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Jozef Širáň

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A NEW LOWER BOUND FOR THE INDUCIBILITY OF A GRAPH

JOZEF ŠIRÁŇ

1. Introduction

Let G be a graph of order p. For an arbitrary graph H let (G, H) denote the number of copies of the graph G in H, i.e., the number of all induced subgraphs of H which are isomorphic to G. Denote by (G, n) the maximum number (G, H) taken over all graphs H of order n. Pippenger and Golumbic [3] have shown the sequence

$$\left\{\binom{n}{p}^{-1}(G, n)\right\}_{n=p}^{\infty}$$

to be nonincreasing; thus there exists the limit

$$\lim_{n\to\infty}\binom{n}{p}^{-1}(G, n)=I(G).$$

In [3] the number $I(G)$ is called the inducibility of the graph G.

Among others, Pippenger and Golumbic [3] have derived the following lower bound for the inducibility of an arbitrary graph G of order p:

$$I(G)\geqq p!(p^p-p)^{-1}. \tag{1}$$

Our aim will be to improve this result.

After having proved some preliminary results in Section 2, Section 3 contains a proof of a new lower bound for $I(G)$, which is in infinitely many cases asymptotically much better than (1).

All graphs considered in this paper are finite, undirected, without loops and multiple edges. Other terminology is essentially the same as that of Harary [2].

2. Some useful lemmas

We start with a result concerning the representation of graphs by the system of their maximal independent sets. For any three sets A, B, C we shall write $\langle A, B, C\rangle$ instead of $(A\cap B)\cup(B\cap C)\cup(C\cap A)$.

Lemma 1. *Let G be a graph and let $\mathcal{S}_m(G)$ be the family of all maximal (in the sense of inclusion) independent sets of G. Then for any $A_1, A_2, A_3 \in \mathcal{S}_m(G)$ the following conditions are fulfilled:*

(i) $A_1 \subseteqq A_2 \Rightarrow A_1 = A_2$;
(ii) *there is a set* $A \in \mathcal{S}_m(G)$ *such that* $\langle A_1, A_2, A_3 \rangle \subseteqq A$.

On the other hand, let $\mathcal{S}$ be a nonempty finite set of nonempty finite sets such that for any $A_1, A_2, A_3 \in \mathcal{S}$ the following conditions are fulfilled:

(i) $A_1 \subseteqq A_2 \Rightarrow A_1 = A_2$;
(ii) *there is a set* $A \in \mathcal{S}$ *such that* $\langle A_1, A_2, A_3 \rangle \subseteqq A$.

Then there exists exactly one graph G with the vertex set $V(G) = \cup \mathcal{S}$ such that $\mathcal{S}_m(G) = \mathcal{S}$.

Proof. We can get the proof by a slight modification of the proof of the necessary and sufficient condition for a hypergraph to be conformal (cf. [1, p. 397]).

Let us continue with some notation. For any set T let $S(T)$ denote the group of all permutations of T. Let $M = \{M_1, M_2, \ldots, M_r\}$ be a family of sets and put $M_0 = M_1 \cup M_2 \cup \ldots \cup M_r$. For any $M_i \in M$ put $M_i^1 = M_i$, $M_i^{-1} = M_0 - M_i$. Given a permutation $h \in S(M)$, and $c = (c_1, c_2, \ldots, c_r) \in \{-1, 1\}^r$, define $hM^c = (hM_1)^{c_1} \cap (hM_2)^{c_2} \cap \ldots \cap (hM_r)^{c_r}$; in the case when h is the identity permutation we simply write M^c.

Consider a fixed graph G of order p. Let $\mathcal{S}_m(G) = M = \{M_1, M_2, \ldots, M_t\}$ be the family of maximal independent sets of G. Denote by K the intersection graph of the family M, i.e. $K = \Omega(M)$ (see [2]), and let Γ_K be the automorphism group of K. Further define $E = \{h \in \Gamma_K;\ M^c = \emptyset \Leftrightarrow hM^c = \emptyset$ for each $c \in \{-1, 1\}^t\}$, the strong automorphism group of the family M, and $E^* = \{h \in \Gamma_K;\ |M^c| = |hM^c|$ for each $c \in \{-1, 1\}^t\}$, the reduced automorphism group of M.

Lemma 2. *Let Γ_G be the automorphism group of G. Then*

$$|\Gamma_G| \Big/ \prod_{c \in P} |S(M^c)| = |E^*|,$$

where $P = \{c;\ |M^c| > 0\}$.

Proof. Consider a mapping f: $\Gamma_G \to E^*$ given by $f(g) = h$ iff $hM_i = g(M_i)$ for any $M_i \in M = \mathcal{S}_m(G)$. It is easily seen that f is a group epimorphism. The kernel of f is created by automorphisms $g \in \Gamma_G$ for which $g(M^c) = M^c$ for any $c \in P$. On the other hand, any permutation g of $V(G)$, the set of all vertices of G, with the property $g(M^c) = M^c$ belongs to Γ_G. Thus, Ker $(f) \cong \prod_{c \in P} S(M^c)$. Lemma 2 follows.

