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Mathematica Slovaca, Vol. 43 (1993), No. 4, 417--425

Persistent URL: http://dml.cz/dmlcz/129147

Mathematica Slovaca
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Slovak Academy of Sciences

Math. Slovaca, **43** (1993), No. 4, 417–425

CONVEX AUTOMORPHISMS OF A LATTICE

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(*Communicated by Tibor Katriňák*)

ABSTRACT. Let L be a lattice which can be decomposed into a direct product of finitely many directly indecomposable factors. There are described bijections f of L onto itself preserving convex sublattices in the sense that A is a convex sublattice of L if and only if so is $f(A)$.

V. I. Marmazeev investigated the lattice $C(L)$ of convex sublattices of a lattice L. Let $L = (L; \wedge, \vee)$ and $L' = (L'; \wedge, \vee)$ be lattices and $f\colon L \to L'$ a bijection. In [2] there is proved that f has the property

$$A \in C(L) \iff f(A) = \{f(a) :\ a \in A\} \in C(L')$$

if and only if for any $a, b \in L$

$$f([a \wedge b,\ a \vee b]) = [f(a) \wedge f(b),\ f(a) \vee f(b)] \tag{$*$}$$

holds.

Marmazeev calls bijections having this property *convex isomorphisms*. The aim of Marmazeev's paper [3] was to describe convex automorphisms of a lattice L (i.e. convex isomorphisms of L onto L) provided that

(α) any bounded chain in L is finite, and

(β) L can be decomposed into a direct product $L = L_1 \times \cdots \times L_n$ such that all L_i are directly indecomposable.

However, the author uses non-explained (non-standard) terms in the formulation of results and there are only short sketches of proofs there. Hence it is not possible to find out what the author's description says.

In the present paper we give a description of convex automorphisms of a lattice L supposing only (β). This is done in Theorem 10.

To prove the main theorem the following two lemmas are useful.

AMS Subject Classification (1991): Primary 06B99.
Key words: Lattice, Convex sublattice, Convex automorphism, Direct product decomposition.

1. LEMMA. *Let a lattice* $L = (L; \wedge, \vee)$ *be a direct product of lattices* $L_1, \dots, L_n$, $L = L_1 \times \dots \times L_n$. *A subset* A *of* L *forms a convex sublattice of* L *if and only if* A *is a direct product* $A = A_1 \times \dots \times A_n$ *of convex sublattices* A_i $(i \in \{1, \dots, n\})$ *of* L_i.

Proof. It is easy to see that if A_i is a convex sublattice of L_i $(i \in \{1, \dots, n\})$, then $A_1 \times \dots \times A_n$ is a convex sublattice of $L_1 \times \dots \times L_n$.

To prove the converse let A be a convex sublattice of $L_1 \times \dots \times L_n$. Denote $A_i = \{x \in L_i\colon$ there exists $(y_1, \dots, y_n) \in A$ with $y_i = x\}$. Evidently A_i is a sublattice of L_i. Obviously $A \subseteq A_1 \times \dots \times A_n$. To prove the opposite inclusion let $b = (b_1, \dots, b_n) \in A_1 \times \dots \times A_n$. For any $i \in \{1, \dots, n\}$ there exists ${}^ic = ({}^ic_1, \dots, {}^ic_n) \in A$ with ${}^ic_i = b_i$. Set $u = {}^1c \wedge \dots \wedge {}^nc$, $v = {}^1c \vee \dots \vee {}^nc$. Evidently $u, v \in A$ and $u \le b \le v$, hence $b \in A$. It remains to show that every A_i is a convex subset of L_i. Let $x, y \in A_i$, $z \in L_i$, $x \le z \le y$. Then there exist $\overline{x}, \overline{y} \in A$, $\overline{x} = (\overline{x}_1, \dots, \overline{x}_n)$, $\overline{y} = (\overline{y}_1, \dots, \overline{y}_n)$ with $\overline{x}_i = x$, $\overline{y}_i = y$. Consider $\overline{z} = (\overline{x}_1 \wedge \overline{y}_1, \dots, \overline{x}_{i-1} \wedge \overline{y}_{i-1}, z, \overline{x}_{i+1} \wedge \overline{y}_{i+1}, \dots, \overline{x}_n \wedge \overline{y}_n)$. Evidently $\overline{x} \wedge \overline{y} \le \overline{z} \le \overline{y}$, hence $\overline{z} \in A$ and $z \in A_i$.

