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ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF THE DIFFERENTIAL EQUATION OF THE FOURTH ORDER

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1. Introduction

In this paper we shall consider the equation

$$y^{(iv)}+q(t)y'+r(t)y=0, \tag{1}$$

where $q(t)$ and $r(t)$ are functions having continuous first derivatives on $[a, \infty)$. We shall investigate the behaviour of solutions of (1) as $t\to\infty$ when the ratios of certain powers of $q(t)$ and $r(t)$ are small (improper integrals on $[a, \infty)$ exist), unlike in other papers (e.g. [1], [2]), where $t^2q(t)$ and $t^3r(t)$ are supposed to be small, or $q(t)$ and $r(t)$ to approach a constant.

The method of proving our theorems lies in reducing equation (1) to the equivalent system of equations to which the following Coddington—Levinson theorem ([1], p. 92) will be applied.

Theorem 1. *Let A be a constant matrix with characteric roots τ_j, $j=1, 2, \dots, n$, all of which are distinct. Let the matrix $V(s)$ be differentiable and satisfy*

$$\int_0^\infty |V'(s)|\,\mathrm{d}s<\infty$$

and let $V(s)\to 0$ as $s\to\infty$. Let the matrix $R(s)$ be integrable and

$$\int_0^\infty |R(s)|\,\mathrm{d}s<\infty.$$

Let the roots of $\det[A+V(s)-\tau E]=0$ *be denoted by $\tau_j(s)$, $j=1, 2, \dots, n$. For a given k, let*

$$D_{kj}(s)=\mathrm{Re}\,[\tau_k(s)-\tau_j(s)].$$

Suppose all j, $1\leq j\leq n$ fall into one of two classes I_1 and I_2, where

$$j\in I_1, \text{ if } \int_0^s D_{kj}(s)\,\mathrm{d}s\to\infty \text{ as } t\to\infty \text{ and}$$

$$\int_{s_1}^{s_2} D_{kj}(s)\,\mathrm{d}s > -K \quad (s_2 \geq s_1 \geq 0)$$

$$j \in I_2,\ \textit{if} \int_{s_1}^{s_2} D_{kj}(s)\,\mathrm{d}s < K \quad (a_2 \geq s_1 \geq 0),$$

where k is fixed and where K is a constant. Let p_k be a characteristic vector of A associated with τ_k, so that

$$Ap_k = \tau_k p_k\,.$$

Then there is a solution $\varphi_k(s)$ of

$$x' = [A + V(s) + R(s)]x \tag{2}$$

and a s_0, $0 \leq s_0 < \infty$, such that

$$\lim_{s \to \infty} \varphi_k(s) \exp\left[-\int_{s_0}^{s} \tau_k(\delta)\,\mathrm{d}\delta\right] = p_k\,.$$

If the hypothesis is satisfied for all k, $1 \leq k \leq n$, then $\varphi_k(s)$, $k = 1, 2, \ldots, n$ form a fundamental system of (2).

The symbol $L[a, \infty)$ will refer to the set of all complexvalued functions which are Lebesque integrable on $[a, \infty)$.

The following theorem is needed.

Theorem 2. (Hinton [3]). *Let $q(s) > 0$ on $[0, \infty)$ and $q\ (s)/q^{1+1/n}(s)$ be in $L[0, \infty)$ for $n = 1, 2, \ldots, n$. Then*

(i) *$q^{1/n}(s)$ is not in $L[0, \infty)$*
(ii) *$[q'(s)/q^{1+1/n}(s)]'$ is in $L[0, \infty)$*
(iii) *$[q'(s)/q^{1+1/2n}(s)]^2$ is in $L[0, \infty)$.*

The system associated with (1) is

$$z' = A(t)z\,, \tag{3}$$

where $z = [y, y', y'', y''']^{\mathrm{T}}$ and

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -r & -q & 0 & 0 \end{bmatrix}$$

If in (3) we change the depedent variable z by setting $w = Tz$, where $T(t)$ is a diagonal and nonsingular matrix, and substitute it in (3), we obtain

$$w' = [T(t)A(t)T^{-1}(t) + T'(t)T^{-1}(t)]w\,. \tag{4}$$

The form of (4) depends on the matrix $T(t)$. If $q(t)\neq 0$, we shall consider $T = \operatorname{dia}[q, q^{2/3}, q^{1/3}, 1]$, and if $r(t)\neq 0$, then $T = \operatorname{dia}[|r|^{3/4}, |r|^{1/2}, |r|^{1/4}, 1]$.

