Mathematica Slovaca

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Mathematica Slovaca, Vol. 48 (1998), No. 2, 187--212

Persistent URL: http://dml.cz/dmlcz/129178

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ON THE ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF SOME FUNCTIONAL-DIFFERENTIAL EQUATIONS

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(*Communicated by Milan Medveď*)

ABSTRACT. The aim of the paper is to give asymptotic estimates of solutions of some retarded linear and sublinear differential equations. We give up conditions under which these estimates can be derived by means of solutions of certain linear differential and linear functional (nondifferential) equations.

0. Introduction

The question of the perturbation of the equation

$$y'(x) = b(x)y(x)\,, \qquad x \in [x_0, \infty)\,, \tag{E_1}$$

has been studied from many points of view. In the first part of this paper, we are going to discuss a similar problem which can be formulated as follows: Find conditions under which the functional-differential equations (FDE) of the form (E_3), resp. (E_4), given below admit solutions which can in a certain sense be compared with the solutions of (E_1).

Similar questions were dealt with in many papers; let us at least mention the recent results given by M. Pituk [10]. We obtain some results concerning the problem posed as a result of the investigation of the FDE of the form (E_2). For discussions relating to this equation, we refer to the paper of M. K. Grammatikopoulos and M. R. Kulenovič [2].

In the second part, we consider certain equations of the form (E_4) such that the conditions derived in the first paragraph are not satisfied. Our aim is to

AMS Subject Classification (1991): Primary 34K15, 34K25; Secondary 39B99.
Key words: functional-differential equation, asymptotic behaviour, transformation, functional equation.

Research partially supported by the grant # 201/93/0452 of the Czech Republic.

describe the asymptotic behavior of solutions of such equations. This part can be viewed as an application of the results derived by F. Neuman [9] and generalizes some partial results obtained by T. Kato and J. B. McLeod in [4].

Throughout this paper, we are going to deal with the scalar FDE

$$y'(x) = \sum_{k=1}^{m} a_k(x) f_k\big(y(\tau_k(x))\big)\,, \qquad x \in [x_0, \infty)\,, \tag{E_2}$$

$$y'(x) = \sum_{k=1}^{m} a_k(x) \big|y(\tau_k(x))\big|^{r_k} \operatorname{sgn} y\big(\tau_k(x)\big) + b(x)y(x)\,, \tag{E_3}$$

where $x \in [x_0, \infty)$, $r_k \in (0,1)$, $k = 1,2,\dots,m$,

$$y'(x) = \sum_{k=1}^{m} a_k(x) y\big(\tau_k(x)\big) + b(x)y(x)\,, \qquad x \in [x_0, \infty)\,, \tag{E_4}$$

and

$$y'(x) = a(x) y\big(\tau(x)\big) + b(x)y(x)\,, \qquad x \in [x_0, \infty)\,. \tag{E_4'}$$

The assumptions we impose on a_k, f_k, τ_k and b can be summarize as follows:

(A_1) $a_k \in C^0\big([x_0,\infty)\big)$, $k = 1,\dots,m$;

(A_1') $a_k(x) \ge 0$ (or $a_k(x) \le 0$) for every $x \in [x_0, \infty)$ and $k = 1,\dots,m$;

(A_2) $f_k \in C^0(\mathbb{R})$, $|f_k(y_1)| \le |f_k(y_2)|$ for any pair $y_1, y_2 \in \mathbb{R}$ such that $|y_1| \le |y_2|$, $k = 1,\dots,m$;

(A_3) $\tau_k \in C^0\big([x_0,\infty)\big)$, $\tau_k(x) < x$ for every $x \in [x_0,\infty)$ and $\lim_{x\to\infty} \tau_k(x) = \infty$, $k = 1,\dots,m$;

(A_4) $b \in C^0\big([x_0^*,\infty)\big)$, where $x_0^* = \min\Big\{ \inf_{x\ge x_0} \tau_1(x), \dots, \inf_{x \ge x_0} \tau_m(x) \Big\}$;

(A_4') $b \in C^0\big([x_0,\infty)\big)$ and $b(x) < 0$ for every $x \in [x_0,\infty)$;

(A_5) (i) $\sum_{k=1}^{m} |a_k(x)| + b(x) \ge 0$ for all x sufficiently large,

(ii) $\sum_{k=1}^{m} |a_k(x)| + b(x) \le 0$ for all x sufficiently large;

(A_5') there exists $\tau \in C^1\big([x_0,\infty)\big)$, $\tau(x) < x$, $\tau'(x) > 0$ for every $x \in [x_0,\infty)$, $\lim_{x\to\infty} \tau(x) = \infty$ such that either

(i) $\tau_k(x) \le \tau(x)$ in case (A_5) (i),

or

(ii) $\tau_k(x) \ge \tau(x)$ in case (A_5) (ii)

for arbitrary $x \in [x_0,\infty)$ and $k = 1,\dots,m$.

Let us remark that methods used below are applicable also for delays τ_k satisfying $\tau_k(x) \le x$; sometimes small modifications in the proofs are necessary.

In order to describe the behavior, as $x \to \infty$, of the unknown function $y(x)$ in terms of the known function $g(x)$, we shall use the following notations:

(a) If $\left|\frac{y(x)}{g(x)}\right|$ is bounded as $x \to \infty$, we write

$$y(x) = O\{g(x)\} \qquad \text{as} \quad x \to \infty,$$

in words, y is of order not exceeding g;

(b) If $\frac{y(x)}{g(x)}$ tends to zero as $x \to \infty$, we write

$$y(x) = o\{g(x)\} \qquad \text{as} \quad x \to \infty,$$

in words, y is of order less than g;

(c) If $\frac{y(x)}{g(x)}$ tends to one as $x \to \infty$, we write

$$y(x) \sim g(x) \qquad \text{as} \quad x \to \infty,$$

in words, y is asymptotic to g.

The main tools of the proof are based on the application of the Schauder fixed point theorem and on some results from the theory of functional (nondifferential) equations in a single variable. In this paper, we shall use the following version of this theorem.

THE SCHAUDER THEOREM. *If S is a convex closed subset of a Banach space B, and S^* a relatively compact subset of S, then every continuous mapping of S into S^* has a fixed point.*

As usual, for B we take the space of continuous and bounded functions on $[x_0, \infty)$ endowed with the sup-norm. Then it is enough to prove the uniform boundedness and equicontinuity of S^* on $[x_0, \infty)$ instead of the relative compactness. For this purpose, it may be useful to apply the following criterion (see [8]).

LEMMA 1. *The family S^* of functions is equicontinuous on $[x_0, \infty)$ if for any $\varepsilon > 0$ there exists a decomposition of the interval $[x_0, \infty)$ into a finite number of subintervals I_j, $j = 1, \dots, n$, such that*

$$|f(x_1) - f(x_2)| < \varepsilon$$

for every $f \in S^*$, $x_1, x_2 \in I_j$, $j = 1, \dots, n$.

1. FDE and linear differential equations

Let f_k, $k = 1, \dots, m$, be functions defined on $\mathbb{R}$. We denote

$$L_\xi := \sup_{r>0} \min\left\{\frac{r}{|f_1(|\xi| + r)|}, \dots, \frac{r}{|f_m(|\xi| + r)|}\right\}.$$

Let us start with the following theorem.

THEOREM 1. *Consider equation* (E_2) *subject to conditions* (A_1), (A_2), (A_3), *and let*

$$\int_{x_0}^{\infty}\left(\sum_{k=1}^{m}|a_k(x)|\right)\,\mathrm{d}x < L_\xi \tag{1}$$

hold for some nonzero $\xi \in \mathbb{R}$. *Then there exists a solution* $y(x)$ *of* (E_2) *satisfying*

$$\lim_{x\to\infty} y(x) = \xi\,. \tag{2}$$

Moreover, if we add condition (A_1') *and assume that there exists a solution* $y(x)$ *of* (E_2) *satisfying* (2) *with* $\xi \in \mathbb{R}$ *such that all the functions* f_k, $k = 1,\dots,m$, *have the same sign and no zeros on a neighbourhood of* ξ, *then*

$$\int_{x_0}^{\infty}\left(\sum_{k=1}^{m}|a_k(x)|\right)\,\mathrm{d}x < \infty\,.$$

