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A NOTE ON THE WEAK CONVERGENCE OF PROBABILITY MEASURES ON $C(K, L)$

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1. Introduction

The aim of this note is to establish a sufficient condition for weak convergence of probability measures on $C(K, L)$, where K is an arbitrary compact metric space. The proof of this result is performed by means of piecewise linear functions and Wichura's theorem, similarly, as it has been done in [4] in the case of $C(\langle 0, 1\rangle)$.

2. Polygonal function

Let $Q=\prod_{j=1}^{n}\langle a_j, b_j\rangle$ be an n-dimensional cube. Let us denote by $\mathcal{D}_Q$ the system of all $(n+1)$-tuples $(P_1, \ldots, P_{n+1})$ which satisfy the following conditions.

(i) There are a variation $j_1, \ldots, j_{n-2}$ of $1, \ldots, n$ and numbers $c_i \in \{a_{j_i}, b_{j_i}\}$, $i=1, \ldots, n-2$ such that for $k=1, \ldots, n-2$ the point P_k is the barycenter of the face

$$F_{j_1 \ldots j_k}(c_1, \ldots, c_k)=\{x \in Q\,;\, x_{j_i}=c_i,\ i=1, \ldots, k\},$$

i. e. $P_k=(x_1, \ldots, x_n)$, where $x_{j_1}=c_1, \ldots, x_{j_k}=c_k$ and $x_j=(a_j+b_j)/2$ for $j \notin \{j_1, \ldots, j_{n-2}\}$.

(ii) P_{n-1}, P are vertices of the cube Q and belong to the face$F_{j_1 \ldots j_{n-2}}(c_1, \ldots, c_{n-2})$.

(iii) P_{n+1} is the barycenter of the cube Q, i. e., $P_{n+1}=((a_1+b_1)/2, \ldots, (a_n+b_n)/2$.

(iv) The points $P_1, \ldots, P_{n+1}$ are linearly independent.

Let us recall that points $P_1, \ldots, P_s$ are said to be linearly independent if the vectors P_j-P_i, $1 \leqq j \leqq s$, $j \neq i$ have this property. We remark that for $n=2$ the system $\mathcal{D}_Q$ is the division of the square Q into triangles which are determined by some side of the square and by its barycenter.

A set $Q \subset \langle 0, 1\rangle^n$ will be called an n-cube (n is a positive integer) if

$$Q=\prod_{j=1}^{n}\langle k_j n^{-1}, (k_j+1)n^{-1}\rangle$$

and k_j are non-negative integers. If we denote by $\mathscr{P}_Q$ the union of all sets $\{P_1, \ldots, P_{n+1}\}$, where $(P_1, \ldots, P_{n+1}) \in \mathscr{D}_Q$, and put

$$(2.1) \qquad \mathscr{P}_n = \bigcup_{Q \in J(n)} \mathscr{P}_Q, \quad J = \bigcup_{Q \in J(n)} Q,$$

where $J(n)$ is some non-empty system of n-cubes, then the following assertion holds.

Proposition. *Let L be a normed linear space and a: $\mathscr{P}_n \to L$. If we put for t belonging to J*

$$(2.2) \qquad Y_n(t, a) = \sum_{j=1}^{n+1} \alpha_j a(P_j)$$

whenever $t = \sum_{j-1}^{n+1} \alpha_j P_j$, $\alpha_j \geqq 0$, $\sum_{j=1}^{n+1} \alpha_{j=1}$ *and* $(P_1, \ldots, P_{n+1}) \in \mathscr{D}_Q$, $Q \in J(n)$, *then*

(i) *The function $Y_n(., a)$ is well defined and continuous on J.*
(ii) *If we put for f: $J \to L$*

$$|||f||| = \sup \{\|f(x)\| ; x \in J\},$$

where $\|.\|$ is the norm on L, then

$$(2.3) \qquad |||Y_n(., a) - Y_n(., b)||| = \max \{\|a(P) - (P)\| ; P \in \mathscr{P}_n\}.$$

To prove the proposition we shall need sone lemmas and the following notations. If $y = (y_1, \ldots, y_s)$ belongs to R^s, then we denote

$$(c, y) = (c, y_1, \ldots, y_s), \quad y^{[1]} = (y_2, \ldots, y_s), \quad (y, c) = (y_1, \ldots, y_s, c)$$

and put

$$Q^{[1]} = \{y^{[1]} ; y \in Q\}$$

for $Q \subset R^s$. As usual, the symbols ∂Q, Q^0, co(Q) denote the boundary, the interior and the convex hull of the set Q, respectively.

