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A PRINCIPAL CONGRUENCE IDENTITY CHARACTERIZING THE VARIETY OF DISTRIBUTIVE LATTICES WITH ZERO

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It is a well-known fact that the product of two congruences Θ, Φ on an algebra A is a congruence on A if and only if they permute, i.e. $\Theta . \Phi = \Phi . \Theta$. Hence, if A is a permutable algebra and x, y, z are elements of A, then

$$\theta(x, y) \subseteq \theta(x, z) . \theta(y, z). \qquad (*)$$

It is easy to prove that if (*) holds for each $x, y, z \in A$ and for every A of a variety $\mathscr{V}$, then $\mathscr{V}$ is permutable, thus the principal congruence identity (*) is equivalent to the permutability of $\mathscr{V}$.

Suppose now that an algebra A has a nullary operation 0. If A is permutable, then (*) implies the validity of the principal congruence identity

$$\theta(x, y) \subseteq \theta(x, 0) . \theta(y, 0) \qquad (**)$$

in A. On the contrary, the identity (**) can be satisfied also in non-permutable varieties, see [1]. The aim of this short note is to prove the following

Theorem. *Let $\mathscr{V}$ be a variety of lattices with the least element* 0. *The following conditions are equivalent*:

(1) *$\mathscr{V}$ is a variety of distributive lattices with* 0;

(2) *$\mathscr{V}$ satisfies the identity* (**).

Proof. (1) $\Rightarrow$ (2): Let A be a distributive lattice with the least element 0. Then clearly

$$\langle x, y \rangle \in \theta(x, 0) . \theta(y, 0).$$

Moreover, we have

$$\langle y, x \vee y \rangle = \langle 0 \vee y, x \vee y \rangle \in \theta(x, 0)$$
$$\langle x \vee y, x \rangle = \langle x \vee y, x \vee 0 \rangle \in \theta(y, 0),$$

thus also

$$\langle y, x \rangle \in \theta(x, 0) . \theta(y, 0).$$

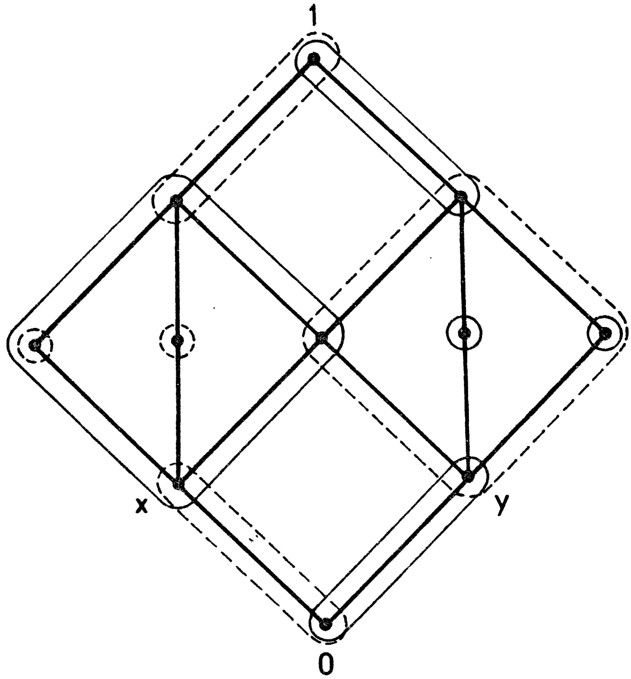

Fig. 1

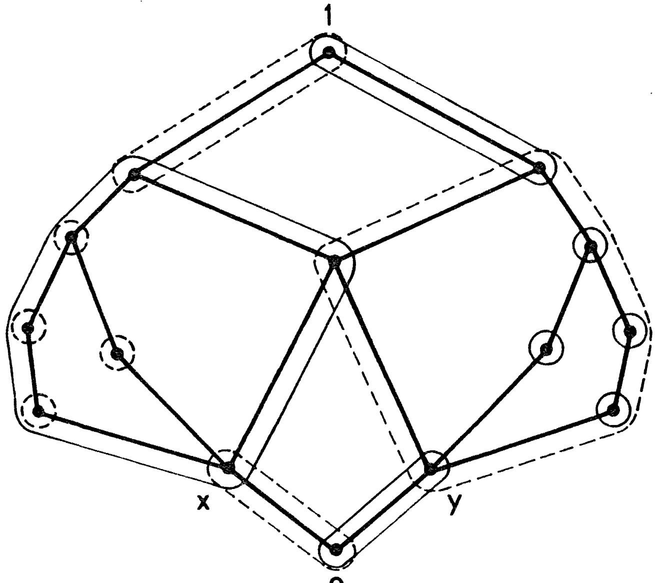

Fig. 2

Since the relations $\theta(x, 0)$, $\theta(y, 0)$ are reflexive and compatible, also the relational product $\theta(x, 0) . \theta(y, 0)$ has this property. Therefore, we infer

$$\langle \varphi(x, y), \varphi(y, x) \rangle \in \theta(x, 0) . \theta(y, 0)$$

for every algebraic function φ over A. It means

$$T(x, y) \subseteq \theta(x, 0) . \theta(y, 0),$$

where $T(x, y)$ is the principal tolerance on A generated by the pair $\langle x, y \rangle$. By [2], we have $T(x, y) = \theta(x, y)$ in every distributive lattice which implies (2).

(2) $\Rightarrow$ (1): Let $\mathscr{V}$ be a variety of lattices with the least element 0 which is not distributive. Then $\mathscr{V}$ contains either the pentagon N_5 or the diamond M_5 as its member. hence, $\mathscr{V}$ contains also either the lattice L_1 or L_2 in Fig. 1 or Fig. 2, respectively. The classes of $\theta(x, 0)$ are in both figures denoted by dotted lines, those of $\theta(y, 0)$ by full lines. It is clear that $\langle 0, 1 \rangle \notin \theta(x, 0) . \theta(y, 0)$ *in these cases but, on the contrary,* $\langle 0, 1 \rangle \in \theta(x, y)$.

REFERENCES

[1] CHAJDA, I.: On the one principal congruence identity, Arch. Math. (Brno), 23 (1987), 187—190.

[2] CHAJDA, I., ZELINKA, B.: Minimal compatible tolerances on lattices, Czech. Math. J., 27 (1977), 452—459.

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ТЖДЕСТВО ГЛАВНЫХ КОНГРУЕНЦИЙ, ХАРАКТЕРИЗИРУЮЩЕЕ МНОГООБРАЗИЕ ДИСТИБУТИВНЫХ РЕШЕТОК С НУЛЕМ

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Резюме

Показывается, что многообразие решеток с нулем является многообразием дистибутивных решеток тогда и только тогда, когда оно удовлетворяет тождеству

$$\theta(x, y) \subseteq \theta(x, 0) . \theta(y, 0).$$