Proof. By (1), we can choose $r^* > 0$ such that

$$\int_{x_0}^{\infty}\left(\sum_{k=1}^{m}|a_k(x)|\right)\,\mathrm{d}x \le \frac{r^*}{|f_j(|\xi| + r^*)|}$$

for every $j = 1,\dots,m$. Put $x_0^* := \min\left\{\inf_{x\ge x_0}\tau_1(x),\dots,\inf_{x\ge x_0}\tau_m(x)\right\}$ and consider the Banach space $B\big([x_0^*,\infty)\big)$ of continuous bounded functions on $[x_0^*,\infty)$ with the sup-norm $\|\ \|$. Further, let

$$S_\xi^{r^*} := \left\{y \in B\big([x_0^*,\infty)\big) \mid \|y-\xi\| \le r^*\right\}.$$

Define the operator $T\colon S_\xi^{r^*} \to B\big([x_0^*,\infty)\big)$ by $Ty = w$, where

$$w(x) = \begin{cases} \xi - \int_{x_0}^{\infty}\left(\sum_{k=1}^{m} a_k(s) f_k\big(y(\tau_k(s))\big)\right)\mathrm{d}s & \text{for } x \in [x_0^*, x_0]\,,\\ \xi - \int_{x}^{\infty}\left(\sum_{k=1}^{m} a_k(s) f_k\big(y(\tau_k(s))\big)\right)\mathrm{d}s & \text{for } x \in [x_0, \infty)\,. \end{cases}$$

Now, we verify the assumptions of the Schauder theorem. Obviously, $S_\xi^{r^*}$ is a convex and closed subset of $B\big([x_0^*,\infty)\big)$. We show that T is a mapping of $S_\xi^{r^*}$ into itself. Indeed,

$$\begin{aligned} |w(x)-\xi| &\le \int_{x_0}^{\infty}\left(\sum_{k=1}^{m}|a_k(s)|\left|f_k\big(y(\tau_k(s))\big)\right|\right)\mathrm{d}s\\ &\le \int_{x_0}^{\infty}\left(\sum_{k=1}^{m}|a_k(s)||f_k(|\xi|+r^*)|\right)\mathrm{d}s \le r^* \end{aligned}$$

for every $x \in [x_0^*, \infty)$.

To prove the continuity of T, we consider a convergent sequence $y_n \in S_\xi^{r^*}$ with limit $y \in S_\xi^{r^*}$, and let $Ty_n = w_n$, $Ty = w$. Then

$$\|w_n - w\| \le \int_{x_0}^{\infty} \left| \sum_{k=1}^{m} \Big(a_k(s) \big(f_k(y_n(\tau_k(s))) - f_k(y(\tau_k(s))) \big) \Big) \right| \, \mathrm{d}s .$$

Since

$$\lim_{n\to\infty} \sum_{k=1}^{m} \Big(a_k(s) \big(f_k(y_n(\tau_k(s))) - f_k(y(\tau_k(s))) \big) \Big) = 0$$

for every $s \in [x_0, \infty)$ and

$$\left| \sum_{k=1}^{m} \Big(a_k(s) \big(f_k(y_n(\tau_k(s))) - f_k(y(\tau_k(s))) \big) \Big) \right| \le 2 |f(|\xi| + r^*)| \sum_{k=1}^{m} |a_k(s)|$$

for every $s \in [x_0, \infty)$, the assumptions of the Lebesgue dominated convergence theorem are satisfied. This implies that $\lim\limits_{n\to\infty} \|w_n - w\| = 0$, hence T is continuous.

Further, $\|Ty\| \le |\xi| + r^*$ for any $y \in S_\xi^{r^*}$, i.e., $TS_\xi^{r^*}$ is uniformly bounded. To show that $TS_\xi^{r^*}$ is equicontinuous on $[x_0, \infty)$, we use Lemma 1. It is easy to see that for any $\varepsilon > 0$ there exists $x^* \in [x_0, \infty)$ such that the inequalities

$$\begin{aligned} |Ty(x_1) - Ty(x_2)| &\le \int_{x_1}^{x_2} \left(\sum_{k=1}^{m} |a_k(s)| \, |f_k(y(\tau_k(s)))| \right) \mathrm{d}s \\ &\le K \int_{x_1}^{x_2} \left(\sum_{k=1}^{m} |a_k(s)| \right) \mathrm{d}s < \varepsilon , \end{aligned}$$

where $K = \max\{ |f_1(|\xi| + r^*)|, \dots, |f_m(|\xi| + r^*)| \}$, hold for any $y \in S_\xi^{r^*}$ and any $x_2 > x_1 > x^*$. Since the interval $[x_0^*, x^*]$ is compact, by applying Lemma 1, we get that $TS_\xi^{r^*}$ is equicontinuous on $[x_0^*, \infty)$, hence, $TS_\xi^{r^*}$ is relatively compact. Then, by the Schauder theorem, there exists $y \in S_\xi^{r^*}$ such that $y = Ty$, in other words, there exists a solution $y(x)$ of (E_2) satisfying (2).

Conversely, let (2) hold for a solution $y(x)$ of (E_2), and denote by $[\alpha, \beta]$ an interval such that $\xi \in [\alpha, \beta]$, and the functions f_k have the same sign and no zeros on $[\alpha, \beta]$. Further, take $x_3 \in [x_0, \infty)$ sufficiently large such that $y(\tau_k(x)) \in [\alpha, \beta]$ for every $x \in [x_3, \infty)$ and $k = 1, \dots, m$ and put

$$\gamma := \min\{ |f_1(\alpha)|, \dots, |f_m(\alpha)| \} .$$

Then, using (A_1'), we get

$$\int\limits_{x_3}^{x}\left(\sum_{k=1}^{m}|a_k(s)|\right)\,\mathrm{d}s=\left|\int\limits_{x_3}^{x}\left(\sum_{k=1}^{m}a_k(s)\right)\,\mathrm{d}s\right|$$
$$\le\frac{1}{\gamma}\left|\int\limits_{x_3}^{x}\left(\sum_{k=1}^{m}a_k(s)f_k\big(y(\tau_k(s))\big)\right)\,\mathrm{d}s\right|$$
$$=\frac{1}{\gamma}|\,y(x)-y(x_3)|\,.$$

Letting $x\to\infty$ we can see that

$$\int\limits_{x_3}^{\infty}\left(\sum_{k=1}^{m}|a_k(s)|\right)\,\mathrm{d}s\le\frac{1}{\gamma}\big(|\xi|+|\beta|\big)<\infty\,.$$

□

Remark 1. Theorem 1 also holds for FDE of some other types, e.g., for equations with an advanced argument.

Now it may be useful to recall the following well-known result, which is stronger than the second part of Theorem 1 for equations (E_2) with $a_k(x)\ge 0$.

THEOREM 2. *Consider equation* (E_2) *subject to conditions* (A_1), (A_2), (A_3), *and let* $a_k(x)\ge 0$ *for every* $x\in[x_0,\infty)$, *and* $yf_k(y)>0$ *for every nonzero* $y\in\mathbb{R}$, $k=1,\dots,m$. *Then equation* (E_2) *has a bounded solution* $y(x)$ *if and only if*

$$\int\limits_{x_0}^{\infty}\left(\sum_{k=1}^{m}a_k(x)\right)\,\mathrm{d}x<\infty\,.$$

P r o o f. The proof is given in [2]. □

LEMMA 2. *Consider equation* (E_3), *resp.* (E_4), *and let conditions* (A_1), (A_3), (A_4) *be satisfied. If*

$$\int\limits_{x_0}^{\infty}\left(\sum_{k=1}^{m}|a_k(x)|\exp\left\{r_k\int\limits_{x_0}^{\tau_k(x)}b(u)\,\mathrm{d}u-\int\limits_{x_0}^{x}b(u)\,\mathrm{d}u\right\}\right)\,\mathrm{d}x<\infty\,,\tag{3}$$

resp.

$$\int\limits_{x_0}^{\infty}\left(\sum_{k=1}^{m}|a_k(x)|\exp\left\{-\int\limits_{\tau_k(x)}^{x}b(u)\,\mathrm{d}u\right\}\right)\,\mathrm{d}x<\infty\,,\tag{4}$$

then every solution $y(x)$ of (E_3), *resp.* (E_4), *satisfies the asymptotic relation*

$$\lim_{x\to\infty}\left(y(x)\exp\left\{-\int_{x_0}^{x} b(u)\,\mathrm{d}u\right\}\right) = L \in \mathbb{R}. \tag{5}$$

Proof. Take equation (E_3) with $r_k \in (0,1]$, i.e., at the same time we prove the assertion for equation (E_4). Put $z(x) = \exp\left\{-\int_{x_0}^{x} b(u)\,\mathrm{d}u\right\}y(x)$. Then equation (E_3) becomes

$$z'(x) = \sum_{k=1}^{m} a_k(x)\exp\left\{r_k\int_{x_0}^{\tau_k(x)} b(u)\,\mathrm{d}u - \int_{x_0}^{x} b(u)\,\mathrm{d}u\right\}\left|z\big(\tau_k(x)\big)\right|^{r_k}\operatorname{sgn} z\big(\tau_k(x)\big).$$

Put $d_0 := x_0$ and $d_j := \sup\limits_{x\in[d_{j-1},\infty)}\{x \mid \tau_k(x') \le d_{j-1}$ for every $x' \in [d_{j-1},x]$ and $k = 1,\dots,m\}$, $j = 1,2,\dots$. Now denote $I_j := [d_{j-1},d_j]$, $m_j := \sup\limits_{x\in I_j}\{|z(x)|\}$ and $M_j := \max\{1,m_1,\dots,m_j\}$, $j = 1,2,\dots$. Notice that $\bigcup\limits_{j=1}^{\infty} I_j = [x_0,\infty)$, and $\tau_k(I_{j+1}) \subset \bigcup\limits_{p=1}^{j} I_j$ for every $k = 1,\dots,m$ and $j = 1,2,\dots$.