Lemma 1. *If $(P_1, \ldots, P_{n+1})$, $(P_1^*, \ldots, P_{n+1}^*)$ belong to $\mathscr{D}_Q$, then*

$$(2.4) \quad \mathrm{co}(P_1, \ldots, P_{n+1}) \cap \mathrm{co}(P_1^*, \ldots, P_{n+1}^*) = \mathrm{co}(\{P_1, \ldots, P_{n+1}\} \cap \{P_1^*, \ldots, P_{n+1}^*\}).$$

Proof. First we prove that

$$(2.5) \quad \mathrm{co}(P_1, \ldots, P_{n+1}) \cap \mathrm{co}(P_1^*, \ldots, P_{n+1}^*) = \mathrm{co}(\mathrm{co}(P_1, \ldots, P_n) \cap \mathrm{co}(P_1^*, \ldots, P_n^*) \cup \{P_{n+1}\}).$$

If y belonging to the left-hand side of (2.5) is an inner point of Q, then according to Lemma 2.2.1 in [2] there is a unique $\beta_0 > 0$ such that the point $G = P_{n+1} + \beta_0(y - P_{n+1})$ belongs to ∂Q. This means that G is an element of the right-hand side of (2.5), hence y has this property, and (2.5) is proved. Now we prove that

(2.6) $$\mathrm{co}(P_1, \ldots, P_n) \cap \mathrm{co}(P_1^*, \ldots, P_n^*) = \mathrm{co}(\{P_1, \ldots, P_n\} \cap \{P_1^*, \ldots, P_n^*\}).$$

Obviously, this equality holds for $n=2, 3$. Let (2, 6) be valid for $n-1 \geqq 3$. If $P_1 = P_1^*$, then assuming $j_1 = 1$, denoting

$$\{W_1, \ldots, W_r\} = \{P_1^{[1]}, \ldots, P_n^{[1]}\} \cap \{P_1^{*[1]}, \ldots, P_n^{*[1]}\}$$

and making use of both (2.5) and the induction assumption we see that

(2.7) $$\mathrm{co}(P_1^{[1]}, \ldots, P_n^{[1]}) \cap \mathrm{co}(P_1^{*[1]}, \ldots, P_n^{*[1]}) = \mathrm{co}(W_1, \ldots, W_r).$$

Obviously, $P_1 = P_1^*$ together with (2.7) implies (2.6). Further, if the number $k = \min\{r;\, P_r = P_r^*\}$ is greater than 1, then

$$\mathrm{co}(P_1, \ldots, P_n) \cap \mathrm{co}(P_1^*, \ldots, P_n^*) = \mathrm{co}(P_k, \ldots, P_n) \cap \mathrm{co}(P_k^*, \ldots, P_n^*) =$$
$$= \mathrm{co}(\{P_k, \ldots, P_n\} \cap \{P_k^*, \ldots, P_n^*\}).$$

which implies (2.6). Finally, combining (2.5) and (2.6) we obtain (2.4).

Lemma 2. *Let Q be a cube and $y \in Q$.*

(i) *There is an $(n+1)$-tuple $(P_1, \ldots, P_{n+1}) \in \mathscr{D}_Q$ such that $y \in \mathrm{co}(P_1, \ldots, P_{n+1})$.*

(ii) *If $(P_1, \ldots, P_{n+1})$, $(P_1^*, \ldots, P_{n+1}^*)$ belong to $\mathscr{D}_Q$ and*

$$y = \sum_{j=1}^{n+1} \alpha_j P_j = \sum_{i=1}^{n+1} \beta_i P_i^*$$

is a convex combination of $\{P_i^*\}$ and $\{P_j\}$, then α_j is positive if and only if there is an index i such that $P_j = P_i^*$, $\alpha_j \beta_i$, $\beta_i > 0$.

