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Ferdinand Gliviak

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ON THE EXISTENCE OF CRITICALLY n-CONNECTED GRAPHS

FERDINAND GLIVIAK

This paper deals with undirected, directed and mixed graphs, too. All graphs will be finite, without loops and multiple edges. The vertex-connectivity and the edge-connectivity of directed or mixed graphs will be used in the sense of the strong connectivity.

Let G be a graph. Then we denote by $V(G)$ the vertex set of G, by $E(G)$ the edge set of G, by $\varkappa(G)$ the vertex-connectivity of G, by $\lambda(G)$ the edge-connectivity of G and by $|A|$ the cardinality of a set A. Let $u \in V(G)$. If G is undirected, then $N_G(u)$ denotes the set of vertices adjacent to u in G. If G is directed, then $O_G(u)$ denotes the set of vertices adjacent to u by an edge going from u and $I_G(u)$ denotes the set of vertices adjacent to u by an edge going to u. The definitions of the notions not presented here can be found in [8].

A graph G is called $\varkappa$-edge-critical, if $\varkappa(G - x) < \varkappa(G)$ for every edge x of G $\varkappa$-vertex-critical, if $\varkappa(G - v) < \varkappa(G)$ for every vertex v of G. Analogously one can define λ-edge-critical and λ-vertex-critical graphs.

One can see that every regular undirected graph of degree $n \geq 2$ and vertex-connectivity n is $\varkappa$-edge, $\varkappa$-vertex, λ-edge and λ-vertex critical. Analogously it can be verified that every directed regular graph of indegree and outdegree $n \geq 2$, vertex-connectivity and edge-connectivity n is $\varkappa$-edge, $\varkappa$-vertex, λ-edge and λ-vertex critical. These four classes of critical undirected or directed graphs were studied in many papers, e. g. [1], [2], [4—7], [9—15]. We shall prove the following theorem on the existence of critical graphs.

Theorem 1. *Let $n \geq p \geq 1$ be given integers. To every undirected (directed) graph G with p vertices there exists an undirected (directed) graph of edge-(strong) connectivity and vertex-(strong) connectivity n that is $\varkappa$-edge, $\varkappa$-vertex, λ-edge and λ-vertex critical and contains G as an induced subgraph.*

The proof of Theorem 1 follows immediately from the following two lemmas.

Lemma 1. *To every undirected graph G with p vertices there exists an undirected, regular graph of degree n, vertex-connectivity and edge-connectivity n containing G as an induced subgraph, where $n \geq p \geq 1$ are given integers.*

Lemma 2. *To every directed graph G with p vertices there exists a directed graph of indegree and outdegree n, vertex-strong-connectivity and edge-strong connectivity n containing G as an induced subgraph, where $n \geq p \geq 1$ are given integers.*

Proof of Lemma 1. Let G be an undirected graph with $p \geqq 1$ vertices and let $n \geq p$. Let G_1 be the graph that arises from G by adding $n - |V(G)|$ isolated vertices. Thus $|V(G_1)| = n$. Let G_1' be a copy of the graph G_1 and let u' be the vertex corresponding to a vertex u of G_1. Let Q be a graph with the vertex set $V(Q) = V(G_1) \cup V(G_1')$ and the edge set $E(Q)$ consisting of the sets $E(G_1)$, $E(G_1')$ and moreover every vertex u of G_1 is joined to any vertex $x \in V(G_1') - N_{G'_1}(u')$ by an unoriented edge (u, x).

From the described construction it follows that Q is a regular graph of degree n containing G as an induced subgraph. Now we prove that $\varkappa(Q) = n$ by finding n paths not having inner vertices in common that join any two vertices of Q (see [8], p. 48).

Let a, b be different vertices of G_1 and let $a', b' \in V(G_1')$ be their copies. Let us put $M_0 = N_{G_1}(a) \cap N_{G_1}(b)$, $M_1 = N_{G_1}(a) - (M_0 \cup \{b\})$, $M_2 = N_{G_1}(b) - (M_0 \cup \{a\})$, $M_3 = V(G_1) - (M_0 \cup M_1 \cup M_2 \cup \{a, b\})$. The vertices a and b are joined by the following n paths not having inner vertices in common: (a, x, b) for every $x \in M_0$; (a, x, x', b) for every $x \in M_1$; (b, x, x', a) for every $x \in M_2$; (a, x', b) for every $x \in M_3$ and finally either the paths (a, b), (a, a', b', b) if $(a, b) \in E(Q)$ or the paths (a, a', b), (a, b', b) if $(a, b) \notin E(Q)$.