Take $t \in I_{j+1}$ arbitrarily. Then

$$\begin{aligned} &z(t) - z(d_j)\\ &= \int_{d_j}^{t}\left(\sum_{k=1}^{m} a_k(x)\exp\left\{r_k\int_{x_0}^{\tau_k(x)} b(u)\,\mathrm{d}u - \int_{x_0}^{x} b(u)\,\mathrm{d}u\right\}\left|z\big(\tau_k(x)\big)\right|^{r_k}\operatorname{sgn} z\big(\tau_k(x)\big)\right)\mathrm{d}x, \end{aligned}$$

and we can carry out the following estimates:

$$\begin{aligned} |z(t)| &\le |z(d_j)| + M_j\int_{d_j}^{t}\left(\sum_{k=1}^{m}|a_k(x)|\exp\left\{r_k\int_{x_0}^{\tau_k(x)} b(u)\,\mathrm{d}u - \int_{x_0}^{x} b(u)\,\mathrm{d}u\right\}\right)\mathrm{d}x\\ &\le M_j\left(1 + \int_{d_j}^{d_{j+1}}\left(\sum_{k=1}^{m}|a_k(x)|\exp\left\{r_k\int_{x_0}^{\tau_k(x)} b(u)\,\mathrm{d}u - \int_{x_0}^{x} b(u)\,\mathrm{d}u\right\}\right)\mathrm{d}x\right). \end{aligned}$$

From here we obtain

$$m_{j+1} \le M_j\left(1 + \int_{d_j}^{d_{j+1}}\left(\sum_{k=1}^{m}|a_k(x)|\exp\left\{r_k\int_{x_0}^{\tau_k(x)} b(u)\,\mathrm{d}u - \int_{x_0}^{x} b(u)\,\mathrm{d}u\right\}\right)\mathrm{d}x\right),$$

hence,

$$m_{j+1} \le m_1 \prod_{p=1}^{j} \left(1 + \int_{d_p}^{d_{p+1}} \left(\sum_{k=1}^{m} |a_k(x)| \exp\left\{ r_k \int_{x_0}^{\tau_k(x)} b(u)\ \mathrm{d}u - \int_{x_0}^{x} b(u)\ \mathrm{d}u \right\} \right) \mathrm{d}x \right).$$

Because of (3), resp. (4), we get $m_j \le M$ for every $j = 1, 2, \dots$, what implies that $z(x)$ is bounded on $[x_0, \infty)$.

Further, let $x_2 > x_1 > x_0$. Then it follows from the previous estimates that

$$|z(x_2) - z(x_1)| \le M \int_{x_1}^{x_2} \left(\sum_{k=1}^{m} |a_k(x)| \exp\left\{ r_k \int_{x_0}^{\tau_k(x)} b(u)\ \mathrm{d}u - \int_{x_0}^{x} b(u)\ \mathrm{d}u \right\} \right) \mathrm{d}x .$$

Condition (3), resp. (4), implies that the left side of this inequality tends to zero as $x_1, x_2 \to \infty$. □

Remark 2. By Lemma 2, every solution $y(x)$ of (E_3), resp. (E_4), is either asymptotic to the function $c \exp\left\{ \int_{x_0}^{x} b(u)\ \mathrm{d}u \right\}$, $c \in \mathbb{R}$, or it is of order less than $\exp\left\{ \int_{x_0}^{x} b(u)\ \mathrm{d}u \right\}$.

Remark 3. Putting

$$z(x) = \exp\left\{ - \int_{x_0}^{x} b(u)\ \mathrm{d}u \right\} y(x),$$

equations (E_3) or (E_4) become the equation of the form (E_2), namely

$$z'(x) = \sum_{k=1}^{m} a_k(x) \exp\left\{ r_k \int_{x_0}^{\tau_k(x)} b(u)\ \mathrm{d}u - \int_{x_0}^{x} b(u)\ \mathrm{d}u \right\} \left| z(\tau_k(x)) \right|^{r_k} \operatorname{sgn} z(\tau_k(x)),$$

where $r_k \in (0, 1)$, or

$$z'(x) = \sum_{k=1}^{m} a_k(x) \exp\left\{ - \int_{\tau_k(x)}^{x} b(u)\ \mathrm{d}u \right\} z(\tau_k(x)),$$

respectively. Then Theorem 1 enables us to establish conditions ensuring the existence of a solution $y(x)$ of (E_3) or (E_4), which is asymptotic to $c \exp\left\{ \int_{x_0}^{x} b(u)\ \mathrm{d}u \right\}$, c being a real.

Theorem 1, Lemma 2 and Remark 3 yield the following two assertions.

Theorem 3. *Consider equations* (E_1) *and* (E_3) *subject to conditions* (A_1), (A_3), (A_4), *and let inequality* (3) *hold. Then for every solution* $y(x)$ *of* (E_3) *there exists a solution* $\overline{y}(x)$ *of* (E_1) *such that either*

$$y(x) \sim \overline{y}(x) \qquad \text{as} \quad x \to \infty\,, \tag{6}$$

or

$$y(x) = o\{\overline{y}(x)\} \qquad \text{as} \quad x \to \infty\,. \tag{7}$$

Conversely, for every solution $\overline{y}(x)$ *of* (E_1) *there exists a solution* $y(x)$ *of* (E_3) *such that* (6) *holds.*

Theorem 4. *Consider equations* (E_1) *and* (E_4) *subject to conditions* (A_1), (A_3), (A_4), *and let inequality* (4) *hold. Then for every solution* $y(x)$ *of* (E_4) *there exists a solution* $\overline{y}(x)$ *of* (E_1) *such that* (6) *or* (7) *is fulfilled. Moreover, if*

$$\int\limits_{x_0}^{\infty} \left(\sum_{k=1}^{m} |a_k(x)| \exp\left\{ - \int\limits_{\tau_k(x)}^{x} b(u)\,\mathrm{d}u \right\} \right) \mathrm{d}x < 1\,, \tag{4'}$$

then for every solution $\overline{y}(x)$ *of* (E_1) *there exists a solution* $y(x)$ *of* (E_4) *such that* (6) *holds.*

Remark 4. A result similar to that of Theorem 4 was obtained, e.g., in [10], where a different approach was used.

In the next statement, we show the strictness of condition (3) for the existence of a solution $y(x)$ of (E_3) asymptotic to a nonzero solution $\overline{y}(x)$ of (E_1).

Corollary 1. *Consider equations* (E_1) *and* (E_3) *subject to conditions* (A_1), (A_1'), (A_3), (A_4). *If there exists a solution* $y(x)$ *of* (E_3) *asymptotic to a nonzero solution* $\overline{y}(x)$ *of* (E_1), *then condition* (3) *follows.*

Proof. The statement follows from the second part of Theorem 1 according to Remark 3. □

The following example shows that condition (4′) in Theorem 4 cannot be improved.

Example 1. ([10; Example 1]) The equation

$$y'(x) = a(x)y(x-1)\,, \qquad x \in [0,\infty),$$

where

$$a(x) = \begin{cases} -2\sin^2 \pi x & \text{for } x \in [2,3]\,, \\ 0\,, & \text{otherwise,} \end{cases}$$

has every solution vanishing on the interval $[3,\infty)$. On the other hand,

$$\int_0^\infty |a(x)|\ \mathrm{d}x = 1\,.$$

Now we illustrate the above results by the following example.

