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# SEMIDOMATIC NUMBERS OF DIRECTED GRAPHS

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In [1] E. J. Cockayne and S. T. Hedetniemi have introduced the concept of the domatic number of an undirected graph. In [2] this concept was transferred to directed graphs. Here we shall define two generalizations of the domatic numbers of directed graphs.

Let $G$ be a directed graph with the vertex set $V(G)$. A subset $D$ of $V(G)$ is called inside-semidominating (or outside-semidominating) in $G$ if to each vertex $x \in V(G) - D$ there exists a vertex $y \in D$ such that the edge $\overrightarrow{xy}$ (or $\overrightarrow{yx}$, respectively) belongs to $G$. An inside-domatic (or outside-domatic) partition of $G$ is a partition of $V(G)$, all of whose classes are inside-semidominating (or outside-semidominating) sets in $G$. The maximum number of classes of an inside-semidomatic (or outside-semidomatic) partition of $G$ is called the inside-semidomatic (or outside-semidomatic) number of $G$ and is denoted by $d^-(G)$ (or $d^+(G)$, respectively). Note that these numbers are defined for all directed graphs, because a partition of $V(G)$ consisting of one class is simultaneously an inside-semidomatic partition of $G$ and an outside-semidomatic one.

Now a dominating set in a directed graph $G$ can be defined as a subset of $V(G)$ which is simultaneously inside-semidominating and outside-semidominating. The domatic number $d(G)$ of $G$ is the maximum number of classes of a domatic partition of $G$, i.e. of a partition, all of whose classes are dominating sets in $G$. This implies the following assertion.

**Proposition 1.** *Let $G$ be a directed graph, let $d^-(G)$, $d^+(G)$, $d(G)$ be its inside-semidomatic, outside-semidomatic and domatic number respectively. Then*

$$d^-(G) \geqq d(G),$$

$$d^+(G) \geqq d(G).$$

Also the following assertion is evident.

**Proposition 2.** *Let $G$ be a directed graph, let $\hat{G}$ be the graph obtained from $G$ by reversing orientations of all edges. Then*

$$d^-(\hat{G}) = d^+(G),$$
$$d^+(\hat{G}) = d^-(G).$$

**Proposition 3.** *Let $G$ be a directed graph, let $\delta^+(G)$ (or $\delta^-(G)$) be the minimum outdegree (or indegree, respectively) of a vertex of $G$. Then*

$$d^-(G) \leqq \delta^+(G) + 1,$$
$$d^+(G) \leqq \delta^-(G) + 1.$$

Proof. Let $d^-(G) = d$ and let $\mathcal{D} = \{D_1, \dots, D_d\}$ be an inside-semidomatic partition of $G$ with $d$ classes. Let $x \in V(G)$; without loss of generality we may suppose that $x \in D_d$. Then in each $D_i$ for $i = 1, \dots, d-1$ there exists a vertex $y_i$ such that $\overrightarrow{xy_i}$ is an edge of $G$. The vertices $y_1, \dots, y_{d-1}$ are pairwise distinct, therefore the outdegree of $x$ is at least $d-1$. As $x$ was chosen arbitrarily, we have $\delta^+(G) \geqq d^-(G) - 1$, which implies the first inequality. The second inequality is dual to the first.

**Proposition 4.** *Let $G$ be a directed graph in which any two vertices are joined by at most one edge, let $n$ be its number of vertices, $n \geqq 2$. Then*

$$d^-(G) \leqq [n/2],$$
$$d^+(G) \leqq [n/2].$$

Proof. Suppose that $d^-(G) > [n/2]$. As $n \geqq 2$, we have $d^-(G) > 1$. Any inside-semidomatic partition $\mathcal{D}$ of $G$ with $d^-(G)$ classes contains at least one class consisting of one vertex. If $u$ is such a vertex, then for each $y \neq u$ there exists the edge $\overrightarrow{yu}$. As $d^-(G) > 1$, there exists a class $D \in \mathcal{D}$ such that $u \notin D$. As $D$ is inside-semidominating and $u \notin D$, there exists $x \in D$ such that $\overrightarrow{ux}$ is an edge of $G$. But then there are both the edges $\overrightarrow{ux}$, $\overrightarrow{xu}$ in $G$, which is a contradiction. The proof for $d^+(G)$ is dual to the preceding.