2. LEMMA. (cf. [1]) *Let* $L = (L; \wedge, \vee)$ *and* $L' = (L'; \wedge, \vee)$ *be lattices and* $f\colon L \to L'$ *a bijection. The following two conditions are equivalent.*

(i) $A \in C(L) \iff f(A) \in C(L')$.

(ii) *There exist lattices* $C = (C; \wedge, \vee)$, $D = (D; \wedge, \vee)$ *and bijections* $g\colon L \to C \times D$, $h\colon L' \to C \times D$ *such that* g *is an isomorphism of* L *onto* $C \times D$, h *is an isomorphism of* L' *onto* $C \times D^d$ *and* $g = f \circ h$ $(D^d$ *is the dual of* $D)$.

3. THEOREM. *Let* L *be a lattice which can be decomposed into a direct product of lattices* L_i, $L = \prod(L_i \mid i \in I)$ $(I$ *is any nonempty set). Further let* π *be a bijection of* I *onto* I, f_i *for each* $i \in I$ *an isomorphism or a dual isomorphism of* L_i *onto* $L_{\pi(i)}$. *Then the mapping* $f\colon L \to L$ *defined by*

$$f(x)_{\pi(i)} = f_i(x_i) \qquad \text{for all} \quad x \in L, \quad i \in I$$

is a convex automorphism of L. *(For* $y \in L$ *and* $j \in I$ *by* y_j *the* j*th component of* y *is meant.)*

P r o o f . Evidently f is a bijection. By Marmazeev's theorem it is sufficient to prove that for any $a, b \in L$ the condition $(*)$ holds.

Let $a, b \in L$, $f(a) = u$, $f(b) = v$. By the definition of f there is $u_{\pi(i)} = f_i(a_i)$, $v_{\pi(i)} = f_i(b_i)$ for each $i \in I$.

a) To prove the inclusion $f\big([a \wedge b,\, a \vee b]\big) \subseteq [f(a) \wedge f(b),\, f(a) \vee f(b)]$ let $s \in f\big([a \wedge b,\, a \vee b]\big)$. Then there exists $c \in [a \wedge b,\, a \vee b]$ with $s = f(c)$. As $a \wedge b \le c \le a \vee b$, we have $a_i \wedge b_i \le c_i \le a_i \vee b_i$ for each $i \in I$, from which there follows $f_i(a_i) \wedge f_i(b_i) \le f_i(c_i) \le f_i(a_i) \vee f_i(b_i)$ for each $i \in I$ (it does not matter, if f_i is an isomorphism or a dual isomorphism). Hence $u_{\pi(i)} \wedge v_{\pi(i)} \le s_{\pi(i)} \le u_{\pi(i)} \vee v_{\pi(i)}$ for each $i \in I$, which means $s \in \big[f(a) \wedge f(b),\, f(a) \vee f(b)\big]$.

b) Now we are going to prove the inclusion $\big[f(a) \wedge f(b),\, f(a) \vee f(b)\big] \subseteq f\big([a \wedge b,\, a \vee b]\big)$. Let $s \in \big[f(a) \wedge f(b),\, f(a) \vee f(b)\big]$. Then $f(a)_{\pi(i)} \wedge f(b)_{\pi(i)} \le s_{\pi(i)} \le f(a)_{\pi(i)} \vee f(b)_{\pi(i)}$ for all $i \in I$.