EXAMPLE 2. We investigate the asymptotic behavior of solutions of the retarded sublinear equation

$$y'(x) = \exp\{ax\}|y(x^s)|^r \operatorname{sgn} y(x^s) + by(x)\,, \qquad x \in [2,\infty)\,,$$

where $a,b \in \mathbb{R}$, $r,s \in (0,1)$. First assume that $a < b$ or $a = b < 0$. Then

$$\int_2^\infty \exp\bigl\{ax + rb(x^s-2) - b(x-2)\bigr\}\ \mathrm{d}x < \infty\,.$$

Thus, according to Theorem 3, for every $c \in \mathbb{R}$ there exists a solution $y(x)$ of the equation investigated, which is asymptotic to $c\exp\{bx\}$. Moreover, those solutions $y(x)$ not having this property are of order less than $\exp\{bx\}$.

If $a > b$ or $a = b > 0$, we get

$$\int_2^\infty \exp\bigl\{ax + rb(x^s-2) - b(x-2)\bigr\}\ \mathrm{d}x = \infty\,,$$

hence, by applying Theorem 2 and Remark 3, this equation has no nonzero solution $y(x)$ of order not exceeding $\exp\{bx\}$.

2. Linear FDE and linear functional equations

Remark 5. The transformation

$$z(x) = \exp\biggl\{-\int_{x_0}^{x} b(u)\ \mathrm{d}u\biggr\}y(x)\,,$$

which has been used so far, makes the coefficient of the unknown function $y(x)$ vanish. Now we consider a rather more general transformation also involving a change of the independent variable as well, namely,

$$z(t) = \exp\biggl\{-\int_{t_0}^{h(t)} b(u)\ \mathrm{d}u\biggr\}y\bigl(h(t)\bigr)\,, \tag{8}$$

where $t_0 = h^{-1}(x_0)$ and h is a C^1-diffeomorphism from $[t_0,\infty)$ onto $[x_0,\infty)$. Sometimes it may be useful to convert equation (E_4) via (8) into an equation

$$z'(t) = \sum_{k=1}^{m} A_k(t) z(\mu_k(t)), \qquad t \in [t_0,\infty), \tag{E_5}$$

where $A_k(t) = a_k(h(t))h'(t)\exp\Big\{-\int\limits_{\tau_k(h(t))}^{h(t)} b(u)\,\mathrm{d}u\Big\}$ and $\mu_k(t) = h^{-1}(\tau_k(h(t)))$ for every $t \in [t_0,\infty)$. Notice that then

$$\lim_{t\to\infty} z(t) = \lim_{t\to\infty} \exp\Bigg\{-\int\limits_{t_0}^{h(t)} b(u)\,\mathrm{d}u\Bigg\} y(h(t)) = \lim_{x\to\infty} \exp\Bigg\{-\int\limits_{x_0}^{x} b(u)\,\mathrm{d}u\Bigg\} y(x).$$

Now consider equation (E_4'). There exist many results on global properties of equation (E_5) (with $m=1$) provided this equation has a constant delay, i.e., it is of the form

$$z'(t) = A(t)z(t-c), \qquad t \in [t_0,\infty), \quad c > 0 \tag{E_5'}$$

(for references, see, e.g., [7]). The problem of the existence of a suitable transformation converting a given equation (E_4') into equation (E_5') was solved in [9]. Let us recall the most important facts.

If $\mu(t) = t - c$, then the relation between τ and μ can be rewritten as

$$h(t-c) = \tau(h(t)), \qquad t \in [t_0,\infty), \ \ c > 0. \tag{9}$$

Putting $\varphi := h^{-1}$, equation (9) becomes

$$\varphi(\tau(x)) = \varphi(x) - c, \qquad x \in [x_0,\infty), \ c > 0. \tag{9'}$$

This equation is called Abel equation and has been very deeply studied within the framework of the theory of functional (nondifferential) equations in a single variable (for more details, see, e.g., [5]). This theory implies that by assuming $\tau \in C^1([x_0,\infty))$, $\tau(x) < x$ and $\tau'(x) > 0$ for every $x \in [x_0,\infty)$, we obtain, e.g., by using the step method, a C^1-solution $\varphi(x)$ of (9′) (and also a C^1-solution $h(t)$ of (9)) with a positive derivative on the interval of definition.

Then the transformation (8), where the function h satisfies (9), converts every solution $y(x)$ of equation (E_4') with continuous coefficients a, b and delay τ into a solution $z(t)$ of equation (E_5').

Using this approach we can, among other things, get further information about the asymptotic behavior of (E_4') with $a(x) < 0$ provided condition (4) is not fulfilled (notice that then, according to Theorem 1 and Remark 3, equation (E_4') has no solution $y(x)$ asymptotic to a nonzero solution $\overline{y}(x)$ of (E_1)).

We are going to generalize the following assertion proved in [6].

Theorem 5. *Consider equation* (E_5'), *where* A *is continuous on* $[t_0,\infty)$, $A(t) < 0$ *for every* $t \in [t_0,\infty)$, $\int_{t_0}^{\infty} |A(t)|\,\mathrm{d}t = \infty$ *and* $\lim_{t\to\infty} \int_{t-c}^{t} |A(s)|\,\mathrm{d}s < \frac{\pi}{2}$. *Then every solution* $z(t)$ *of* (E_5') *tends to zero as* $t \to \infty$.

Now it can be easily proved the following proposition.

Proposition 1. *Consider equation* (E_4'), *where the functions* a *or* b *are continuous on* $[x_0,\infty)$ *or* $[\tau(x_0),\infty)$, *respectively,* $a(x) < 0$ *for every* $x \in [x_0,\infty)$, *and* τ *is a continuously differentiable function with a positive derivative on* $[x_0,\infty)$ *fulfilling* $\tau(x) < x$ *for every* $x \in [x_0,\infty)$. *If*

$$\int_{x_0}^{\infty} \left(|a(x)| \exp\left\{ - \int_{\tau(x)}^{x} b(u)\,\mathrm{d}u \right\} \right) \mathrm{d}x = \infty$$

and

$$\lim_{x\to\infty} \int_{\tau(x)}^{x} \left(|a(s)| \exp\left\{ - \int_{\tau(s)}^{s} b(u)\,\mathrm{d}u \right\} \right) \mathrm{d}s < \frac{\pi}{2},$$

then (7) *is satisfied for any solution* $y(x)$ *of* (E_4') *and any solution* $\overline{y}(x)$ *of* (E_1).

Proof. It is enough to show that the statement can be covered by Theorem 5. Using (8) we get

$$A(t) = a\big(h(t)\big)h'(t) \exp\left\{ - \int_{\tau(h(t))}^{h(t)} b(u)\,\mathrm{d}u \right\},$$

i.e., the integral conditions of Proposition 1 can be converted to those of Theorem 5 using the integral substitution. □

Example 3. Consider the equation

$$y'(x) = \frac{a}{x} y(\lambda x) + \frac{b}{x} y(x), \qquad x \in [1,\infty),$$

where $a, b \in \mathbb{R}$, $a < 0$, $\lambda \in (0,1)$. Since

$$\int_{1}^{\infty} \left(\frac{|a|}{x} \exp\left\{ - \int_{\lambda x}^{x} \frac{b}{u}\,\mathrm{d}u \right\} \right) \mathrm{d}x = |a|\lambda^b \int_{1}^{\infty} \frac{\mathrm{d}x}{x} = \infty,$$

there is no solution $y(x)$ of this equation such that $y(x) \sim x^b$ as $x \to \infty$.

If $a > 0$, then with respect to Theorem 2 and Remark 3, we have no nonzero solution $y(x)$ of this equation such that $y(x) = O\{x^b\}$ as $x \to \infty$.

If $a < 0$, then we may apply Proposition 1. Since

$$\lim_{x\to\infty} \int_{\lambda x}^{x} \left(\left|\frac{a}{s}\right| \exp\left\{ -\int_{\lambda s}^{s} \frac{b}{u}\, \mathrm{d}u \right\} \right) \mathrm{d}s = |a| \lambda^{b} \ln \lambda^{-1} ,$$

we get that if $-\frac{\pi}{2} < a\lambda^b \ln \lambda^{-1} < 0$, then every solution $y(x)$ of the equation investigated satisfies $y(x) = o\{x^b\}$ as $x \to \infty$.

Notice that the equation

$$y'(x) = -\frac{1}{x} y\left(\exp\left\{-\frac{\pi}{2}\right\} x\right), \qquad x \in [1, \infty),$$

(i.e., $a = -1$, $b = 0$, $\lambda = \exp\{-\frac{\pi}{2}\}$) has a solution $y(x) = \sin \ln x$, which is not of order less than one. In other words, the constant $\frac{\pi}{2}$ in Proposition 1 cannot be improved.