2. Theorems

$\alpha. q(t)>0$

Theorem 3. *If $q''/q^{4/3}$, $r^2/q^{7/3}$, and $r'/q^{4/3}$ are in $L[a, \infty)$, then there are four linearly independent solutions z_k, $k=1, 2, 3, 4$ of (3) and $t_0\geq a$ such that*

$$Tz_1q^{-1}\exp\left[\int_{t_0}^{t}\frac{r(\delta)}{q(\delta)}\,\mathrm{d}\delta\right]\rightarrow p_1$$

$$Tz_kq^{-1/3}\exp\left[-\int_{t_0}^{t}\tau_kq^{1/3}+\frac{1}{3}\frac{r(\delta)}{q(\delta)}\,\mathrm{d}\delta\right]\rightarrow p_k$$

$k=2, 3, 4$ and $\tau_1=0$, $\tau_2=-1$, $t_{3,4}=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$, $p_1=(1, 0, 0, 0)^T$, $p_k = (\bar{\tau}_k, 1, \tau_k, -\bar{\tau}_k)^T$, $k=2, 3, 4$.

Theorem 4. *If $q''/q^{4/3}$ and r/q are in $L[a, \infty)$, then there are linearly independent solutions z_k, $k=1, 2, 3, 4$ of (3) and $t_0\geq a$ such that*

$$Tz_1q^{-1}\rightarrow p_1$$

$$Tz_kq^{-1/3}\exp\left[-\tau_k\int_{t_0}^{t}q^{1/3}(\delta)\,\mathrm{d}\delta\right]\rightarrow p_k,\quad k=2,3,4$$

where τ_k, p_k are the same as in Theorem 3.

$\beta. q(t)<0$

Theorem 5. *If the hypotheses of Theorem 3 hold, then there are linearly independent solutions z_k, $k=1, 2, 3, 4$ of (3) and $t_0\geq a$ such that*

$$Tz_1q^{-1}\exp\left[\int_{t_0}^{t}\frac{r(\delta)}{q(\delta)}\,\mathrm{d}\delta\right]\rightarrow p_1$$

$$Tz_kq^{-1/3}\exp\left[\int_{t_0}^{t}\left[\tau_kq^{1/3}(\delta)-\frac{1}{3}\frac{r(\delta)}{q(\delta)}\right]\mathrm{d}\delta\right]\rightarrow p_k,$$

for $k=2, 3, 4$ and where $\tau_1=0$, $\tau_2=1$, $\tau_{3,4} = -\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$, $p_1 = (1, 0, 0, 0,)^T$, $p_k = (1, \tau_k, \tau_k^2, 1)^T$.

Theorem 6. *If the hypotheses of Theorem 4 hold, then there exist four linearly independent solutions z_k, $k=1, 2, 3, 4$ and $t_0 \geqslant a$ such that*

$$Tz_1 q^{-1} \rightarrow p_1$$

$$Tz_k q^{-1/3} \exp\left[\tau_k \int_{t_0}^{t} q^{1/3}(\delta)\, \mathrm{d}\delta\right] \rightarrow p_k,$$

where τ_k, p_k are the same as in Theorem 5.

γ. $r(t) > 0$

Theorem 7. *If $q'/r^{3/4}$, $r''/r^{5/4}$, and $q^2/r^{5/4}$ are in $L[a, \infty)$, then there are four linearly independent solutions z_k, $k=1, 2, 3, 4$ of (3) and $t_0 \geqslant a$ such that*

$$Tz_k r^{3/8} \exp\left[-\int_{t_0}^{t}\left[\tau_k r^{1/4}(\delta) + \frac{1}{4}\frac{q(\delta)}{r(\delta)}\tau_k^2\right] \mathrm{d}\delta\right] \rightarrow p_k,$$

$k=1, 2, 3, 4$, *where τ_k are the roots of $\tau^4+1=0$ and $p_k = (1, \tau_k, \tau_k^2, \tau_k^3)^T$.*

Theorem 8. *If $r''/r^{5/4}$ and q/r are in $L[a, \infty)$, then there are four linearly independent solutions z_k, $k=1, 2, 3, 4$ of (3) and $t_0 \geqslant a$ such that*

$$Tz_k r^{3/8} \exp\left[-\tau_k \int_{t_0}^{t} r^{1/4}(\delta)\, \mathrm{d}\delta\right] \rightarrow p_k, \quad k=1, 2, 3, 4,$$

where τ_k and p_k are the same as in Theorem 7.

δ. $r(t) < 0$

Theorem 9. *If the hypotheses of Theorem 7 hold, then there are four linearly independent solutions z_k, $k=1, 2, 3, 4$ of (3) and $t_0 \geqslant a$ such that*

$$Tz_k(-r)^{3/8} \exp\left[-\int_{t_0}^{t}\left[\tau_k(-r)^{1/4} + \frac{1}{4}\frac{q}{r}\tau_k^2\right] \mathrm{d}\delta\right] \rightarrow p_k,$$

where τ_k are the roots of $\tau^4-1=0$, and $p_k = (1, \tau_k, \tau_k^2, \tau_k^3)^T$.

Theorem 10. *If the hypotheses of Theorem 8 hold, then there are four linearly independent solutions z_k, $k=1, 2, 3, 4$ of (3) and $t_0 \geqslant a$ such that*

$$Tz_k(-r)^{3/8} \exp\left[-\tau_k \int_{t_0}^{t} [-r(\delta)]^{1/4}\, \mathrm{d}\delta\right] \rightarrow p_k,$$

where τ_k, p_k are the same as in Theorem 9.