Since f is a bijection, there exists $c \in L$ such that $s = f(c)$ and the last inequalities can be rewritten as $f_i(a_i) \wedge f_i(b_i) \le f_i(c_i) \le f_i(a_i) \vee f_i(b_i)$. Applying f_i^{-1}, which is also an isomorphism or a dual isomorphism, we get $a_i \wedge b_i \le c_i \le a_i \vee b_i$ for each $i \in I$. Consequently $c \in [a \wedge b,\, a \vee b]$ and $s = f(c) \in f\big([a \wedge b,\, a \vee b]\big)$.

In the sections 4–8 we will suppose that f is a convex automorphism of a lattice $L = L_1 \times \cdots \times L_n$, where L_i are directly indecomposable lattices. Without loss of generality we can also assume that every L_i has more than one element. The denotation $\overline{n}$ will be used for the set $\{1, \dots, n\}$.

4. Lemma. *Let $a = (a_1, \dots, a_n)$ be any fixed element of L, $f(a) = u = (u_1, \dots, u_n)$. For every $i \in \overline{n}$ there exists $k \in \overline{n}$ such that*

$$f\big(\{a_1\} \times \cdots \times L_i \times \cdots \times \{a_n\}\big) = \{u_1\} \times \cdots \times U_k \times \cdots \times \{u_n\}$$

for a convex sublattice U_k or L_k. Moreover the mapping $f_i\colon L_i \to U_k$ derived from

$$f\colon \{a_1\} \times \cdots \times L_i \times \cdots \times \{a_n\} \to \{u_1\} \times \cdots \times U_k \times \cdots \times \{u_n\}$$

in a natural way is an isomorphism or a dual isomorphism.

P r o o f . Let $L'_i = \{a_1\} \times \cdots \times L_i \times \cdots \times \{a_n\}$. As L'_i is a convex sublattice of L, $f(L'_i)$ is a convex sublattice of L, too. By 1, $f(L'_i) = U_1 \times \cdots \times U_n$, where U_j is a convex sublattice of L_j for each $j \in \overline{n}$. Evidently the restriction of f to L'_i, $f'_i = f \restriction L'_i$, is a convex isomorphism of L'_i onto $f(L'_i)$, hence there exist lattices C, D and isomorphisms $g\colon L'_i \to C \times D$, $h\colon f(L'_i) \to C \times D^d$ with $g = f'_i \circ h$, by 2. Since L_i is directly indecomposable, so is L'_i and consequently either C or D is a one-element set.

a) Suppose C has only one element. Let $\overline{g}$ and $\overline{h}$ be mappings $L'_i \to D$ and $f(L'_i) \to D^d$, respectively, assigned to g and h. Evidently $\overline{g}$, $\overline{h}$ are isomorphisms and $\overline{g} = f'_i \circ \overline{h}$. Since $\overline{g}$ is also an isomorphism of $L'^{\,d}_i$ onto D^d and $\overline{h}^{-1}$ is an isomorphism of D^d onto $f(L'_i)$, their composition $\overline{g} \circ \overline{h}^{-1} = f'_i$ is an isomorphism of $L'^{\,d}_i$ onto $f(L'_i)$ and a dual isomorphism of L'_i onto $f(L'_i)$. Using the fact that L'_i is directly indecomposable and hence so is $f(L'_i)$, we obtain that there is an index $k \in \overline{n}$ such that each U_j for $j \neq k$ is a one-element lattice. Because $a = (a_1, \dots, a_n) \in L'_i$, we have $f(a) = u = (u_1, \dots, u_n) \in f(L'_i)$, from which it follows that $f(\{a_1\} \times \cdots \times L_i \times \cdots \times \{a_n\}) = \{u_1\} \times \cdots \times U_k \times \cdots \times \{u_n\}$. Moreover since f'_i is a dual isomorphism of L'_i onto $f(L'_i)$, f_i is a dual isomorphism of L_i onto U_k.

b) Assume D is a one-element lattice. Analogously as in a) it can be proved that $f(L'_i) = \{u_1\} \times \cdots \times U_k \times \cdots \times \{u_n\}$ for a $k \in \overline{n}$ the mapping $f_i\colon L_i \to U_k$ is an isomorphism.