Lemma 3. *Let y_c, $c \in P$ be natural numbers such that $y_c \geqq |M^c|$. There exists a graph H with the family of its maximal independent sets $\mathcal{S}_m(H) = Y = \{Y_1, Y_2, \ldots, Y_t\}$ such that $|Y^c| = y_c$.*

Proof. It suffices to prove the above statement for a family of numbers y_c such that $y_c = |M^c|$ for all but one $c_0 \in P$, for which $y_{c_0} = |M^{c_0}| + 1$. Choose an element v such that $v \notin \bigcup_{i=1}^{t} M_i$ and put $Y = \{Y_1, Y_2, \ldots, Y_t\}$, where $Y^{c_0} = M^{c_0} \cup \{v\}$ and $Y^c = M^c$ for each $c \in P - \{c_0\}$. To show the existence of a graph H satisfying $\mathcal{S}_m(H) = Y$ it suffices to verify (ii) of Lemma 1. If at most one of the sets Y_i, Y_j, Y_k contains v, then $\langle Y_i, Y_j, Y_k \rangle = \langle M_i, M_j, M_k \rangle$ and (ii) is fulfilled. Let $v \in Y_i \cap Y_j$. Then $Y^{c_0} \subseteq Y_i \cap Y_j$ and $M^{c_0} \subseteq M_i \cap M_j$. There is a set M_r such that $\langle M_1, M_j, M_k \rangle \subseteq M_r$. But $M_r = \bigcup_{c_r = 1} M^c$ and $M^{c_0} \subseteq M_r$, therefore $(c_0)_r = 1$ and $\langle Y_i, Y_j, Y_k \rangle \subseteq \subseteq Y_r$. Lemma 3 follows by induction.

Note that graphs G, H from Lemma 3 have isomorphic strong groups of automorphisms of their families of maximal independent sets.

Lemma 4. *Let a_c, $c \in P$ be positive real numbers such that $\sum_{c \in P} a_c = 1$. There exists a sequence $\{H_n\}_{n=1}^{\infty}$ of graphs H_n of order h_n with $\mathcal{S}_m(H_n) = Y_n = \{Y_{n,1}, Y_{n,2}, \ldots, Y_{n,t}\}$ for which*

$$\lim_{n \to \infty} \frac{|Y_n^c|}{h_n} = a_c \quad \text{for each } c \in P.$$

Proof. Let $\{q_n\}_{n=1}^{\infty}$ be an increasing sequence of natural numbers such that $[a_c q_1] \geqq p$ for any $c \in P$ {recall that p is the order of our graph G, and $[z]$ denotes the greatest integer not exceeding z). According to Lemma 3, there exists a graph H_n with $\mathcal{S}_m(H_n) = Y_n = \{Y_{n,1}, Y_{n,2}, \ldots, Y_{n,t}\}$ for which $|Y^c| = [a_c q_n]$. It is easily seen that there is an integer $n_0$such that for any $n > n_0$ we have

$$\frac{a_c q_n - 1}{q_n} \leqq \frac{|Y_n^c|}{h_n} \leqq \frac{a_c q_n}{q_n - |P|},$$

which implies $h_n^{-1}|Y_n^c| \to a_c$ as $n \to \infty$. This completes the proof of Lemma 4.

3. The main result

Theorem. *Let G be a graph of order p with $\mathcal{S}_m(G) = M = \{M_1, M_2, \ldots, M_t\}$, Γ_G the automorphism group of G, and E the strong automorphism group of the family M. Further put $P = \{c \in \{-1, 1\}^t;\ |M^c| \geqq 1\}$, $B = \{c \in P;\ |hM^c| = 1$ for each*

$h \in E\}$. *Then, for any positive real numbers* a_c *such that* $\sum_{c \in P} a_c = 1$ *the following estimation is valid:*

$$I(G) \geqq \frac{p!}{|\Gamma_G|} \cdot \left(1 - \sum_{c \in B} a_c^p\right)^{-1} \sum_{h \in E} \prod_{c \in P} a_c^{|hM^c|}.$$

Proof. Let G be a graph satisfying all assumptions of our Theorem and let a_c, $c \in P$ be positive real numbers for which $\sum_{c \in P} a_c = 1$. Let $\{H_n\}_{n=1}^{\infty}$ be the same sequence of graphs as constructed in Lemma 4, i.e. $\mathcal{S}_m(H_n) = Y_n = \{Y_{n,1}, Y_{n,2}, \ldots, Y_{n,t}\}$ and $h_n^{-1}|Y_n^c| \to a_c$ for each $c \in P$.