3. Corollaries and examples

Corollary 1. *If $q''/q^{4/3}$, r/q are in $L[a, \infty)$ and $q \neq 0$, then there exists a solution $y(x)$ of (1) in the form*

$$y(x)=\left[c_1+c_2q^{-2/3}\exp\left(-\int_{t_0}^{t}q^{1/3}\,d\delta\right)+q^{-2/3}\exp\left(\frac{1}{2}\int_{t_0}^{t}q^{1/3}\,d\delta\right)\cdot\right.$$

$$\left.\cdot\left(c_3\cos\frac{\sqrt{3}}{2}\int_{t_0}^{t}q^{1/3}\,d\delta+c_4\sin\frac{\sqrt{3}}{2}\int_{t_0}^{t}q^{1/3}\,d\delta\right]\cdot[1+o(1)].$$

Corollary 2. *If $r''/r^{5/4}$, q/r are in $L[a,\infty)$ and $r\neq 0$, then there exists a solution $y(x)$ of (1) in the form*

$$y(x)=r^{-9/8}\left[\exp\left(\frac{\sqrt{2}}{2}\int_{t_0}^{t}r^{1/4}\,d\delta\right)\left(c_1\cos\frac{\sqrt{2}}{2}\int_{t_0}^{t}r^{1/4}\,d\delta+\right.\right.$$

$$+c_2\sin\frac{\sqrt{2}}{2}\int_{t_0}^{t}r^{1/4}\,d\delta\right)+\exp\left(-\frac{\sqrt{2}}{2}\int_{t_0}^{t}r^{1/4}\,d\delta\right)\cdot$$

$$\left.\cdot\left(c_3\cos\frac{\sqrt{2}}{2}\int_{t_0}^{t}r^{1/4}\,d\delta+c_4\sin\frac{\sqrt{2}}{2}\int_{t_0}^{t}r^{1/4}\,d\delta\right)\right][1+o(1)],$$

if $r(t)>0$, and

$$y(x)=(-r)^{-9/8}\left[c_1\exp\left(-\int_{t_0}^{t}(-r)^{1/4}\,d\delta\right)+c_2\exp\left(\int_{t_0}^{t}(-r)^{1/4}\,d\delta\right)+\right.$$

$$\left.+c_3\cos\int_{t_0}^{t}(-r)^{1/4}\,d\delta+c_4\sin\int_{t_0}^{t}(-r)^{1/4}\,d\delta\right][1+o(1)],$$

if $r(t)<0$, where c_1, c_2, c_3, c_4 are arbitrary numbers.

Example 1. Let $r(t)$ be in $L[b,\infty)$ and a be an arbitrary nonzero number. Then the differential equation

$$y^{(iv)}+a^4y'+r(t)y=0$$

satisfies the assumptions of Corollary 1, and therefore its solution is

$$y(x)=\left[c_1+c_2e^{-at}+c_3e^{\frac{1}{2}at}\cos\frac{\sqrt{3}}{2}at+c_4e^{\frac{1}{2}at}\sin\frac{\sqrt{3}}{2}at\right]\cdot[1+o(1)].$$

Example 2. If $t^{\alpha}q$ is in $L[b,\infty)$, $\alpha<4$, and a is an arbitrary nonzero number, then equation

$$y^{(iv)}+q(t)y'+\frac{a}{t^{\alpha}}y=0$$

satisfies the assumptions of Corollary 2, and therefore its solution is in the form of the above Corollary.

4. Proofs of theorems

We begin the proofs of theorems with two lemas.

Lema 1. *Let $q(t)>0$ on $[a,\infty)$ and $\int_a^\infty q^{1/3}\,dt=\infty$. Let $T(t)=\mathrm{dia}\,[q(t), q^{2/3}(t), q^{1/3}(t), 1]$. If we make the change of variable $s=\omega(t)=\int_a^t q^{1/3}(\delta)\,d\delta$ in (4), then it leads to*

$$x'(s)=[A_0+A_1f(s)+A_2g(s)]x(s), \tag{5}$$

where $x(s)=w(\alpha(s))$, $f(s)=r(\alpha(s))/q^{4/3}(\alpha(s))$, $g(s)=q'(\alpha(s))/q^{4/3}(\alpha(s))$, $\alpha(s)$ is an inverse function of $s=\omega(t)$ and

$$A_0=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\0&-1&0&0\end{bmatrix}\quad A_1=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\-1&0&0&0\end{bmatrix}\quad A_2=\begin{bmatrix}1&0&0&0\\0&2/3&0&0\\0&0&1/3&0\\0&0&0&0\end{bmatrix}$$