5. LEMMA. *Under the assumptions and denotations as in the previous lemma there is $U_k = L_k$.*

P r o o f . f^{-1} is also a convex automorphism of L and $f^{-1}(u) = a$, so using 4 for f^{-1} and elements u and $f^{-1}(u) = a$ we get

$$f^{-1}(\{u_1\} \times \cdots \times L_k \times \cdots \times \{u_n\}) = \{a_1\} \times \cdots \times B_j \times \cdots \times \{a_n\}$$

for a $j \in \overline{n}$. Applying f^{-1} to

$$\{u_1\} \times \cdots \times U_k \times \cdots \times \{u_n\} \subseteq \{u_1\} \times \cdots \times L_k \times \cdots \times \{u_n\}$$

we obtain

$$\{a_1\} \times \cdots \times L_i \times \cdots \times \{a_n\} \subseteq \{a_1\} \times \cdots \times B_j \times \cdots \times \{a_n\}.$$

Since L_i contains more than one element, it must be $i = j$, $L_i = B_j$. Hence $f(L'_i) = f(\{a_1\} \times \cdots \times L_i \times \cdots \times \{a_n\}) = f(\{a_1\} \times \cdots \times B_j \times \cdots \times \{a_n\}) = \{u_1\} \times \cdots \times L_k \times \cdots \times \{u_n\}$, so that $U_k = L_k$.

6. COROLLARY. *Let* $a = (a_1, \dots, a_n)$ *be any fixed element of* L, $f(a) = u = (u_1, \dots, u_n)$. *Then there exists a permutation* π *of the set* $\overline{n}$ *such that for each* $i \in \overline{n}$

$$f(\{a_1\} \times \cdots \times L_i \times \cdots \times \{a_n\}) = \{u_1\} \times \cdots \times L_{\pi(i)} \times \cdots \times \{u_n\}$$

and moreover the mapping $f_i \colon L_i \to L_{\pi(i)}$ *derived from*

$$f \colon \{a_1\} \times \cdots \times L_i \times \cdots \times \{a_n\} \to \{u_1\} \times \cdots \times L_{\pi(i)} \times \cdots \times \{u_n\}$$

in a natural way is an isomorphism or a dual isomorphism.

Proof. Using denotation as in 4, set $\pi(i) = k$. To verify that π is a permutation, it is sufficient to show that $\pi(i_1) = \pi(i_2)$ implies $i_1 = i_2$. Now if $\pi(i_1) = \pi(i_2) = k$, then

$$\begin{aligned} f(\{a_1\} \times \cdots \times L_{i_1} \times \cdots \times \{a_n\}) &= \{u_1\} \times \cdots \times L_k \times \cdots \times \{u_n\} \\ &= f(\{a_1\} \times \cdots \times L_{i_2} \times \cdots \times \{a_n\}). \end{aligned}$$

Recall that f is a bijection. Hence it must be

$$\{a_1\} \times \cdots \times L_{i_1} \times \cdots \times \{a_n\} = \{a_1\} \times \cdots \times L_{i_2} \times \cdots \times \{a_n\}.$$

By assumption L_{i_1}, L_{i_2} are not one-element sets, therefore $i_1 = i_2$. In view of 4 and 5 the further assertions are evident.

We want to make clear if the permutation π and the mappings f_i depend on the choice of $a \in L$ or not.