Now we turn our attention to equations (E_4), resp. (E_4') with $b(x) < 0$. Condition $(4')$ or (4) is satisfied in such a case only for equations (E_4) or (E_4') with very small absolute value of $a_k(x)$ or $a(x)$, respectively. Notice that certain cases of equations of the form (E_4') with $a(x), b(x) < 0$ can be covered by Proposition 1 (see Example 3). In what follows, we wish to derive estimates of solutions of (E_4), resp. (E_4') under assumptions different from those of Proposition 1.

Suppose that assumptions (A_1), (A_3), (A_4'), (A_5) are satisfied, and let φ be an increasing C^1-diffeomorphism fulfilling $(9')$. Then we denote

$$\lambda_k(s) := \exp\{\varphi(\tau_k(\varphi^{-1}(s))) - s\}, \qquad s \in [\varphi(x_0), \infty), \tag{10}$$

and

$$\lambda_k := \begin{cases} \sup\limits_{s\in[\varphi(x_0),\infty)} \lambda_k(s) & \text{in case } (\mathrm{A}_5)(\mathrm{i}), \\ \inf\limits_{s\in[\varphi(x_0),\infty)} \lambda_k(s) & \text{in case } (\mathrm{A}_5)(\mathrm{ii}). \end{cases} \tag{10'}$$

In the next theorem, we shall require the existence of $\alpha \in \mathbb{R}$ such that the inequality

$$\sum_{k=1}^{m} \left(\lambda_k^{\alpha} |a_k(x)|\right) \le \alpha\varphi'(x) - b(x) \tag{11}$$

is satisfied for every $x \ge x_0$, x being sufficiently large. Therefore we denote

$$A := \{\alpha \in \mathbb{R} \mid \alpha \text{ satisfies (11) for all } x \text{ sufficiently large}\}.$$

If A is nonempty (notice that this holds, e.g., if $a_k(x) = O\{b(x)\}$ as $x \to \infty$), we put

$$\alpha^* := \inf A .$$

Moreover, if $\alpha^* > -\infty$, we introduce the function

$$\varepsilon(x) := \sum_{k=1}^{m} \left(\lambda_k^{\alpha^*} |a_k(x)|\right) - \alpha^* \varphi'(x) + b(x)$$

and

$$\overline{\varepsilon}(x) := \max\{\varepsilon(x), 0\}, \qquad x \in [x_0, \infty).$$

Notice that then α^* satisfies

$$\sum_{k=1}^{m} \left(\lambda_k^{\alpha^*} |a_k(x)|\right) \le \alpha^* \varphi'(x) - b(x) + \overline{\varepsilon}(x) \tag{11'}$$

for every $x \in [x_0, \infty)$.

Using this notation we have:

THEOREM 6. *Consider equation* (E_4) *subject to conditions* (A_1), (A_3), (A_1'), $(\mathrm{A}_5)(\mathrm{i})$, $(\mathrm{A}_5')(\mathrm{i})$. *If* $\alpha \in A$, *then every solution* $y(x)$ *of* (E_4) *satisfies*

$$y(x) = O\{\exp\{\alpha\varphi(x)\}\} \qquad \text{as} \quad x \to \infty.$$

Moreover, if $\int_{x_0}^{\infty} \overline{\varepsilon}(x)\, \mathrm{d}x$ *converges, then every solution* $y(x)$ *of* (E_4) *satisfies*

$$y(x) = O\{\exp\{\alpha^*\varphi(x)\}\} \qquad \text{as} \quad x \to \infty; \quad \alpha^* \ge 0.$$

THEOREM 7. *Consider equation* (E_4) *subject to conditions* (A_1), (A_3), (A_4'), $(\mathrm{A}_5)(\mathrm{ii})$, $(\mathrm{A}_5')(\mathrm{ii})$. *If* $\alpha \in A$, *then every solution* $y(x)$ *of* (E_4) *satisfies*

$$y(x) = O\{\exp\{\alpha\varphi(x)\}\} \qquad \text{as} \quad x \to \infty.$$

Moreover, if $\alpha^* > -\infty$, $\alpha^*\varphi'(x) - b(x) \ge 0$ *for every* $x \in [x_0, \infty)$, *and* $\int_{x_0}^{\infty} \overline{\varepsilon}(x)\, \mathrm{d}x$ *converges, then every solution* $y(x)$ *of* (E_4) *satisfies*

$$y(x) = O\{\exp\{\alpha^*\varphi(x)\}\} \qquad \text{as} \quad x \to \infty; \quad \alpha^* \le 0.$$

P r o o f . Since the technique of the proof does not depend on whether or not conditions $(\mathrm{A}_5)(\mathrm{i})$, $(\mathrm{A}_5')(\mathrm{i})$ or $(\mathrm{A}_5)(\mathrm{ii})$, $(\mathrm{A}_5')(\mathrm{ii})$ are assumed, we prove this statement under (A_1), (A_3), (A_4'), (A_5), (A_5').

It is enough to prove the O-estimate with α^* because the proof of the corresponding O-estimate with $\alpha \in A$ is involved.

We carry out the transformation

$$z(t) = \exp\{-\alpha^* t\} y(h(t)),$$

where $h = \varphi^{-1}$ is an increasing C^1-diffeomorphism fulfilling (9), converting equation (E_4) into an equation

$$z'(t) = \sum_{k=1}^{m}\Big(a_k\big(h(t)\big)h'(t)\exp\{\alpha^*\big(\mu_k(t)-t\big)\}z\big(\mu_k(t)\big)\Big) + \big(b\big(h(t)\big)h'(t)-\alpha^*\big)z(t)\,,$$

$t \in [\varphi(x_0), \infty)$, where $\mu_k(t) = h^{-1}\big(\tau_k\big(h(t)\big)\big)$, $k = 1,\dots,m$. Further,

$$\begin{aligned} &z'(t)\exp\left\{\alpha^* t - \int_{x_0}^{h(t)} b(u)\,\mathrm{d}u\right\} \\ &\qquad + z(t)\exp\left\{\alpha^* t - \int_{x_0}^{h(t)} b(u)\,\mathrm{d}u\right\}\big(\alpha^* - b\big(h(t)\big)h'(t)\big) \\ &= \sum_{k=1}^{m}\left(a_k\big(h(t)\big)h'(t)\exp\Big\{\alpha^*\big(\mu_k(t)-t\big)\Big\}\exp\left\{\alpha^* t - \int_{x_0}^{h(t)} b(u)\,\mathrm{d}u\right\}z\big(\mu_k(t)\big)\right), \end{aligned}$$

i.e.,

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left[z(t)\exp\left\{\alpha^* t - \int_{x_0}^{h(t)} b(u)\,\mathrm{d}u\right\}\right] \\ &= \sum_{k=1}^{m}\left(a_k\big(h(t)\big)h'(t)\exp\Big\{\alpha^*\big(\mu_k(t)-t\big)\Big\}\exp\left\{\alpha^* t - \int_{x_0}^{h(t)} b(u)\,\mathrm{d}u\right\}z\big(\mu_k(t)\big)\right). \end{aligned}$$

Now we denote by (d_j), $j = 1,2,\dots$, an increasing consequence of reals, where $d_0 := \varphi(x_0)$ and $d_j := \sup_{t\in[d_{j-1},\infty)} \{t \mid \mu_k(t') \le d_{j-1}$ for every $t' \in [d_{j-1}, t]$ and $k = 1,\dots,m\}$, $j = 1,2,\dots$. Further, let $I_j := [d_{j-1}, d_j]$, $m_j := \sup_{t\in I_j}\{|z(t)|\}$ and $M_j := \max\{m_1,\dots,m_j\}$, $j = 1,2,\dots$.

