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UNIFORMLY DISTRIBUTED SEQUENCES OF POSITIVE INTEGERS IN BAIRE'S SPACE

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ABSTRACT. Topological properties of the set of all uniformly distributed sequences of positive integers in Baire's space S of all sequences of positive integers are investigated in this paper.

Introduction

In [6] the concept of uniformly distributed sequences of positive integers $\bmod m$ $(m \geq 2)$ and uniformly distributed sequences of positive integers in $\mathbb{Z}$ is introduced (see also [3], p. 305).

Let $a = \{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers. Denote by $A(j,m,N)$ the *number of terms among* $a_1,\ldots,a_N$ *that satisfy the congruence* $a_i \equiv j \pmod m$. The sequence a is said to be *uniformly distributed* $\bmod m$ if

$$\lim_{N\to\infty} \frac{A(j,m,N)}{N} = \frac{1}{m} \qquad (j = 1,2,\ldots,m) \tag{1}$$

and a is said to be *uniformly distributed in* $\mathbb{Z}$ if (1) is satisfied for every integer $m \geq 2$.

We recall the notion of *Baire's space* S *of all sequences of positive integers.* This means the metric space S endowed with the metric d defined on $S \times S$ in the following way:

Let $x = \{x_k\}_{k=1}^{\infty} \in S$, $y = \{y_k\}_{k=1}^{\infty} \in S$. If $x = y$, then $d(x,y) = 0$, if $x \neq y$, then

$$d(x,y) = \frac{1}{\min\{n:\ x_n \neq y_n\}}.$$

The space (S,d) is a complete metric space (cf. [1], pp. 185,190; [5], pp. 95–96).

The aim of this paper is the study of topological properties of the class of all such sequences in S that are uniformly distributed $\bmod m$ (uniformly distributed in $\mathbb{Z}$). Let us remark that the study of the class of uniformly distributed

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mod 1 sequences of real numbers from the topological point of view is contained in [2], pp. 72–74 (see also [4]) and from the metric point of view in [3], pp. 313–316 (see also [6]).

Uniformly distributed sequences of positive integers in the space S

Denote by U_m and U the *class of all uniformly distributed sequences of positive integers* $\bmod m$ and the *class of all uniformly distributed sequences of positive integers in* $\mathbb{Z}$, respectively. We shall study topological properties of sets U_m $(m \geq 2)$, U as subsets of the metric space S.

From the definition of the previous classes of sequences we get

$$U = \bigcap_{m=2}^{\infty} U_m . \tag{2}$$

The following theorem shows that the sets U_m $(m \geq 2)$ are "small' from the topological point of view.

Theorem 1. *The set* U_m $(m \geq 2)$ *is a dense set of the first Baire category in* S.

Proof. The density of U_m in S follows from the well-known fact that if two sequences differ only in a finite number of terms, then either each of them is uniformly distributed $\bmod m$ or none of them is uniformly di tributed $\bmod m$.

Define for $x = \{x_k\}_{k=1}^{\infty} \in S$ and fixed m, n the function g_n in the following way:

$$g_n(x) = \frac{1}{n} \sum_{k=1}^{n} e^{2\pi i \frac{x_k}{m}} \qquad (x = \{x_k\}_{k=1}^{\infty} \in S).$$

Evidently we have $|g_n(x)| \leq 1$ for each $x \in S$. The function g_n maps S into the metric space $\mathbb{C}$ of all complex numbers with the metric ρ, $\rho(z, z') = |z - z'|$, $z, z' \in \mathbb{C}$.

Denote by S^* the set of all $x = \{x_k\}_{k=1}^{\infty} \in S$ for which there exists the limit $\lim_{n\to\infty} g_n(x) \in \mathbb{C}$. Put $g(x) = \lim_{n\to\infty} g_n(x)$ for $x \in S^*$. Then the function g maps S^* into $\mathbb{C}$.

We shall show that:

(a) The function g_n (n is fixed) is a continuous function on S.
(b) The function g is discontinuous at each point $x \in S^*$ (S^* is regarded as a metric subspace of S).

Proof of (a). Let $a = \{a_k\}_{k=1}^{\infty} \in S$. Let us form the ball

$$K(a, \tfrac{1}{n}) = \{x \in S:\ d(x,a) < \tfrac{1}{n}\}.$$

If x belongs to $K(a, \frac{1}{n})$, then $x_k = a_k$ $(k = 1, \dots, n)$ and therefore $g_n(x) = g_n(a)$. The assertion (a) follows.

Proof of (b). Let $b = \{b_k\}_{k=1}^{\infty} \in S^*$. We shall show that the function $g: S^* \to \mathbb{C}$ is discontinuous at b.

We have two possibilities: 1) $|g(b)| < 1$ 2) $|g(b)| = 1$.

In the case 1) we put $\varepsilon_0 = 1 - |g(b)| > 0$. It suffices to prove that in each ball $K(b,\delta) = \{x \in S^*:\ d(x,b) < \delta\}$ of the subspace S^* of S there is a point $y = \{y_k\}_{k=1}^{\infty}$ such that $|g(y) - g(b)| \geq \varepsilon_0$.