(iii) *If Q, Q^* are n-cubes and $y \in Q \cap Q^*$, then there exist $(P_1, \ldots, P_{n+1}) \in \mathscr{D}_Q$, $(P_1^*, \ldots, P_{n+1}^*) \in \mathscr{D}_{Q^*}$ such that*

(2.8) $$y = \sum_{j=1}^{r} \alpha_j W_j, \quad \alpha_j \geqq 0, \quad \sum_{j=1}^{r} \alpha_{j=1}$$
$$\{W_1, \ldots, W_r\} = \{P_1, \ldots, P_{n+1}\} \cap \{P_1^*, \ldots, P_{n+1}^*\}.$$

Proof. Let $Q = \prod_{j=1}^{n} \langle a_j, b_j \rangle$.

(i) Let the assertion hold for $n-1 \geqslant 2$. If $y \in \partial Q$, we may assume that $y_1 = a_1$. Choosing points $(\bar{P}_2, \ldots, \bar{P}_{n+1}) \in \mathscr{D}_Q[1]$ such that $y^{[1]} \in \mathrm{co}(P_2, \ldots, P_{n+1})$ and putting

$$P_1 = (a_1, \bar{P}_{n+1}), \;\; P_j = (a_1, \bar{P}_j) j = 2, \ldots, n, \;\; P_{n+1} = ((a_1 + b_1)/2, \bar{P}_{n+1})$$

we see that (i) holds. Further, if $y \in Q^0$, then according to Lemma 2.2.1 in [2] the halfline

$$\{T+\alpha(y-T);\ \alpha\geqq 0,\ T=((a_1+b_1)/2, \ldots, (a_n+b_n)/2)\}$$

intersects the boundary of Q for a unique $\alpha_0>1$. Denoting $G=T+\alpha_0(y-T)$ we can find $(P_1, \ldots, P_{n+1}) \in \mathscr{D}_Q$ such that $G\in \mathrm{co}(P_1, \ldots, P_{n+1})$ and therefore y belongs to this set.

(ii) The proof follows from Lemma 1 and from the fact that coefficients in any convex combination of linearly independent points are uniquely determined.

(iii) Let the assertion hold for $n-1\geqq 2$, $Q^*=\prod_{j=1}^{n}\langle a_j^*, b_j^*\rangle$ and $\langle a_{j_i}^*, b_{j_i}^*\rangle = \langle a_{j_i}, b_{j_i}\rangle$ $i=1, \ldots, r$ for some $1\leqq r\leqq n-1$. To avoid complications with notations, we assume that $\{j_1, \ldots, j_r\} \subset \{2, \ldots, n\}$. According to the assumptions there are $(\bar{P}_2, \ldots, \bar{P}_{n+1}) \in \mathscr{D}Q[1]$, $(\bar{P}_2^*, \ldots, \bar{P}_{n+1}^*) \in \mathscr{D}_{Q^*}[1]$ such that the relation (2.8) holds for $y^{[1]}$. Let us denote

$$P_1=(y_1, \bar{P}_{n+1}),\ P_j=(y_1, \bar{P}_j)\ j=2, \ldots, n,\ P_{n+1}=((a_1+b_1)/2, \bar{P}_{n+1}),$$

$$P_1^*=(y_1, \bar{P}_{n+1}^*),\ P_j^*=(y_1, \bar{P}_j^*)\ j=2, \ldots, n,\ P_{n+1}^*=((a_1^*+b_1^*)/2, \bar{P}_{n+1}^*).$$

Since any convex conbination $\sum_{i-1}^{n+1}\alpha_i P_i$ with $\alpha_{n+1}>0$ belongs to Q^0, the lemma is proved.