Choose $t \in I_{j+1}$ arbitrarily. Integrating the last relation we get

$$\left[z(s)\exp\left\{ \alpha^* s - \int_{x_0}^{h(s)} b(u)\,\mathrm{d}u \right\} \right]_{d_j}^{t}$$

$$= \sum_{k=1}^{m} \int_{d_j}^{t} \Bigg(a_k\big(h(s)\big)h'(s)\exp\Big\{\alpha^*\big(\mu_k(s)-s\big)\Big\} \cdot$$

$$\cdot \exp\left\{ \alpha^* s - \int_{x_0}^{h(s)} b(u)\,\mathrm{d}u \right\} z\big(\mu_k(s)\big) \Bigg)\,\mathrm{d}s\,,$$

hence,

$$z(t) = \exp\left\{ \alpha^*(d_j - t) + \int_{h(d_j)}^{h(t)} b(u)\,\mathrm{d}u \right\} z(d_j) + \exp\left\{ \int_{x_0}^{h(t)} b(u)\,\mathrm{d}u - \alpha^* t \right\} \cdot$$

$$\cdot \sum_{k=1}^{m} \int_{d_j}^{t} \Bigg(a_k\big(h(s)\big)h'(s)\exp\Big\{\alpha^*\big(\mu_k(s)-s\big)\Big\} \cdot$$

$$\cdot \exp\left\{ \alpha^* s - \int_{x_0}^{h(s)} b(u)\,\mathrm{d}u \right\} z\big(\mu_k(s)\big) \Bigg)\,\mathrm{d}s.$$

This implies

$$|z(t)| \le m_j \exp\left\{ \alpha^*(d_j - t) + \int_{h(d_j)}^{h(t)} b(u)\,\mathrm{d}u \right\} + M_j \exp\left\{ \int_{x_0}^{h(t)} b(u)\,\mathrm{d}u - \alpha^* t \right\} \cdot$$

$$\cdot \left| \sum_{k=1}^{m} \int_{d_j}^{t} \Bigg(a_k\big(h(s)\big)h'(s)\exp\Big\{\alpha^*\big(\mu_k(s)-s\big)\Big\} \cdot \right.$$

$$\left. \cdot \exp\left\{ \alpha^* s - \int_{x_0}^{h(s)} b(u)\,\mathrm{d}u \right\} \Bigg)\,\mathrm{d}s \right|. \qquad (12)$$

To estimate the last sum of integrals, we remark that (10) and (10′) imply that $\exp\{\mu_k(s) - s\} = \lambda_k(s)$, $\lambda_k \in (0,1)$, and $\lambda_k^{\alpha^*}(s) \le \lambda_k^{\alpha^*}$ for every $s \in [d_0, \infty)$

and $k = 1, \dots, m$. Then using (11′) and the assumptions of Theorem we get

$$\left| \sum_{k=1}^{m} \int_{d_j}^{t} \left(a_k\big(h(s)\big)h'(s)\lambda_k^{\alpha^*}(s) \exp\left\{ \alpha^* s - \int_{x_0}^{h(s)} b(u)\,\mathrm{d}u \right\} \right) \mathrm{d}s \right|$$

$$\leq \int_{d_j}^{t} \sum_{k=1}^{m} \left(\lambda_k^{\alpha^*} \left|a_k\big(h(s)\big)\right| h'(s) \exp\left\{ \alpha^* s - \int_{x_0}^{h(s)} b(u)\,\mathrm{d}u \right\} \right) \mathrm{d}s$$

$$\leq \int_{d_j}^{t} \left(\Big(\alpha^* - b\big(h(s)\big)h'(s)\Big) \exp\left\{ \alpha^* s - \int_{x_0}^{h(s)} b(u)\,\mathrm{d}u \right\} \right) \mathrm{d}s$$

$$+ \int_{d_j}^{t} \left(\bar{\varepsilon}\big(h(s)\big)h'(s) \exp\left\{ \alpha^* s - \int_{x_0}^{h(s)} b(u)\,\mathrm{d}u \right\} \right) \mathrm{d}s$$

$$\leq \left[\exp\left\{ \alpha^* s - \int_{x_0}^{h(s)} b(u)\,\mathrm{d}u \right\} \right]_{d_j}^{t} + \exp\left\{ \alpha^* t - \int_{x_0}^{h(t)} b(u)\,\mathrm{d}u \right\} \int_{d_j}^{d_{j+1}} \bar{\varepsilon}\big(h(s)\big)h'(s)\,\mathrm{d}s\,.$$

Substituting this into (12) we have

$$|z(t)| \leq M_j \exp\left\{ \alpha^*(d_j - t) + \int_{h(d_j)}^{h(t)} b(u)\,\mathrm{d}u \right\} + M_j$$

$$- M_j \exp\left\{ \alpha^*(d_j - t) + \int_{h(d_j)}^{h(t)} b(u)\,\mathrm{d}u \right\} + M_j \int_{d_j}^{d_{j+1}} \bar{\varepsilon}\big(h(s)\big)h'(s)\,\mathrm{d}s$$

$$= M_j \left(1 + \int_{d_j}^{d_{j+1}} \bar{\varepsilon}\big(h(s)\big)h'(s)\,\mathrm{d}s \right).$$

Since $t \in I_{j+1}$ was arbitrary, we get

$$m_{j+1} \leq M_j \left(1 + \int_{h(d_j)}^{h(d_{j+1})} \bar{\varepsilon}(u)\,\mathrm{d}u \right),$$

hence

$$m_{j+1} \leq m_1 \prod_{p=1}^{j} \left(1 + \int_{h(d_p)}^{h(d_{p+1})} \bar{\varepsilon}(u)\,\mathrm{d}u \right).$$

Thus, according to the convergence of the infinite product (as $j \to \infty$), we have

$$|z(t)| = \exp\{-\alpha^* t\}\big|y(\varphi^{-1}(t))\big| \le M$$

for every $t \ge d_0$.

As was remarked above, the proof of the first part of the assertion is quite similar to this one just presented (with $\alpha \in A$ instead of α^* and $\bar{\varepsilon}(x) = 0$ for every $x \in [x_0, \infty)$). □

EXAMPLE 4. Consider the equation

$$y'(x) = \frac{1+x}{x \ln \frac{x}{2}} y\Big(\frac{x}{2}\Big) + y\Big(\frac{x}{\mathrm{e}}\Big) - y(x), \qquad x \in [3, \infty). \tag{13}$$

Putting $\tau(x) = \frac{x}{2}$ we can verify the validity of (A_5')(i). Then equation (9′) becomes

$$\varphi\Big(\frac{x}{2}\Big) = \varphi(x) - c$$

having $\varphi(x) = \ln x$ as a required solution (with $c = \ln 2$). Thus inequality (11) acquires the form

$$\Big(\frac{1}{2}\Big)^{\alpha} \frac{1+x}{x \ln \frac{x}{2}} + \Big(\frac{1}{\mathrm{e}}\Big)^{\alpha} \le \frac{\alpha}{x} + 1,$$

which implies that $A = (0, \infty)$ and $\alpha^* = 0$. Now, according to the first part of Theorem 6, every solution $y(x)$ of (13) is of order not exceeding any positive power of x. On the other hand, the second part of Theorem 6 does not yield that every solution $y(x)$ of (13) satisfies $y(x) = O\{1\}$ as $x \to \infty$ (i.e., $y(x)$ is bounded) because

$$\bar{\varepsilon}(x) = \frac{1+x}{x \ln \frac{x}{2}}$$

and $\int_3^\infty \bar{\varepsilon}(x)\,\mathrm{d}x = \infty$. Notice that (13) in fact admits unbounded solutions, namely $y(x) = c \ln x$.

COROLLARY 2. *Consider equation* (E_4) *subject to conditions* (A_1), (A_3), (A_4'), (A_5), (A_5'). *Then we have:*

(i) *if* $\sum_{k=1}^{m} |a_k(x)| + b(x) \le 0$ *for all* x *sufficiently large, then every solution* $y(x)$ *of* (E_4) *is bounded;*

(ii) *if* $\sum_{k=1}^{m} |a_k(x)| + b(x) \ge 0$ *for all* x *sufficiently large and*

$$\int_{x_0}^{\infty} \Big(\sum_{k=1}^{m} |a_k(x)| + b(x)\Big)\,\mathrm{d}x < \infty,$$

then every solution $y(x)$ *of* (E_4) *is bounded;*

(iii) *if* $\sum_{k=1}^{m} |a_k(x)| + Lb(x) \le 0$ *for some* $L \in (0,1)$ *and every* $x \in [x_0,\infty)$ *sufficiently large, and if the function* $\frac{\varphi'(x)}{-b(x)}$ *is bounded for an increasing* C^1*-diffeomorphism* φ *fulfilling* (9′), *then every solution* $y(x)$ *of* (E_4) *tends to zero.*

Proof. The assumptions of (i), resp. (ii), implies that $\alpha^* \le 0$. Further, put $\lambda := \min\{\lambda_1,\dots,\lambda_m\}$, and let $\alpha_1 < 0$, $\overline{L} \in (L,1)$ be such that

$$\lambda^{\alpha_1} \sum_{k=1}^{m} |a_k(x)| \le \overline{L}(-b(x))$$

for every $x \in [x_0,\infty)$ sufficiently large. Assuming $\frac{\varphi'(x)}{-b(x)} \le M$ for every $x \in [x_0,\infty)$ we put

$$\alpha_2 := \frac{\overline{L}-1}{M} < 0 .$$

From here we get

$$\alpha_2\varphi'(x) \ge \left(\overline{L}-1\right)\left(-b(x)\right), \qquad x \in [x_0,\infty).$$

If we denote $\overline{\alpha} := \max\{\alpha_1,\alpha_2\}$, we have

$$\sum_{k=1}^{m} \lambda_k^{\overline{\alpha}} |a_k(x)| \le \lambda^{\overline{\alpha}} \sum_{k=1}^{m} |a_k(x)| \le -\overline{L}b(x) \le \overline{\alpha}\varphi'(x) - b(x),$$

x being sufficiently large, and this implies $\overline{\alpha} \in A$.