Proof. First at all we see that if we put $T=\mathrm{dia}\,[q, q^{2/3}, q^{1/3}, 1]$ in (4), we obtain

$$TAT^{-1}=\begin{bmatrix}0&q^{1/3}&0&0\\0&0&q^{1/3}&0\\0&0&0&q^{1/3}\\-rq^{-1}&-q^{1/3}&0&0\end{bmatrix}=A_0q^{1/3}+A_1r/q$$

$$T'T^{-1}=\begin{bmatrix}q'q^{-1}&0&0&0\\0&(2/3)q'q^{-1}&0&0\\0&0&(1/3)q'q^{-1}&0\\0&0&0&0\end{bmatrix}=A_2q'q^{-1}$$

and the equation (4) will have the form

$$w'(t)=\left(A_0q^{1/3}(t)+A_1\frac{r(t)}{q(t)}+A_2\frac{q'(t)}{q(t)}\right)w(t). \tag{6}$$

The function $s=\omega(t)=\int_a^t q^{1/3}(\delta)\,d\delta$ has a derivative $\omega'(t)=q^{1/3}(t)>0$, hence it increases on $[a,\infty)$. This means that $s=\omega(t)$ has an inverse function $t=\alpha(s)$ defined on $[0,\infty)$, since $\int_a^\infty q^{1/3}(t)\,dt=\infty$. Putting $t=\alpha(s)$ into (6) we get

$$w'(\alpha(s))q^{1/3}(\alpha(s))=\left[A_0q^{1/3}(\alpha(s))+A_1\frac{r(\alpha(s))}{q(\alpha(s))}+A_2\frac{q'(\alpha(s))}{q(\alpha(s))}\right]w(\alpha(s))\Big).$$

Consequently, if the last equation is divided by $q^{1/3}(\alpha(s))$, we obtain (5).

Similarly we can prove the following lema

Lema 2. *Let $r(t)>0$ on $[a,\infty)$ and $\int_a^\infty r^{1/4}(t)\,dt=\infty$. Let $T(t)=\operatorname{dia}[r^{3/4}(t), r^{1/2}(t), r^{1/4}(t), 1]$. Then making the change of the variable $s=\omega(t)=\int_a^t r^{1/4}(\delta)\,d\delta$ in (4), we get*

$$x'(s)=[B_0+B_1h(s)+B_2k(s)]x(s), \tag{7}$$

where $x(s)=w(\alpha(s))$, $h(s)=q/r^{3/4}(\alpha(s))$, $k(s)=r'/r^{5/4}(\alpha(s))$, $\alpha(s)$ is an inverse function of $s=\omega(t)$ and

$$B_0=\begin{bmatrix}0&1&0&0\\0&0&1&0\\0&0&0&1\\-1&0&0&0\end{bmatrix}\quad B_1=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&-1&0&0\end{bmatrix}\quad B_2=\begin{bmatrix}3/4&0&0&0\\0&1/2&0&0\\0&0&1/4&0\\0&0&0&0\end{bmatrix}$$

Proof of Theorem 3. We show that all hypotheses of Theorem 1 are fulfilled for the equation (5). The characteristic equation of A_0 is $\tau^4+\tau=0$ so the characteristic roots $\tau_1=0$, $\tau_2=-1$, $\tau_{3,4}=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$ of A_0 are distinct. The vectors $p_1=(1,0,0,0)^T$, $p_k=(\bar{\tau}_k, 1, \tau_k, \bar{\tau}_k)^T$, $k=2,3,4$ are characteristic vectors of A_0 coresponding to τ_k.

Denote $V(s)=A_1f(s)+A_2g(s)$, i.e. $R(s)=0$ in Theorem 1. In order to be $\int_0^\infty |V'(s)|\,ds<\infty$ it is sufficient to prove that $\int_0^\infty |f'(s)|\,ds<\infty$ and $\int_0^\infty |g'(s)|\,ds<\infty$. In both integrals we put $\alpha(s)=t$. Then from the definition of the functions $f(s)$ and $g(s)$ there follows

$$\int_0^\infty |f'(s)|\,ds=\int_0^\infty |[r(\alpha(s))/q^{4/3}(\alpha(s))]'|\,ds\leq$$

$$\leq\int_0^\infty |[r'(\alpha(s))q^{4/3}(\alpha(s))\alpha'(s)]/[q^{4/3}(\alpha(s))]^2|\,ds+$$

$$+\frac{4}{3}\int_0^\infty |[q^{1/3}(\alpha(s))q'(\alpha(s))r(\alpha(s))\alpha'(s)]/[q^{4/3}(\alpha(s))]^2|\,ds=$$

$$=\int_a^\infty |r'(t)/q^{4/3}(t)|\,dt+\frac{4}{3}\int_a^\infty |[r(t)q'(t)]/q^{7/3}(t)|\,dt.$$

The first integral is in $L[a,\infty)$ by hypothesis. By applying the Cauchy inequality to the second integral we get

$$\textstyle\int_a^\infty |[r(t)q'(t)]/q^{7/3}(t)|\,dt=\int_a^\infty |r(t)/q^{7/6}|\cdot|q'(t)/q^{7/6}(t)|\,dt\leq$$

$$\leqslant\left[\int_a^\infty |r^2(t)/q^{7/3}(t)|\,\mathrm{d}t\right]^{1/2}\cdot\left[\int_a^\infty |q'(t)/q^{7/6}(t)|^2\,\mathrm{d}t\right]^{1/2}.$$

From Theorem 2 it follows that if $q''/q^{4/3}$ is in $L[a,\infty)$, then $[q'/q^{7/6}]^2$ is in $L[a,\infty)$ and hence both above integrals are in $L[a,\infty)$. Therefore $\int_0^\infty |f'(s)|\,\mathrm{d}s<\infty$.