Proof of Proposition. If $Q\in j(n)$ and $t\in Q$, then according to Lemma 2 the mapping Y is well defined. Further, if we denote for $t\in \mathrm{co}(P_1, \ldots, P_{n+1})$

$$s(t)=(\alpha_1, \ldots, \alpha_{n+1})$$

whenever $t=\sum_{j=1}^{n+1}\alpha_j P_j$, $\sum_{j-1}^{n+1}\alpha_j=1$, then s is a continuouns mapping, which implies continuity of Y_n. The last assertion follows from the inequality

$$\|Y_n(t, a)-Y_n(t, b)\|\leqslant \sum_{j\ 1}^{n+1}\alpha_j\|a(P_j)-b(P_j)\|,$$

where the equality sign can be written for $t\in \mathscr{P}_n$.

3. Weak convergence of probability measures

Let K be a compact metric space and L be a normed linear space. We shall denote by $C(K, L)$ the linear space of all continuous L-valued function on K with the norm $|||f|||=\max\{\|f(k)\|;\ k\in K\}$, $\mathscr{S}$ the σ-algebra generated by closed subsets of $C(K, L)$ and v_δ: $C(K, L)\to\langle 0, \infty)$, the modulus of continuity defined by the formula

$$v_\delta(f)=\{\sup\|f(t)-f(s)\|;\ \mathscr{S}(t, s)\leqslant\delta,\ s,\ t\in K\},$$

where $\mathscr{S}$ is the metric on K.

Let X, X_n $(n \geqq 1)$ be $C(K, L)$-valued $\mathscr{S}$-measurable random variables. We shall establish a sufficient conditon for

$$\mathscr{L}(X_n) \to \mathscr{L}(X), \tag{3.1}$$

where $\mathscr{L}(X)$ is the probability distribution induced by the mapping X, and the convergence in (3.1) is the usual weak convergence of probability measures on metric spaces (cf. [1]). We remark, that if $k \in K$, then $X_n(k)(\omega) = (X_n(\omega))(k)$, hence $X_n(k)$ is an L-valued random variable. Similarly $(X_n(k_1), \dots, X_n(k_r))$ is an L^r-valued random variable for any $k_1, \dots, k_r$ belonging to K.

Theorem. *Let*

$$\mathscr{L}(X_n(k_1), \dots, X_n(k_r)) \to \mathscr{L}(X(k_1), \dots, X(k_r)) \tag{3.2}$$

for every finite subset $\{k_1, \dots, k_r\}$ *of some dense subset* U *of* K. *If*

$$\lim_{\delta \to 0} \limsup_{n \geqq 1} P[v_\delta(X_n) \geqq \varepsilon] = 0 \tag{3.3}$$

for each ε *positive, then*

$$\mathscr{L}(X_n) \to \mathscr{L}(X). \tag{3.4}$$

Proof. According to Proposition 1 in [4] it is sufficient to construct mappings T_n: $C(K, L) \to C(K, L)$ such that the conditions

$$\mathscr{L}(T_q(X_n)) \to \mathscr{L}(T_q(X)), \tag{3.5}$$

$$\lim_{m \to \infty} \limsup_{n \geqq 1} P[|||X_n - T_m(X_n)||| \geqq \varepsilon] = 0, \tag{3.6}$$

$$\lim_{m \to \infty} P[|||X - T_m(X)||| \geqq \varepsilon] = 0 \tag{3.7}$$

are fulfilled for any positive integer q and any positive number ε.

Let N be the set of all positive integers and H be the cube $\langle 0, 1 \rangle^N$ with the metric $\mathscr{T}(x, y) = \sum_{j=1}^{\infty} |x_j - y_j| 2^{-j}$. Since K is a separable metric space, there is a continuous mapping e: $K \to H$ which is a homeomorphism from K on $e(K)$ (cf. [3], §22).