Now all the assertions follow from Theorem 7 with the respect to the unboundedness of φ. □

Remark 6. Corollary 2 gives conditions under which the zero solution of (E_4) is stable, resp. asymptotic stable. It might be interesting to compare these results with those obtained by the methods of the stability theory of FDE. These methods applied to equation (E_4) yield that the zero solution of (E_4) is uniformly asymptotic stable if $a_k(x)$, $b(x)$ are continuous bounded functions satisfying $b(x) \le -\delta < 0$, $\sum_{k=1}^{n} |a_k(x)| < L\delta$ for a suitable $L \in (0,1)$, and $\tau_k(x) = x - r_k(x)$ are continuous delays with bounded $r_k(x)$ (see [3; p. 154]).

Considering these delays, we can formulate the results of Corollary 2 in a very similar way. Indeed, if $\tau_k(x) = x - r_k(x)$, $0 \le r_k(x) \le r$, then $\tau(x) = x - r$, and equation (9′) with $c = r$ admits the identity function $\varphi(x) = x$ as a solution.

Now Theorem 7 implies that every solution $y(x)$ of (E_4) is of order not exceeding $\exp\{-\alpha x\}$, $\alpha < 0$.

In what follows, we consider equation (E_4') with $b(x) < 0$. Using similar ideas as in the previous part we obtain conditions under which every solution $y(x)$ of (E_4') can be approximated by a solution of a certain functional equation.

First we consider the equation

$$\alpha(t) = \alpha(t-1) + \ln \frac{a\big(h(t)\big)}{-b\big(h(t)\big)}, \qquad t \in [t_0, \infty), \tag{14}$$

where $a, b \in C^n\big([x_0,\infty)\big)$, $a(x) \neq 0$, $b(x) < 0$ for every $x \in [x_0, \infty)$, and h is a C^n-diffeomorphism from $[t_0,\infty)$ onto $[x_0,\infty)$, $n = 0,1,2,\dots$. This equation has a C^n-solution $\alpha(t)$ defined on $[t_0 - 1, \infty)$ which depends on an arbitrary function (see [5]). For

$$\alpha_r(t) := \mathrm{Re}\ \alpha(t), \quad \widetilde{\alpha_r'}(t) := \max\{-\alpha_r'(t), 0\}, \qquad t \in [t_0, \infty),$$

we have the following theorem:

THEOREM 8. *Consider equation* (E_4'), *where* $a, b, \tau \in C^n\big([x_0,\infty)\big)$, n *being specified later,* $a(x) \neq 0$, $b(x) < 0$, $\tau(x) < x$ *and* $\tau'(x) > 0$ *for every* $x \in [x_0, \infty)$. *Further, let* φ *be an increasing* C^n*-diffeomorphism fulfilling* (9′) (*with* $c = 1$), $h := \varphi^{-1}$ *on* $\big[\varphi(x_0), \infty\big)$, *and let* α *be a* C^n*-function fulfilling* (14). *Then we have:*

(i) *if* $n = 2$, *and there exists* $t' \in \big[\varphi(x_0), \infty\big)$ *such that*

$$\alpha_r'(t) - b\big(h(t)\big)h'(t) \geq 0 \qquad \textit{for every } t \geq t',$$

$$\frac{\widetilde{\alpha_r'}(t)}{\alpha_r'(t) - b\big(h(t)\big)h'(t)} \qquad \textit{is nonincreasing for every } t \geq t',$$

and

$$\int_{t'}^{\infty} \frac{\widetilde{\alpha_r'}(t)}{\alpha_r'(t) - b\big(h(t)\big)h'(t)}\, \mathrm{d}t < \infty,$$

then every solution $y(x)$ *of* (E_4') *satisfies*

$$y(x) = O\Big\{\exp\{\alpha_r(\varphi(x))\}\Big\} \qquad \textit{as} \quad x \to \infty;$$

(ii) *if* $n = \infty$, *and there exist* $t'' \in [\varphi(x_0), \infty)$ *and suitable real constants* $K > 0$, $\rho > 1$ *such that*

$$\left| \left(\frac{1}{-b(h(t))h'(t)} \right)^{(m)} \right| < \frac{K^{m+1} m^m}{t^{m+\rho}},$$

$$\left| \left(\frac{\alpha'(t)}{-b(h(t))h'(t)} \right)^{(m)} \right| < \frac{K^{m+1} m^m}{t^{m+\rho}}$$

for every $t \geq t''$ *and* $m = 0, 1, 2, \dots$, *then no solution* $y(x)$ *of* (E_4') *satisfies*

$$y(x) = o\Big\{\exp\{\alpha_r(\varphi(x))\}\Big\} \qquad as \quad x \to \infty,$$

except the trivial one.

P r o o f . The idea of the proof of part (i) is similar to that used in the proof of Theorem 6. The main difference consists in using a different technique in estimating the integral in (12).

Using (9) and (14) we get that the transformation

$$z(t) = \exp\{-\alpha(t)\} y(h(t))$$

converts equation (E_4') into the form

$$z'(t) = -b(h(t))h'(t)z(t-1) + \Big(b(h(t))h'(t) - \alpha'(t)\Big)z(t), \qquad t \in [\varphi(x_0), \infty). \tag{15}$$

This can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[z(t) \exp\left\{ \alpha(t) - \int_{x_0}^{h(t)} b(u)\,\mathrm{d}u \right\} \right]$$

$$= -\,b(h(t))h'(t) \exp\left\{ \alpha(t) - \int_{x_0}^{h(t)} b(u)\,\mathrm{d}u \right\} z(t-1).$$

Denote by (d_j), $j = 0, 1, 2, \dots$ an increasing consequence of reals, where $d_0 := t'$ and $d_j := d_0 + j d_0$, $j = 1, 2, \dots$. Further, let $I_j := [d_{j-1}, d_j]$, $M_j := \sup_{t \in I_j}\{|z(t)|\}$, $j = 1, 2, \dots$.

Choose $t \in I_{j+1}$ arbitrarily. Then

$$z(t) = \exp\left\{ \alpha(d_j) - \alpha(t) + \int_{h(d_j)}^{h(t)} b(u)\,\mathrm{d}u \right\} z(d_j) + \exp\left\{ \int_{x_0}^{h(t)} b(u)\,\mathrm{d}u - \alpha(t) \right\} \cdot$$
$$\cdot \int_{d_j}^{t} \left(-b\big(h(s)\big)h'(s) \exp\left\{ \alpha(s) - \int_{x_0}^{h(s)} b(u)\,\mathrm{d}u \right\} z(s-1) \right) \mathrm{d}s\,,$$

which implies

$$\begin{aligned} |z(t)| \le M_j \exp&\left\{ \alpha_r(d_j) - \alpha_r(t) + \int_{h(d_j)}^{h(t)} b(u)\,\mathrm{d}u \right\} \\ &+ M_j \exp\left\{ \int_{x_0}^{h(t)} b(u)\,\mathrm{d}u - \alpha_r(t) \right\} \cdot \\ &\cdot \int_{d_j}^{t} \left(-b\big(h(s)\big)h'(s) \exp\left\{ \alpha_r(s) - \int_{x_0}^{h(s)} b(u)\,\mathrm{d}u \right\} \right) \mathrm{d}s\,. \end{aligned} \tag{16}$$

To estimate this integral, we write

$$\begin{aligned} &\int_{d_j}^{t} \left(-b\big(h(s)\big)h'(s) \exp\left\{ \alpha_r(s) - \int_{x_0}^{h(s)} b(u)\,\mathrm{d}u \right\} \right) \mathrm{d}s \\ &= \left[\exp\left\{ \alpha_r(s) - \int_{x_0}^{h(s)} b(u)\,\mathrm{d}u \right\} \right]_{d_j}^{t} \\ &\qquad - \int_{d_j}^{t} \left(\alpha_r'(s) \exp\left\{ \alpha_r(s) - \int_{x_0}^{h(s)} b(u)\,\mathrm{d}u \right\} \right) \mathrm{d}s \\ &\le \left[\exp\left\{ \alpha_r(s) - \int_{x_0}^{h(s)} b(u)\,\mathrm{d}u \right\} \right]_{d_j}^{t} \\ &\qquad + \int_{d_j}^{t} \left(\widetilde{\alpha_r'}(s) \exp\left\{ \alpha_r(s) - \int_{x_0}^{h(s)} b(u)\,\mathrm{d}u \right\} \right) \mathrm{d}s\,. \end{aligned}$$