Similarly, from Theorem 2 it follows that $[q'/q^{4/3}]'$ is in $L[a,\infty)$ and so $\int_0^\infty |g'(s)|\,\mathrm{d}s<\infty$. Consequently $\int_0^\infty |V'(s)|\,\mathrm{d}s<\infty$.

From Theorem 2 we also get

$$\int_0^\infty \left|f^2(s)\right|\,\mathrm{d}s=\int_0^\infty |[r(\alpha(s))/q^{4/3}(\alpha(s))]^2\,\mathrm{d}s=\int_a^\infty |r^2(t)/q^{7/3}(t)|\,\mathrm{d}t<\infty$$

and

$$\int_0^\infty |g^2(s)|\,\mathrm{d}s=\int_a^\infty |[q'(t)/q^{4/3}(t)]^2q^{1/3}(t)|\,\mathrm{d}t=$$

$$=\int_a^\infty |[q'(t)/q^{7/6}(t)]^2|\,\mathrm{d}t<\infty.$$

Since $g'(s)$ and $g^2(s)$ are in $L[0,\infty)$, then $g(s)\to 0$ as $s\to\infty$. Similarly we obtain $f(s)\to 0$ as $s\to\infty$, and therefore $V(s)\to 0$ as $s\to\infty$.

Let us calculate the characteristic roots of

$$A_0+V(s)=\begin{bmatrix} g(s) & 1 & 0 & 0\\ 0 & (2/3)g(s) & 1 & 0\\ 0 & 0 & (1/3)g(s) & 1\\ -f(s) & -1 & 0 & 0\end{bmatrix}$$

The characteristic equation of $A_0+V(s)$ is

$$P(\tau)\equiv\tau^4-2g\tau^3+\frac{11}{9}g^2\tau^2+\left(1-\frac{2}{9}g^3\right)\tau+f-g=0. \tag{9}$$

Since $f(s)$, $g(s)\to 0$ as $s\to\infty$, we get that $P(\tau)\to\tau^4+\tau$ as $s\to\infty$. Hence the roots of (9) converge to the roots of $\tau^4+\tau=0$. Thus we may write for $s\in[0,\infty)$

$$\tau(s)=\tau+\delta(s), \tag{10}$$

where $\delta(s)\to 0$ as $s\to\infty$.

In order to find whether the hypothesis of Theorem 1 is fulfilled we show that the function $\delta(s)$ may be written as a sum

$$\delta(s)=\beta(s)+\gamma(s),$$

where $\beta(s)$ and $\gamma(s)\to 0$ as $s\to\infty$ and $\gamma(s)$ is in $L[0,\infty)$. Substituting $\tau(s) = \tau+\beta(s)+\gamma(s)$ into (9), we get

$$P[\tau+\beta(s)+\gamma(s)]=$$

$$=\gamma^4+[4(\tau+\beta)-2g]\gamma^3+\left[6(\tau+\beta)^2-6g(\tau+\beta)+\frac{11}{9}g^2\right]\gamma^2+$$

$$+\left[4(\tau+\beta)^3-6g(\tau+\beta)^2+\frac{22}{9}g^2(\tau+\beta)+\left(1-\frac{2}{9}g^3\right)\right]\gamma+P(\tau+\beta),$$

$$P(\tau+\beta)=\beta^4+(4\tau-2g)\beta^3+ \tag{11}$$

$$+\left(6\tau^2-6g\tau+\frac{11}{9}g^2\right)\beta^2+\left(4\tau^3-6g\tau^2+\frac{22}{9}g^2\tau+1-\frac{2}{9}g^3\right]\beta-2g\tau^3+$$

$$+\frac{11}{9}g^2\tau^2-\frac{2}{9}g^3\tau+f-g.$$

Then the equation (9) may be written as

$$\gamma\Big\{\gamma^3+[4(\tau+\beta)-2g]\gamma^2+\left[6(\tau+\beta)^2-6g(\tau+\beta)+\frac{11}{9}g^2\right]\gamma+ \tag{12}$$

$$+4(\tau+\beta)^3-6g(\tau+\beta)^2+\frac{22}{9}g^2(\tau+\beta)+\left(1-\frac{2}{9}g^3\right)\Big\}=-P(\tau+\beta).$$

Let us denote the expression in the complex bracket as $A(s)$. Since $f, g, \beta, \gamma\to 0$ as $s\to\infty$, then

$$\lim_{s\to\infty} A(s)=4\tau^3+1$$

Thus for every $\varepsilon>0$ there is a number $s'\in[0,\infty)$ such that

$$|A(s)-(4\tau^3+1)|<\varepsilon \quad \text{for } s\in[s'\ \infty),$$

from which it follows that $|A(s)|>|4\tau^2+1|-\varepsilon\geq 1-\varepsilon$. (If $\tau_1=0$, then $|4\tau_1^3+1|=1$, for other values of τ_k there is $|4\tau_k^3+1|=3$.) For $\varepsilon=1/2$ we have $|A(s)\geq 1/2$ on $[s',\infty)$.