The vertices a and b' are joined by the following n paths not having inner vertices in common:
(a, x, x', b') for every $x \in M_0$; (a, x, b') for every $x \in M_1$; (a, x', b') for every $x \in M_2$; (a, x', x, b') for every $x \in M_3$ and finally if $(a, b) \in E(Q)$, then the paths (a, b, b'), (a, a', b') and if $(a, b) \notin E(Q)$, then the paths (a, b'), (a, a', b, b'). One can find n paths not having inner vertices in common that join the vertices a' and b' or the vertices a and a'. Thus we have $\varkappa(Q) = n$, hence the equality $\lambda(Q) = n$ follows by the well-known inequalities $\varkappa(Q) \leq \lambda(Q) \leq n$, where n is the minimum degree of Q, (see [8]). Lemma 1 follows.

Proof of Lemma 2. Let $n \geq p \geq 1$ be given integers. Let G be a graph with p vertices. Let G_1 be the graph arisen from G by adding $n - |V(G)|$ isolated vertices to G. Let G_1' be a copy of G_1 and let u' be the vertex corresponding to the vertex u of G_1. Let Q be a graph with the vertex set $V(Q) = V(G_1) \cup V(G_1')$ and the edge set $E(Q) = E(G_1) \cup E(G_1') \cup A \cup B$, where A is the set of directed edges outgoing from any vertex u of G_1 to a vertex $x \in V(G_1') - O_{G'_1}(u')$ and B is the set of directed edges outgoing from any vertex v' of G_1' to a vertex $x \in V(G_1) - O_{G_1}(v)$.

Directly from the construction it follows that Q is a regular directed graph of indegree and outdegree n containing G as an induced subgraph. We shall prove that $\varkappa(Q) = n$ by finding n oriented paths not having inner vertices in common that join any ordered pair of vertices of Q.

Let a, b be different vertices of G_1 and let $a', b' \in V(G_1')$ be their copies. Let us put $M_0 = O_{G_1}(a) \cap I_{G_1}(b)$, $M_1 = O_{G_1}(a) - (M_0 \cup \{b\})$, $M_2 = I_{G_1}(b) - (M_0 \cup \{a\})$, $M_3 = V(G_1) - (M_0 \cup M_1 \cup M_2 \cup \{a, b\})$. The following n directed paths not having inner vertices in common join the vertex a with the vertex b in Q:
(a, x, b) for every $x \in M_0$; (a, x, x', b) for every $x \in M_1$; (a, x', x, b) for every $x \in M_2$; (a, x', b) for every $x \in M_3$ and finally either the paths (a, b), (a, a', b', b) if $\overrightarrow{(a, b)} \in E(Q)$ or the paths (a, b', b) and (a, a', b) if $\overrightarrow{(a, b)} \notin E(Q)$.

The vertices a and b' are joined by the following n directed paths not having inner vertices in common: (a, x, x', b') for every $x \in M_0$; (a, x, b') for every $x \in M_1$; (a, x', b') for every $x \in M_2$; (a, x', x, b') for every $x \in M_3$ and finally if $\overrightarrow{(a, b)} \in E(Q)$, then the paths (a, b, b'), (a, a', b') and if $\overrightarrow{(a, b)} \notin E(Q)$, then the paths (a, b'), (a, a', b, b'). One can find n directed paths not having inner vertices in common that join the pair of vertices $[a, a']$ or $[a', a]$ or $[a', b']$ or $[b', a]$ analogously as in the previous cases. Thus we have $\varkappa(Q) = n$. It follows that $\lambda(Q) = n$ by the inequalities $\varkappa(Q) \leq \lambda(Q) \leq \min(n_1, n_2)$, where n_1 and n_2 are the minimum indegree and the minimum outdegree of Q, respectively. The inequalities mentioned above can be proved for strong-connectivity similarly as the same inequalities for undirected graphs (see [8], p. 43). This completes the proof.

Corollary 1. *Let $n \geq p \geq 1$ be given integers. To every mixed graph G with p vertices there exists a mixed graph of edge-strong-connectivity and vertex-strong-connectivity n that is $\varkappa$-edge, $\varkappa$-vertex, λ-edge and λ-vertex critical and contains G as an induced subgraph.*

Proof. Let G be a mixed graph having p vertices. Let $n \geq p$. Let G^* be the graph arisen from G by replacing every its undirected edge by the pair of directed edges with opposite orientation. Let Q^* be the directed graph constructed to G^* and the integer n by Lemma 2. If we replace every pair of opposite oriented edges of Q^* by an undirected edge, then we get a graph Q with the desired properties, which can be verified analogously as in Lemma 2.

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ul. T. Vansovej č. 36
917 00 Trnava