Choose an s such that

$$\frac{1}{s} < \delta. \tag{3}$$

Put $y_k = b_k$ $(k = 1, 2, \dots, s)$ and $y_{s+l} = lm$ $(l = 1, 2, \dots)$, $y = \{y_k\}_{k=1}^{\infty}$. Then for $n = s + v$ we get

$$g_n(y) = g_{s+v}(y) = \frac{1}{n}\sum_{k=1}^{s} e^{2\pi i \frac{y_k}{m}} + \frac{1}{n}\sum_{k=s+1}^{s+v} e^{2\pi i \frac{y_k}{m}} =$$

$$\frac{1}{n}\sum_{k=1}^{s} e^{2\pi i \frac{b_k}{m}} + \frac{1}{n}\sum_{j=1}^{v} e^{2\pi i j} = o(1) + \frac{v}{n} = o(1) + \frac{n-s}{n}.$$

Hence $\lim_{n\to\infty} g_n(y) = g(y) = 1$ and so $y \in S^*$. Further, according to (3) the point y belongs to $K(b,\delta)$ and

$$|g(y) - g(b)| = |1 - g(b)| \geq 1 - |g(b)| = \varepsilon_0 > 0.$$

In the case 2) we have $|g(b)| = 1$. It suffices to show that in any ball $K(b,\delta)$ $(\delta > 0)$ there is a point y such that

$$|g(y) - g(b)| = 1. \tag{4}$$

Let $\delta > 0$. Choose s such that (3) holds. Let $z = \{z_k\}_{k=1}^{\infty}$ be a fixed sequence from U (e.g. we can choose $z_k = k$, $k = 1, 2, \dots$). Define $y = \{y_k\}_{k=1}^{\infty}$ in the following way: $y_k = b_k$ $(k = 1, 2, \dots, s)$, $y_k = z_k$ for $k > s$.

On account of the well-known criterion for uniformly distributed sequences of positive integers $\bmod m$ (cf. [3], p. 306, Theorem 1.2) we have

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} e^{2\pi i \frac{y_k}{m}} = 0.$$

Hence $g(y) = \lim_{n\to\infty} g_n(y) = 0$. Therefore $U_m \subset S^*$ and (4) evidently holds.

According to (a),(b) the function g is a limit function of the sequence $\{g_n\}_{n=1}^{\infty}$. The functions g_n $(n = 1, 2, \dots)$ are continuous on S and therefore $g_n|S^*$ $(n = 1, 2, \dots)$ are continuous on S^*. The function g being a function in the first Baire class on S^* has the following property: The set D_g of all discontinuity points of g in S^* is a set of the first Baire category in S^* (cf. [7], p. 185). Hence S^* is a set of the first Baire category in S^* and therefore in S, too.

Since $U_m \subset S^*$, the theorem follows. ∎

The following two theorems are immediate consequences of Theorem 1.

Theorem 2. *The set U is a dense set of the first Baire category in S.*

Theorem 3. *The set W of all sequences of positive integers that are uniformly distributed* mod m *for no $m \geq 2$ is a residual set in the space S.*

P r o o f. It follows from Theorem 1 that the set $\bigcup_{m=2}^{\infty} U_m$ is a set of the first Baire category in S. Therefore the set

$$W = S \setminus \bigcup_{m=2}^{\infty} U_m = \bigcap_{m=2}^{\infty} (S \setminus U_m)$$

is residual in S. ∎

We shall show that the set U belongs to the second Borel class in S.

Theorem 4. *The set U_m $(m \geq 2)$ is an $F_{\sigma\delta}$-set in S.*

According to (2) we get from Theorem 4:

Corollary. *The set U is an $F_{\sigma\delta}$-set in S.*

P r o o f o f T h e o r e m 4. It is proved in [3] (Theorem 1.2, p. 306) that a sequence $\{a_k\}_{k=1}^{\infty}$ of positive integers is uniformly distributed mod m if and only if we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} e^{2\pi i h \frac{a_k}{m}} = 0$$

for every $h = 1, 2, \dots, m-1$.

Put for $x = \{x_k\}_{k=1}^{\infty} \in S$ and fixed $h \in \{1, 2, \dots, m-1\}$

$$f_{n,h}(x) = \frac{1}{n} \sum_{k=1}^{n} e^{2\pi i h \frac{x_k}{m}}.$$

We can show that $f_{n,h}$ is a continuous function on S. (This can be shown analogously as the continuity of g_n in the proof of Theorem 1). Therefore on account of the quoted Theorem 1.2 from [3] we get

$$U_m = \bigcap_{h=1}^{m-1} \bigcap_{k=1}^{\infty} \bigcup_{s=1}^{\infty} \bigcap_{n=s}^{\infty} D(n,h,k), \tag{5}$$

where

$$D(n,h,k) = \left\{ x = \{x_j\}_{j=1}^{\infty} \in S : \ |f_{n,h}(x)| \le \frac{1}{k} \right\}.$$

The continuity of $f_{n,h}$ implies that $D(n,h,k)$ is a closed set in S. The assertion follows at once from (5). ■

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