Then integrating by parts we have

$$\begin{aligned}
&\int_{d_j}^{t}\left(\widetilde{\alpha_r'}(s)\exp\left\{\alpha_r(s)-\int_{x_0}^{h(s)} b(u)\,\mathrm{d}u\right\}\right)\mathrm{d}s \\
&=\left[\exp\left\{\alpha_r(s)-\int_{x_0}^{h(s)} b(u)\,\mathrm{d}u\right\}\frac{\widetilde{\alpha_r'}(s)}{\alpha_r'(s)-b\big(h(s)\big)h'(s)}\right]_{d_j}^{t} \\
&\quad+\int_{d_j}^{t}\left(\exp\left\{\alpha_r(s)-\int_{x_0}^{h(s)} b(u)\,\mathrm{d}u\right\}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{-\widetilde{\alpha_r'}(s)}{\alpha_r'(s)-b\big(h(s)\big)h'(s)}\right)\right)\mathrm{d}s \\
&\le\left[\exp\left\{\alpha_r(s)-\int_{x_0}^{h(s)} b(u)\,\mathrm{d}u\right\}\frac{\widetilde{\alpha_r'}(s)}{\alpha_r'(s)-b\big(h(s)\big)h'(s)}\right]_{d_j}^{t} \\
&\quad+\exp\left\{\alpha_r(t)-\int_{x_0}^{h(t)} b(u)\,\mathrm{d}u\right\}\left[-\frac{\widetilde{\alpha_r'}(s)}{\alpha_r'(s)-b\big(h(s)\big)h'(s)}\right]_{d_j}^{t} \\
&=\left[\exp\left\{\alpha_r(s)-\int_{x_0}^{h(s)} b(u)\,\mathrm{d}u\right\}\right]_{d_j}^{t}\frac{\widetilde{\alpha_r'}(d_j)}{\alpha_r'(d_j)-b\big(h(d_j)\big)h'(d_j)}\,.
\end{aligned}$$

Substituting this into (16) we have

$$|z(t)|\le M_j\left(1+\frac{\widetilde{\alpha_r'}(d_j)}{\alpha_r'(d_j)-b\big(h(d_j)\big)h'(d_j)}\right),\qquad t\in I_{j+1}\,,$$

hence,

$$\begin{aligned}
M_{j+1}&\le M_j\left(1+\frac{\widetilde{\alpha_r'}(d_j)}{\alpha_r'(d_j)-b\big(h(d_j)\big)h'(d_j)}\right)\\
&\le M_1\prod_{p=1}^{j}\left(1+\frac{\widetilde{\alpha_r'}(d_p)}{\alpha_r'(d_p)-b\big(h(d_p)\big)h'(d_p)}\right),
\end{aligned}$$

$j=1,2,\dots$. Now applying Cauchy's integral criterion we see that the sequence $(M_j)_{j=1}^{\infty}$ is bounded as $j\to\infty$, and this proves (i).

Let the assumptions of (ii) be fulfilled. Rewrite equation (15) as

$$r(t)z'(t)=z(t-1)-s(t)z(t)\,,\qquad t\in\big[\varphi(x_0),\infty\big)\,,\tag{17}$$

where

$$r(t) = \frac{1}{-b\big(h(t)\big)h'(t)} \qquad \text{and} \qquad s(t) = 1 - \alpha'(t)r(t)\,.$$

This equation was investigated by N. G. de Bruijn [1], and we recall here the statement relevant to the proof of (ii):

Let $r, s \in C^\infty\big([\varphi(x_0), \infty)\big)$*, and let the positive real constants* K*,* ρ*,* $\rho > 1$ *satisfy*

$$|r^{(m)}(t)| < \frac{K^{m+1}m^m}{t^{m+\rho}}\,,$$
$$\left|\big(s(t)-1\big)^{(m)}\right| < \frac{K^{m+1}m^m}{t^{m+\rho}}$$

for every t *sufficiently large and* $m = 0, 1, 2, \dots$ *. Then equation* (17) *has no nontrivial solution* $z(t)$ *tending to zero as* $t \to \infty$*.*

The assertion of (ii) is now covered by this result. □

Remark 7. If $|a(x)| \ge -b(x)$, then it is possible to choose a solution $\alpha(t)$ of (14) such that $\alpha'_r(t) \ge 0$. Then $\widetilde{\alpha'_r}(t) = 0$, and we can omit the assumptions of Theorem 8 (i) and only specify $n = 1$ in Theorem 8 (i).

Remark 8. Putting $\psi(x) = \exp\big\{\alpha\big(\varphi(x)\big)\big\}$ we get that the function $\psi(x)$ fulfils the linear functional equation

$$\psi(x) = \frac{a(x)}{-b(x)}\psi\big(\tau(x)\big)\,, \qquad x \in [x_0, \infty)\,.$$

Then Theorem 8 implies that, under certain assumptions, every solution $y(x)$ of (E'_4) can be approximated by a solution $\psi(x)$ of this functional equation.

EXAMPLE 5. Consider the equation

$$y'(x) = axy(\lambda x) + by(x)\,, \qquad x \in [1, \infty)\,, \tag{18}$$

$a, b \in \mathbb{R}$, $b < 0$, $\lambda \in (0, 1)$. Since equation (9′) (with $c = 1$) becomes

$$\varphi(\lambda x) = \varphi(x) - 1\,, \qquad x \in [1, \infty)\,,$$

the function $\varphi(x) = \dfrac{\ln x}{\ln \lambda^{-1}}$ is the required C^∞-diffeomorphism, hence $h(t) = \lambda^{-t}$ is a C^∞-diffeomorphism fulfilling (9). Then equation (14) becomes

$$\alpha(t) = \alpha(t-1) + \ln\frac{a\lambda^{-t}}{-b}\,, \qquad t \in [0, \infty)\,,$$

and admits an infinitely differentiable solution $\alpha(t) = \big(\frac{t^2}{2} + \frac{t}{2}\big)(\ln \lambda^{-1}) + t \ln \frac{a}{-b}$.

Now it is easy to verify that the assumptions of (i) and (ii) are valid. Indeed,

$$\alpha_r'(t) - b\lambda^{-t} \ge 0 \qquad \text{for every } t \ge 0\,,$$

$$\frac{\widetilde{\alpha_r'}(t)}{\alpha_r'(t) - b\lambda^{-t}} \qquad \text{is nonincreasing for every } t \ge 0\,,$$

and

$$\int_0^\infty \frac{\widetilde{\alpha_r'}(t)}{\alpha_r'(t) - b\lambda^{-t}}\, \mathrm{d}t < \infty\,.$$

Similarly,

$$\left(\frac{1}{-b\big(h(t)\big)h'(t)}\right)^{(m)} = \frac{\ln^{m-1}\lambda}{b}\lambda^t$$

and

$$\left(\frac{\alpha'(t)}{-b\big(h(t)\big)h'(t)}\right)^{(m)} = \frac{\lambda^t}{-b}\Big((\ln^m\lambda)\Big(t+\tfrac{1}{2}\Big) + (\ln^{m-1}\lambda)\Big(m - \ln\tfrac{a}{-b}\Big)\Big)\,.$$

Since

$$\frac{|\ln^{m-1}\lambda|}{-b}\lambda^t t^{m+\rho} < K^{m+1}m^m$$

and

$$\frac{\lambda^t}{-b}\left|(\ln^m\lambda)\Big(t+\tfrac{1}{2}\Big) + (\ln^{m-1}\lambda)\Big(m - \ln\tfrac{a}{-b}\Big)\right| t^{m+\rho} < K^{m+1}m^m$$

for a suitable $K > 0$, $\rho > 1$, and for every $t \ge 1$ and $m = 0,1,2,\dots$, the assumptions of (ii) are valid as well.

Summarizing this, every solution $y(x)$ of (18) satisfies

$$y(x) = O\left\{\exp\left\{\frac{\ln^2 x}{2\ln\lambda^{-1}} + \frac{\ln x}{2} + \frac{\ln\frac{|a|}{-b}}{\ln\lambda^{-1}}\ln x\right\}\right\} \qquad \text{as} \quad x\to\infty\,,$$

and no nontrivial solution $y(x)$ of (18) satisfies

$$y(x) = o\left\{\exp\left\{\frac{\ln^2 x}{2\ln\lambda^{-1}} + \frac{\ln x}{2} + \frac{\ln\frac{|a|}{-b}}{\ln\lambda^{-1}}\ln x\right\}\right\} \qquad \text{as} \quad x\to\infty\,.$$

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Received June 1, 1995
Revised January 10, 1996

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