For any $c \in P$, $h \in E$ and n choose a set $T(n, h, c) \subseteq Y_n^c$ such that $|T(n, h, c)| = |hM^c|$. Define $V(n, h, i) = \bigcup_{c_i = 1} T(n, h, c)$ for each i, $1 \leqq i \leqq t$. Since $\bigcup_{c_i=1} T(n, h, c) \subseteq \bigcup_{c_i=1} Y_n^c = Y_{n,i}$, we deduce that $V(n, h, i)$ are independent sets in H_n. Our construction guarantees that families M and $\{V(n, h, i), 1 \leqq i \leqq t\}$ have not only the same strong automorphism groups but also that the subgraph $L(n, h)$ of H_n induced by the set $\bigcup_{i=1}^{t} V(n, h, i)$ is isomorphic to G. Further, $L(n, h) = L(n, g)$ for some $h, g \in E$ if and only if $hg^{-1} \in E^*$, the reduced automorphism group of the family M. Thus, in the way above we have constructed exactly

$$|E^*|^{-1} \sum_{h \in E} \prod_{c \in P} \binom{|Y_n^c|}{|hM^c|}$$

different copies $L(n, h)$ of the graph G in H_n.

To obtain a little more copies of G, we shall change the structure of the graph H_n as follows. Define the graph H_n' by: $V(H_n') = V(H_n)$, $E(H_n') \supseteq E(H_n)$, whereby the set of added edges $E(H_n') - E(H_n)$ will be constructed in the following way: For any $c \in B$ and any Y_n^c choose a graph $H_{n,c}$ such that $V(H_{n,c}) = Y_n^c$ and $(G, H_{n,c}) = (G, |Y_n^c|)$ (for the notation see the introduction of the paper). Now put $E(H_n') - E(H_n) = \bigcup_{c \in B} E(H_{n,c})$. It is easily seen that all sets $V(n, h, i)$ are independent in the new graph H_n'. Thus we have

$$(G, H_n') \geqq |E^*|^{-1} \sum_{h \in E} \prod_{c \in P} \binom{|Y_n^c|}{|hM^c|} + \sum_{c \in B} (G, |Y_n^c|).$$

This inequality implies

$$\frac{(G, h_n)}{\binom{h_n}{p}} \geqq |E^*|^{-1} \binom{h_n}{p}^{-1} \sum_{h \in E} \prod_{c \in P} \binom{|Y_n^c|}{|hM^c|} + \sum_{c \in B} \frac{(G, |Y_n^c|)}{\binom{|Y_n^c|}{p}} \cdot \frac{\binom{|Y_n^c|}{p}}{\binom{h_n}{p}}.$$

For the limiting procedure we may suppose that $|Y_n^c| \to \infty$ for any $c \in P$. After a short computation, the last inequality, together with the definition of inducibility, Lemma 4, Lemma 2 and some other simple relations lead to the desired result. Our Theorem follows.

4. Concluding remarks

Denote by $J(G)$ the supremum of the right-hand side of the inequality of our Theorem, taken over all $|P|$-tuples of positive real numbers a_c such that $\sum_{c \in P} a_c = 1$. If $-G$ denotes the complement of the graph G, then $I(G) = I(-G)$, but in general $J(G) \neq J(-G)$.

Corollary. $I(G) \geqq \max\{J(G), J(-G)\}$ *for any graph* G.

Note that the lower bound in the above corollary is in many cases the best possible. For example, if $G = K_n$, $K_{n,n}$ or $K_{n,n+1}$, then $I(G) = \max\{J(G), J(-G)\}$, cf. [3].

It remains to compare our lower bound for $I(G)$ with that of [3]. Putting $a_c = p^{-1}|M^c|$, $|M^c| = m_c$, and looking at the proof of Theorem, we easily obtain

$$J(G) \geqq \frac{p!}{\prod_{c \in P} m_c!} (p^p - |B|)^{-1} \prod_{c \in P} (m_c)^{m_c}.$$

Now it is clear that in the case $|B| < p$ our estimation is even asymptotically better than that of [3]. If $|B| = p$, then $E = E^*$ and $m_c = 1$ for each $c \in P$, therefore

$$J(G) \geqq p!(p^p - p)^{-1}.$$

We see that in any case our estimation is at least as good as (1).

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Katedra matematiky a deskr. geometrie
Stavebnej fakulty SVŠT
Radlinského 11
813 68 Bratislava

НОВАЯ НИЖНЯЯ ГРАНИЦА ДЛЯ ИНДУЦИРУЕМОСТИ ГРАФА

J. Širáň

Резюме

Индуцируемостью графа G с k вершинами называют предел

$$I(G) = \lim_{n\to\infty} (G, n) \Big/ \binom{n}{k},$$

где (G, n) обозначает наибольшее число индуцированных подграфов, изоморфных графу G, которое может содержать некоторый n-вершинный граф. В статье доказана новая нижняя граница для $I(G)$, основанная на распределении независимых множеств вершин в графе G.