7. LEMMA. *Let* $a = (a_1, \dots, a_n)$ *be any fixed element of* L, $f(a) = u = (u_1, \dots, u_n)$. *Then for any* $i, j \in \overline{n}$, $i < j$

$$f(\{a_1\} \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times \{a_n\}) = \{u_1\} \times \cdots \times L_k \times \cdots \times L_l \times \cdots \times \{u_n\}$$

holds, where $\{k, l\} = \{\pi(i), \pi(j)\}$.

Proof. Applying f to

$$\{a_1\} \times \cdots \times L_i \times \cdots \times \{a_n\} \subseteq \{a_1\} \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times \{a_n\}$$

and to

$$\{a_1\} \times \cdots \times L_j \times \cdots \times \{a_n\} \subseteq \{a_1\} \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times \{a_n\},$$

we get

$$\{u_1\} \times \cdots \times L_{\pi(i)} \times \cdots \times \{u_n\} \subseteq f\big(\{a_1\} \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times \{a_n\}\big)\,, \tag{1}$$

$$\{u_1\} \times \cdots \times L_{\pi(j)} \times \cdots \times \{u_n\} \subseteq f\big(\{a_1\} \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times \{a_n\}\big)\,, \tag{2}$$

by 6. Since $\{a_1\} \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times \{a_n\}$ is a convex sublattice of L, so is $f\big(\{a_1\} \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times \{a_n\}\big)$. Therefore by 1 $f\big(\{a_1\} \times \dots \cdots \times L_i \times \cdots \times L_j \times \cdots \times \{a_n\}\big) = U_1 \times \cdots \times U_n$, with U_t being a convex sublattice of L_t for each $t \in \overline{n}$.

Now $U_1 \times \cdots \times U_n \supseteq \{u_1\} \times \cdots \times L_{\pi(i)} \times \cdots \times \{u_n\}$ by (1) and $U_1 \times \cdots \times U_n \supseteq \{u_1\} \times \cdots \times L_{\pi(j)} \times \cdots \times \{u_n\}$ by (2). So $L_{\pi(i)} = U_{\pi(i)}$ and $L_{\pi(j)} = U_{\pi(j)}$ and we have

$$f\big(\{a_1\} \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times \{a_n\}\big) = U_1 \times \cdots \times L_k \times \cdots \times L_l \times \cdots \times U_n\,,$$

where $\{k, l\} = \big\{\pi(i),\, \pi(j)\big\}$.

Consider the mapping f^{-1}, which is also a convex automorphism of L, and elements u and $f^{-1}(u) = a$. Evidently there holds an analogous statement as in 6, the belonging permutation is π^{-1}. Using analogous considerations as above we get

$$f^{-1}\big(\{u_1\} \times \cdots \times L_k \times \cdots \times L_l \times \cdots \times \{u_n\}\big) = V_1 \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times V_n\,,$$

where V_t is a convex sublattice of L_t, for each $t \in \overline{n} - \{i,\, j\}$.
Applying f to

$$\{a_1\} \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times \{a_n\} \subseteq V_1 \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times V_n$$

($a_t \in V_t$ for each $t \in \overline{n} - \{i,\, j\}$ because $a = f^{-1}(u) \in f^{-1}\big(\{u_1\} \times \cdots \times L_k \times \cdots \times L_l \times \cdots \times \{u_n\}\big) = V_1 \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times V_n$), we get

$$U_1 \times \cdots \times L_k \times \cdots \times L_l \times \cdots \times U_n \subseteq \{u_1\} \times \cdots \times L_k \times \cdots \times L_l \times \cdots \times \{u_n\}\,,$$

so that $U_1 = \{u_1\}, \dots, U_n = \{u_n\}$. Hence

$$f\big(\{a_1\} \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times \{a_n\}\big) = \{u_1\} \times \cdots \times L_k \times \cdots \times L_l \times \cdots \times \{u_n\}$$

and the proof is finished.

8. Corollary. *The permutation* π *and the mappings* $f_i\colon L_i \to L_{\pi(i)}$ *mentioned in* 6 *do not depend on the choice of the element* $a \in L$.