From (12) it follows that

$$|P(\tau+\beta)=|\gamma A(s)|\geq\frac{1}{2}|\gamma(s)|$$

and then

$$|\gamma(s)|\leq 2|P(\tau+\beta(s))|, \tag{13}$$

from which we get that $\gamma(s)$ is in $L[0,\infty)$ if $P(\tau+\beta(s))$ is in $L[0,\infty)$.

Choose β in (11) such that

$$4\tau^3\beta+\beta-2g\tau^3+f-g=0,$$

i.e.

$$\beta(s)=\frac{2g(s)\tau^3+g(s)-f(s)}{4\tau^3+1}. \tag{14}$$

Then we obtain

$$P(\tau+\beta)=\beta^4+[4\tau-2g]\beta^3+\left[6\tau^2-6g\tau+\frac{11}{9}g^2\right]\beta^2+ \tag{15}$$
$$+\left[-6g\tau^3+\frac{22}{9}g^2\tau-\frac{2}{9}g^3\right]\beta+\frac{11}{9}g^2\tau^2-\frac{2}{9}g^3\tau.$$

Substituting $\beta(s)$ from (14) into (15) we get that each term of $P(\tau+\beta)$ contains f^2 or g^2 or fg. Since f^2 and g^2 are in $L[0,\infty)$, then fg is in $L[0,\infty)$ too and consequently $P(\tau+\beta)$ is in $L[0,\infty)$. Hence from (13) it follows that $\gamma(s)$ is in $L[0,\infty)$.

From the above we get that the roots $\tau_k(s)$ of $P(\tau)=0$ may be written as

$$\tau_k(s)=\tau_k+\frac{2g(s)\tau_k^3+g(s)-f(s)}{4\tau_k^3+1}+\gamma_k(s), \tag{16}$$

where $\tau_1=0$, $\tau_2=-1$, $\tau_{3,4}=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$, the second term in (16) converges to zero as $s\to\infty$ and $\gamma_k(s)$ is $L[0,\infty)$, $\gamma_k(s)\to 0$ as $s\to\infty$.

Then $D_{jk}(s)=\mathrm{Re}\,[\tau_j(s)-\tau_k(s)]$ for all $j,k=1,2,3,4$ may have the following forms

a) $D_{jk}(s)=G(s)$

b) $D_{jk}(s)=c+F(s)+G(s)$

c) $D_{jk}(s)=-c+F(s)+G(s)$,

where $c>0$, $F(s)$, $G(s)$ are functions such that $F(s)\to 0$, $G(s)\to 0$ as $s\to\infty$ and $G(s)$ is in $L[0,\infty)$.

a) In this case $j\in I_2$, because from the hypotheses stating that $G(s)$ is continuous on $[0,\infty)$ and $G(s)$ is $L[0,\infty)$ it follows that there exists $K>0$ such that $\int_{s_1}^{s_1} D_{jk}(s)\,ds<K$ for all $s_2\geq s_1\geq 0$.

b) Since $F(s)\to 0$ as $s\to\infty$, then there exists a number $s'\in[0,\infty)$ such that for every number $s>s'$ there is $c+F(s)+G(s)\geq c/2+G(s)$. Then $\int_0^\infty D_{jk}(s)\,ds=$ $=\int_0^\infty [c+F(s)+G(s)]\,ds=\infty$ since $\int_0^\infty [c/2+G(s)]\,ds=\infty$, and $\int_{s_1}^{s_2} D_{jk}(s)\,ds>$ $-K$ for all $s_2\geq s_1\geq 0$ and $K>0$, i.e. $j\in I_1$.

c) Similarly as in case b) it follows from the condition $F(s)\to 0$ as $s\to\infty$ that there is a number $s''\in[0,\infty)$ such that $-c+F(s)+G(s)<-c/2+G(s)$ on $[s'',\infty)$, from which it follows that $\int_0^\infty D_{jk}(s)\,ds=\int_0^\infty[-c+F(s)+G(s)]\,ds$ $=-\infty$ and $\int_{s_1}^{s_2}[-c+F(s)+G(s)]\,ds<K$ for some $K>0$ and every $s_2>s_1>0$, i.e. $j\in I_2$.

