv 1 \pmod 2$.

(d) $c_1 \equiv 1 \pmod 4$, $c_2 \equiv 3 \pmod 4$, $p \equiv 0 \pmod 2$.

(d′) $c_1 \equiv 3 \pmod 4$, $c_2 \equiv 1 \pmod 4$, $p \equiv 0 \pmod 2$.

Proof. From Theorem 2 we have $c_1 \not\equiv 2 \pmod 4$, $c_2 \not\equiv 2 \pmod 4$. Let the vertices of P be $v_0, \ldots, v_p$, let its edges be $v_i v_{i+1}$ for $i = 0, \ldots, p-1$. Let v_0 belong to C_1 and v_p to C_2. Let (a) occur. If $c_2 \equiv 1 \pmod 4$, we colour C_2 by the colours 1 and 2 so that v_p has the colour 1 and so have the vertices adjacent to it and any vertex distinct from v_p is adjacent to vertices of both the colours (see the proof of Theorem 3). The vertices of P will be coloured so that v_i has the colour 1 if and only if $i \equiv p \pmod 4$ or $i \equiv p + 1 \pmod 4$; otherwise it has the colour 2. Then we take a total domatic colouring of C_1 by the colours 1 and 2 in which v_0 has the same colour as in the colouring of P. We obtain a total domatic colouring of G by two colours. If $c_2 \equiv 3 \pmod 4$, then we colour C_2 in such a way that v_p has the colour 1 and the vertices adjacent to it have the colour 2 and any vertex distinct from v_p is adjacent to vertices of both the colours. The vertices of P will be coloured so that v_i has the colour 1 if and only if $i \equiv p \pmod 4$ or $i \equiv p - 1 \pmod 4$; otherwise it has the colour 2. The vertices of C_1 will be coloured as in the preceding case. If $c_2 \equiv 0 \pmod 4$, we take total domatic colourings of both C_1, C_2 by two colours and we colour the vertices of P in an arbitrary one of the described ways. The case (a′) is analogous. In the case (b) we take the colourings of C_1 and C_2 such as for C_2 in the case (a) for $c_2 \equiv 1 \pmod 4$, and such that v_0 is coloured by 1. Then we must colour the vertices of P so that v_1 has another colour than v_0 and v_{p-1} has another colour than v_p. If p is odd, this is possible in such a way that v_i has the colour 1 if and only if $i \equiv 0 \pmod 4$ or $i \equiv 3 \pmod 4$ and it has the colour 2 otherwise. If p is even, this is evidently not possible. In the case (c) we take the colourings of C_1

and C_2 such as for C_2 in the case (a) for $c_2 \equiv 3 \pmod 4$ and so that v_0 is coloured by 1. Then we must colour the vertices of P so that v_1 has the same colour as v_0 and v_{p-1} has the same colour as v_p. If p is odd, this is possible in such a way that v_i has the colour 1 if and only if $i \equiv 0 \pmod 4$ or $i \equiv 1 \pmod 4$ and it has the colour 2 otherwise. If p is even, this is evidently not possible. In the case (d) we proceed analogously. The vertices of P must be coloured so that v_1 has another colour than v_0 and v_{p-1} has the same colour as v_p. If p is even, this is possible in such a way that v_i has the colour 1 if and only if $i \equiv 0 \pmod 4$ or $i \equiv 3 \pmod 4$ and the colour 2 otherwise. If p is odd, this is evidently not possible. The case (d′) is analogous. Thus we have exhausted all cases and the assertion is proved. □

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ТОТАЛЬНОЕ ДОМАТИЧЕСКОЕ ЧИСЛО КАКТУСОВ

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Резюме

Подмножество D множества $V(G)$ вершин неориентированного графа G называется тотальным доминантным, если для каждой вершины $x \in V(G)$ существует вершина $y \in D$, смежная с x в G. Максимальное число классов разбиения множества $V(G)$, все классы которого являются тотальными доминантными множествами в G, называется тотальным доматическим числом графа G. Здесь оно исследуется для кактусов, то есть графов, в которых каждое ребро принадлежит по крайней мере одному контуру.