Proof. Obviously it is sufficient to prove that if elements a, a' differ in one component, then the belonging permutations π and π' and mappings $f_t\colon L_t \to L_{\pi(t)}$, $f'_t\colon L_t \to L_{\pi'(t)}$ $(t \in \overline{n})$ are equal.

Let $a = (a_1, \dots, a_i, \dots, a_n)$, $a' = (a_1, \dots, a'_i, \dots, a_n)$, hence a, a' differ in the i th component. Further let $f(a) = u = (u_1, \dots, u_n)$. First we show that $\pi = \pi'$. Since $a' \in \{a_1\} \times \cdots \times L_i \times \cdots \times \{a_n\}$, there is

$$f(a') \in f\big(\{a_1\} \times \cdots \times L_i \times \cdots \times \{a_n\}\big) = \{u_1\} \times \cdots \times L_{\pi(i)} \times \cdots \times \{u_n\}, \quad (3)$$

by 4, 5, 6. Therefore $f(a')$ differs from $u = f(a)$ only in the $\pi(i)$ th component, $f(a') = \big(u_1, \dots, u'_{\pi(i)}, \dots, u_n\big)$. By 4, 5, 6 used for a' instead of a we obtain

$$f\big(\{a_1\} \times \cdots \times L_i \times \cdots \times \{a_n\}\big) = \{u_1\} \times \cdots \times L_{\pi(i)} \times \cdots \times \{u_n\} \qquad (4)$$

and comparing (3) and (4) we have $\pi(i) = \pi'(i)$. Now let $j \in \overline{n}$, $j \neq i$, e.g. $j > i$. Use 7 first for a, then for a'. We get that the set $f\big(\{a_1\} \times \cdots \times L_i \times \cdots \times L_j \times \cdots \times \{a_n\}\big)$ is equal to $\{u_1\} \times \cdots \times L_k \times \cdots \times L_l \times \cdots \times \{u_n\}$ ($\{k,l\} = \{\pi(i), \pi(j)\}$) and also to $\{u_1\} \times \cdots \times L_{k'} \times \cdots \times L_{l'} \times \cdots \times \{u_n\}$ ($\{k',l'\} = \{\pi'(i), \pi'(j)\}$). Hence $\{k,l\} = \{k',l'\}$ and as $\pi(i) = \pi'(i)$, it must be also $\pi(j) = \pi'(j)$. We have proved $\pi = \pi'$.

If $x_i \in L_i$, then $f_i(x_i)$ is the $\pi(i)$ th component of $f(a_1, \dots, x_i, \dots, a_n)$ and $f'_i(x_i)$ is the $\pi'(i)$ th component of the same element $f(a_1, \dots, x_i, \dots, a_n)$. But we already know that $\pi(i) = \pi'(i)$, so that $f_i(x_i) = f'_i(x_i)$. We have proved that $f_i = f'_i$. Now let $j \in \overline{n}$, $j \neq i$. Without loss of generality we can suppose that $j > i$. Take any $x_j \in L_j$. Using 4, 5, 6 for the element a we get that $f(a_1, \dots, a_i, \dots, x_j, \dots, a_n)$ has the same components as $f(a)$, with the exception of the $\pi(j)$ th component, which is $f_j(x_j)$. If we use 4, 5, 6 for a', we obtain that $f(a_1, \dots, a'_i, \dots, x_j, \dots, a_n)$ has such components as $f(a')$ besides the $\pi(j)$ th component, which is $f'_j(x_j)$. Use once more 4 and 5, now for $(a_1, \dots, a_i, \dots, x_j, \dots, a_n)$. We get

$$f\big(\{a_1\} \times \cdots \times L_i \times \cdots \times \{x_j\} \times \cdots \times \{a_n\}\big) = \{u_1\} \times \cdots \times \big\{f_j(x_j)\big\} \times \cdots \times \{u_n\}$$

(the $\pi(i)$ th factor is $L_{\pi(i)}$, $\big\{f_j(x_j)\big\}$ is the $\pi(j)$ th factor).