Thus all assumptions of Theorem 1 are fulfilled. Then, because of it, there are four linearly independent solutions $x_k(s)$, $k=1, 2, 3, 4$ of (5) *such that*

$$x_k(s)\exp\left[-\int_{s_0}^{s}\tau_k(\delta)\,d\delta\right]\to p_k .$$

Substituting $\tau_k(s)$ from (16) into the last expression we have

$$x_1(s)\exp\left[-\int_{s_0}^{s}[g(\delta)-f(\delta)+\gamma_1(\delta)]\,d\delta\right]\to p_1,$$

and after substituting $f(\delta)$ and $g(\delta)$ we get

$$x_1(s)\exp\left[-\int_{s_0}^{s}\left[\frac{q'(\alpha(\delta))-r(\alpha(\delta))}{q^{4/3}(\alpha(\delta))}+\gamma_1(\delta)\right]d\delta\right]\to p_1 .$$

If we denote $\int_{s_0}^{\infty}\gamma_1(\delta)\,d\delta=K$ and put $\alpha(\delta)=\xi$, i.e. $\delta=\omega(\xi)$, where $\omega(\xi)$ $=\int_a^{\xi}q^{1/3}(\vartheta)\,d\vartheta$, into the preceding formula, we get

$$x_1[\omega(t)]\cdot K\cdot\exp\left[-\int_{t_0}^{t}\frac{q'(\delta)-r(\delta)}{q^{4/3}(\delta)}\,q^{1/3}(\delta)\,d\delta\right]\to p_1 .$$

Dividing the fraction under the integral sign into two parts we finally have

$$Tz_1q^{-1}\exp\left[\int_{t_0}^{t}\frac{r(\delta)}{q(\delta)}\,d\delta\right]\to p_1,$$

where $Tz_1=w_1(t)=x_1[\omega(t)]e^{t_0}e^{-K}$.

Similarly for $k=2, 3, 4$ we have

$$Tz_kq^{-1/3}\exp\left[-\int_{t_0}^{t}\left[\tau_kq^{1/3}(\delta)+\frac{1}{3}\frac{r(\delta)}{q(\delta)}\right]d\delta\right]\to p_k .$$

Proof of Theorem 4. Let us denote in the equation (5) $V(s)=A_2g(s)$, $R(s)=A_1f(s)$ and now apply Theorem 1.

The matrix A_0 is the same as in Theorem 3, so its characteristic equation has distinct roots. Further $V'(s)$ is in $L[0,\infty)$ if and only if $g'(s)$ is in $L[0,\infty)$. From the proof of Theorem 3 it follows that $q''/q^{4/3}$ in $L[0, \infty)$ is sufficient, which is fulfilled by hypothesis. Similarly

$$\int_0^\infty |g^2(s)|\, \mathrm{d}s = \int_a^\infty \left| \left[\frac{q'(t)}{q^{7/6}(t)} \right]^2 \right| \mathrm{d}t < \infty$$

and so $V(s) \to 0$ as $s \to \infty$. From this hypothesis it also follows that

$$\int_0^\infty |R(s)|\, \mathrm{d}s = \int_a^\infty \left| A_1 \frac{r(t)}{q(t)} \right| \mathrm{d}t < \infty .$$

The characteristic equation of

$$A_0 + V(s) = \begin{bmatrix} f(s) & 1 & 0 & 0 \\ 0 & (2/3)g(s) & 1 & 0 \\ 0 & 0 & (1/3)g(s) & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

is

$$P(\tau) \equiv \tau^4 - 2g\tau^3 + \frac{11}{9} g^2 \tau^2 + \left(1 - \frac{2}{9} g^3\right)\tau - g = 0. \tag{17}$$

By the same consideration as in Theorem 3 we get that the roots of (17) may be written as

$$\tau_k(s) = \tau_k + \frac{2\tau_k^3 + 1}{4\tau_k^3 + 1} g(s) + \gamma_k(s), \quad k = 1, 2, 3, 4,$$

e.i. $\tau_1(s) = g(s) + \gamma_1(s)$, $\tau_k(s) = \tau_k + \frac{1}{3} g(s) + \gamma_k(s)$, $k = 2, 3, 4$ and $\gamma_k(s)$ is in $L[0, \infty)$. Then, by Theorem 1, we obtain that there are four linearly independent solutions $z_k(t)$ of (3) such that

$$Tz_1 q^{-1} \to p_1$$

$$Tz_k q^{-1/3} \exp\left[-\tau_k \int_{t_0}^t q^{1/3}(\delta)\, \mathrm{d}\delta \right] \to p_k, \quad k = 2, 3, 4.$$

Proof of Theorem 5. If $q(t) < 0$ instead of (1), we shall consider the equation $y^{(iv)} - \bar{q}(t) y' + r(t) y = 0$, where $\bar{q}(t) = -q(t) > 0$. Hence Lema 1 and Theorem 3 will remain if we replace $A_0(a_{ik})$ by $\bar{A}_0(|a_{ik}|)$, $q(t)$ by $-\bar{q}(t)$, characteristic roots of A_0 by characteristic roots of $\bar{A}_0$, which are $\tau_1 = 0$, $\tau_2 = 1$, $\tau_{3,4} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$, characteristic vectors of A_0 by characteristic vectors of $\bar{A}_0$ which are $p_1 \equiv (1, 0, 0, 0)$, $p_k = (1, \tau_k, \tau_k^2, 1)$, $k = 2, 3, 4$, characteristic vectors of A_0