Let n be a positive integer and Q be an n-cube. Let us put

$$\tilde{Q} = \{x \in H; (x_1, \dots, x_n) \in Q\}$$

and denote by $J(n)$ the system of all n-cubes satisfying the relation

$$\tilde{Q} \cap e(K) \neq \emptyset.$$

Further, let $\pi_j(y)$ be the j-th member of the sequence (or vector) y. For every point $P \in \langle 0, 1 \rangle^n$ we denote by P^∞ the point from H defined by the formula

$$\pi_j(P^\infty) = \left\langle \begin{array}{ll} \pi_j(P) & j=1, \ldots, n \\ 0 & j>n, \end{array} \right.$$

and choose $\bar{P} \in e(U)$ such that

$$(3.8) \qquad \tau(P^\infty, \bar{P}) < n^{-1} + \inf\{\tau(P^\infty, e(k)); k \in U\}.$$

Now we are able to define the mentioned mapping T_n. Let $g \in C(K, L)$. If x belongs to

$$\tilde{J} = \bigcup_{Q \in J(n)} \tilde{Q},$$

then denoting $\pi^{[n]}(x) = (\pi_1(x), \ldots, \pi_n(x))$ we obtain from Lemma 2 that

$$(3.9) \qquad \pi^{[n]}(x) = \sum_{j-1}^{n+1} \alpha_j P_j$$

for some $(P_1, \ldots, P_{n+1}) \in \mathscr{D}_Q$, $Q \in K(n)$ and the combination (3.9) is convex. Taking into account the proposition on piecewise linear functions we see that the function

$$\tilde{g}(x) = \sum_{j=1}^{n+1} \alpha_j g(e^{-1}(\bar{P}_j))$$

is well defined and continuous on $\tilde{J}$, hence the function

$$T_n(g)(k) = \tilde{g}(e(k))$$

belongs to $C(K, L)$. Making use of (2.3) we see that T_n is a continuous linear operator, which implies its $\mathscr{S}$-measurability. Now if $k \in K$ and $\pi^{[n]}(e(k)) \in \mathrm{co}(P_1, \ldots, P_{n+1})$, $(P_1, \ldots, P_{n+1}) \in \mathscr{D}_Q$, $Q \in J(n)$, then the inequality (3.8) implies $\tau(\bar{P}_j, e(k)) < 5/n$, hence

$$(3.10) \qquad |||T_n(g) - g||| \leqq \sup\{\|g(e^{-1}(x)) - g(e^{-1}(y))\|; \tau(x, y) \leqq 5/n\}.$$

Since the set $e(K)$ is compact and e^{-1} is a continuous mapping, taking into account both (3.10) and (3.3) we obtain (3.6) and (3.7).

Finally, let $V_1, \ldots, V_r$ be an ordering of the set $\mathscr{P}_q$ (cf. (2.1)). If we definne a mapping F: $L^r \to C(K, L)$ by the formula

$$F(1_1, \ldots, 1_r) = Y_q(\pi^{[q]}(e(.)) \cdot b), \quad b(V_i) = 1_i,$$

then

$$(3.11) \qquad T_q(X) = F(X(e^{\ 1}(\bar{V}_1)), \ldots, X(e^{-1}(\bar{V}_r))).$$

Since (2.3) implies continuity of F, both (3.11) and (3.2) imply (3.5), which completes the proof.

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ЗАМЕТКА О СЛАБОЙ СХОДИМОСТИ ВЕРОЯТНОСТНЫХ МЕР НА $C(K, L)$

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Резюме

Пусть $\mathscr{L}_n \to \mathscr{L}$ обозначает, что вероятностные меры $\{\mathscr{L}_n\}$ слабо сходятся к вероятности $\mathscr{L}$. Пусть $C(K, L)$ — нормированное линейное пространство непрерывных отображений метрического компакта K в нормированное линейное пространство L. Если случайные величины $\{X_n\}$, X, принимающие значения в $C(K, L)$, такие, что распределения $\{\mathscr{L}(X_n)\}$, $\mathscr{L}(X)$ удовлетворяют условиям (3.2) и (3.3), то $\mathscr{L}(X_n) \to \mathscr{L}(X)$.