Evidently $(a_1, \dots, a'_i, \dots, x_j, \dots, a_n) \in \{a_1\} \times \cdots \times L_i \times \cdots \times \{x_j\} \times \cdots \times \{a_n\}$, so $f(a_1, \dots, a'_i, \dots, x_j, \dots, a_n) \in \{u_1\} \times \cdots \times \big\{f_j(x_j)\big\} \times \cdots \times \{u_n\}$. We know that the $\pi(j)$ th component of $f(a_1, \dots, a'_i, \dots, x_j, \dots, a_n)$ is $f'_j(x_j)$, therefore $f'_j(x_j) \in \big\{f_j(x_j)\big\}$, which means $f'_j(x_j) = f_j(x_j)$. We have proved also $f_j = f'_j$ for each $j \in \overline{n}$, $j \neq i$.

9. THEOREM. *Let L be a lattice, that is a direct product of finitely many directly indecomposable lattices L_i, $L = L_1 \times \dots \times L_n$. Further let f be a convex automorphism of L. Then there exist a permutation π of $\overline{n}$ and mappings $f_i\colon L_i \to L_{\pi(i)}$, all being isomorphisms or dual isomorphisms, such that for every $x \in L$*

$$f(x)_{\pi(i)} = f_i(x_i)$$

holds.

Proof. First suppose that every L_i has more than one element. Let $a = (a_1, \dots, a_n)$ be any fixed element of L. Take the permutation π and the mappings $f_i\colon L_i \to L_{\pi(i)}$ belonging to the element a by 6. Now let $x = (x_1, \dots, x_n)$ be any element of L. It is sufficient to show that for each $i \in \overline{n}$ there holds $f(x)_{\pi(i)} = f_i(x_i)$. In view of 8 the permutation π and the mappings f_i belong also to x, so that if $f(x) = y = (y_1, \dots, y_n)$, then for any $i \in \overline{n}$ $y_{\pi(i)} = f_i(x_i)$, by 6.

Now assume that some of L_i are one-element lattices. Without loss of generality we can suppose that one-element lattices are those for $i \in \overline{n} - \overline{k}$ ($k \in \overline{n}$, $k < n$). Evidently f determines a convex automorphism f' of $L' = L_1 \times \dots \times L_k$. Hence there exist a permutation π' of $\overline{k}$ and mappings $f_i\colon L_i \to L_{\pi'(i)}$ for $i \in \overline{k}$, all being isomorphisms or dual isomorphisms, such that for every $x' \in L'$ there holds $f'(x')_{\pi'(i)} = f_i(x'_i)$. Now extend π' to a permutation π of $\overline{n}$ and define $f_i\colon L_i \to L_{\pi(i)}$ for $i \in \overline{n} - \overline{k}$ in a natural way (i.e. if $L_i = \{a_i\}$, $L_{\pi(i)} = \{a_{\pi(i)}\}$, then $f_i(a_i) = a_{\pi(i)}$). It is easy to see that such a π and $f_i\colon L_i \to L_{\pi(i)}$ are as we need. The proof is complete.

The following theorem is obtained by combining the preceding one with 3.

10. THEOREM. *Let L be a lattice which can be decomposed into a direct product $L = L_1 \times \dots \times L_n$, where all L_i are directly indecomposable. Convex automorphisms of L are just the mappings obtained as follows: we take a permutation π of the set $\overline{n}$ such that there exist bijections $f_i\colon L_i \to L_{\pi(i)}$, each of them being either an isomorphism or a dual isomorphism, and we set for any $x \in L$ $f(x)_{\pi(i)} = f_i(x_i)$.*

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Received February 10, 1992

Revised July 9, 1992

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