$+ f(s)A_1 + g(s)A_2$ by characteristic vectors of $\bar{A}_0 + \bar{f}(s)A_1 + \bar{g}(s)A_2$, which may be written as

$$\tau_k(s)=\tau_k+\frac{2\bar{g}(s)\tau_k^3-\bar{f}(s)-\bar{g}(s)}{4\tau_k^3-1}+\gamma_k(s),$$

where

$$\bar{f}(s)=\frac{r(\alpha(s))}{\bar{q}^{4/3}(\alpha(s))},\quad \bar{g}(s)=\frac{\bar{q}'(\alpha(s))}{\bar{q}^{4/3}(\alpha(s))}.$$

Therefore there is a fundamental system $x_k(s)$ of

$$x'=(A_0+\bar{f}(s)A_1+\bar{g}(s)A_2)x$$

such that

$$x_k(s)\exp\left[-\int_{s_0}^{s}\tau_k(\delta)\,\mathrm{d}\delta\right]\to p_k.$$

Substituting $\alpha(\delta)=\xi$ and putting $-\bar{q}=q$ into the last expression we get the conclusion of the Theorem.

Proof of Theorem 6 is analogous to the proof of Theorem 4.

Proof of Theorem 7. Denote $V(s) = B_1h(s)+B_2k(s)$. Since

$$\int_0^\infty |h'(s)|\,\mathrm{d}s=\int_0^\infty |[q(\alpha(s))/r^{3/4}(\alpha(s))]'|\,\mathrm{d}s\leq\int_a^\infty |q'(t)/r^{3/4}(t)|\,\mathrm{d}t+$$

$$+\frac{4}{3}\int_a^\infty |[q(t)r'(t)]/r^{7/4}(t)|\,\mathrm{d}t\leq\int_a^\infty |q'(t)/r^{3/4}(t)|\,\mathrm{d}t+$$

$$+\frac{4}{3}\left[\int_a^\infty [q(t)/r^{5/8}(t)]^2\,\mathrm{d}t\right]^{1/2}\cdot\left[\int_a^\infty [r'(t)/r^{9/8}(t)]^2\,\mathrm{d}t\right]^{1/2}<\infty.$$

Likewise $k'(s)$, $k^2(s)$, $h^2(s)$ are in $L[0,\infty)$, and so $V'(s)$ is in $L[0,\infty)$ and $V(s)\to 0$ as $s\to\infty$.

The characteristic equation of $B_0+V(s)$ is

$$\text{(19)}\qquad P(\tau)\equiv\tau^4+\frac{3}{2}k\tau^3+\frac{5}{8}k^2\tau^2+\left(h-\frac{3}{42}k^3\right)\tau-\frac{3}{4}hk+1=0.$$

Similarly as in Theorem 3 the roots of (19) may be written as

$$\text{(20)}\qquad \tau_k(s)=\tau_k-\frac{3}{8}k(s)+\frac{1}{4}h(s)\tau_k^2+\gamma_k(s),$$

where τ_k are the roots of $\tau^4+1=0$ and $\gamma_k(s)$ are in $L[0,\infty)$ The functions $D_{jk}(s)$ are equal to c or $c+G(s)$, where $c\neq 0$ and $G(s)$ is in $L[0,\infty)$. Thus $j\in I_1$, resp. $j\in I_2$ for all k. All hypotheses of Theorem 1 are fulfilled and hence there is a fundamental system $x_k(s)$ of (7) such that

$$x_k(s)\exp\left[-\int_{s_0}^{s}\tau_k(\delta)\,d\delta\right]\to p_k\,.$$

After substituting $s=\omega(t)=\int_{t_0}^{t} r^{1/4}(\delta)\,d\delta$ we get the conclusion of this Theorem.

The proofs of the Theorems 8, 9, 10 are analogous to the proofs of the Theorems 4, 5, 6.

Proof of Corollary 1. Since $T=\operatorname{dia}[q, q^{2/3}, q^{1/3}, 1]$, $z_k=[y_k, y_k', y_k'', y_k''']^T$, then from Theorem 4 it follows that $\operatorname{dia}[q, q^{2/3}, q^{1/3}, 1]\cdot[y_1, y_1', y_1'', y_1''']^T\cdot q^{-1}\to(1, 0, 0, 0)^T$, i.e. $y_1\to 1$ and so $y_1=1+o(1)$.

Similarly for $k=2, 3, 4$ we get

$$y_k=-\tau_k^2 q^{-2/3}\exp\left(\tau_k\int_{t_0}^{t} q^{1/3}\,d\delta\right)(1+o(1)).$$

If we take Re y_3 and Im y_4, we get the assertion of the Corollary. This assertion is valid for $q(t)<0$ too.

The proof of Corollary 2 is analogous.

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АСЫМПТОТИЧЕСКОЕ ПОВЕДЕНИЕ РЕШЕНИЙ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ ЧЕТВЕРТОГО ПОРІДКА

Йосеф Ровдер

Резюме

В работе рассматриваются асимптотические поведения решений уравнения (1) при $t\to\infty$, если несобственные интегралы из некоторых дробей функций q и r являются конечными.