**Corollary 1.** *If a directed graph $G$ contains a source (or a sink), then $d^+(G) = 1$ (or $d^-(G) = 1$, respectively).*

**Theorem 1.** *Let $d_1$, $d_2$, $n$ be three positive integers such that $d_1 \leqq n/2$, $d_2 \leqq n/2$. Then there exists a tournament $T$ with $n$ vertices such that $d^-(T) = d_1$, $d^+(T) = d_2$.*

Proof. First suppose $d_1 \leqq d_2$, $d_1 < n/2$. Let $U = \{u_i \mid i = 1, \dots, d_2\}$, $V = \{v_i \mid i = 1, \dots, d_2\}$, $W = \{w_i \mid i = 1, \dots, n - 2d_2 - 1\}$ and $Z = \{z\}$ be pairwise disjoint sets. ($W$ is empty if $n = 2d_2 + 1$.) Put $V(T) = U \cup V \cup W \cup Z$ and construct a tournament $T$ with the vertex set $V(T)$. The edge set of $T$ will contain the edges $\overrightarrow{u_iu_j}$ for $i < j$, $\overrightarrow{v_iv_j}$ for $i < j$, $\overrightarrow{u_iv_j}$ for $i \geqq j$, $\overrightarrow{v_iu_j}$ for $i > j$, $\overrightarrow{u_iw_j}$ for all $i$ and $j$, $\overrightarrow{w_iv_j}$ for all $i$ and $j$, $\overrightarrow{w_iw_j}$ for $i < j$, $\overrightarrow{zv_i}$ for $i \leqq d_1 - 1$, $\overrightarrow{v_iz}$ for $i \geqq d_1$, $\overrightarrow{u_iz}$ for each $i$, $\overrightarrow{w_iz}$ for each $i$.

Now we shall prove that $d^+(T) = d_2$. Put $D_i^+ = \{u_i, v_i\}$ for $i = 1, \dots, d_2 - 1$ and $D_{d_2}^+ = \{u_{d_2}, v_{d_2}\} \cup W \cup Z$. Consider $D_i^+$ for fixed $i \leqq d_2 - 1$ and let $x \in V(T) - D_i^+$.

Then $x \in D_j^+$ for $j \neq i$. If $j < i$, then either $x = u_j$ and there exists the edge $\overrightarrow{v_ix} = \overrightarrow{v_iu_j}$, $v_i \in D_i^+$, or $x = v_j$ and there exists the edge $\overrightarrow{u_ix} = \overrightarrow{u_iv_j}$, $u_i \in D_i^+$. If $j > i$, then either $x = u_j$ and there exists the edge $\overrightarrow{u_ix} = \overrightarrow{u_iu_j}$, $u_i \in D_i^+$, or $x = v_j$ and there exists the edge $\overrightarrow{v_ix} = \overrightarrow{v_iv_j}$, $v_i \in D_i^+$, or $x \in W \cup Z$ and there exists the edge $\overrightarrow{u_ix}$, $u_i \in D_i^+$. Now consider $D_{d_2}^+$. If $x \in V(T) - D_{d_2}^+$, then $x \in D_j^+$ for $j \leqq d_2 - 1$. Again either $x = u_j$ and there exists the edge $\overrightarrow{v_{d_2}u_j} = \overrightarrow{v_{d_2}x}$, $v_{d_2} \in D_{d_2}^+$, or $x = v_j$ and there exists the edge $\overrightarrow{u_{d_2}x} = \overrightarrow{u_{d_2}v_j}$, $u_{d_2} \in D_{d_2}^+$. Hence $\mathscr{D}^+ = \{d_1^+, \ldots, D_{d_2}^+\}$ is an outside-semidomatic partition of $T$. As the indegree of $u_{d_2}$ is $d_2 - 1$, we have $d^+(T) = d_2$.

Now let $D_i^- = D_i^+$ for $i = 1, \ldots, d_1 - 1$ and $D_{d_1}^- = \bigcup_{j=d_1}^{d_2} D_j^+$. Consider $D_i^-$ for fixed $i \leqq d_1 - 1$. Let $x \in V(T) - D_i^-$; then $x \in D_j^-$ for $j \neq i$. If $j < i$, then either $x = u_j$ and there exists the edge $\overrightarrow{xu_i} = \overrightarrow{u_ju_i}$, $u_i \in D_i^-$, or $x = v_j$ and there exists the edge $\overrightarrow{xv_i} = \overrightarrow{v_jv_i}$, $v_i \in D_i^-$. If $j > i$, then either $x = u_j$ and there exists the edge $\overrightarrow{xv_i} = \overrightarrow{u_jv_i}$, $v_i \in D_i^-$, or $x = v_j$ and there exists the edge $\overrightarrow{xu_i} = \overrightarrow{v_ju_i}$, $u_i \in D_i^-$, or $x \in W \cup Z$ and there exists the edge $\overrightarrow{xv_i}$, $v_i \in D_i^-$. Now consider $D_{d_1}^-$. If $x \in V(T) - D_{d_1}^-$, then $x \in D_j^-$ for $j \leqq d_1 - 1$. Again either $x = u_j$ and there exists the edge $\overrightarrow{xu_{d_1}} = \overrightarrow{u_ju_{d_1}}$, $u_{d_1} \in D_{d_1}^-$, or $x = v_j$ and there exists the edge $\overrightarrow{xv_{d_1}} = \overrightarrow{v_jv_{d_1}}$, $v_{d_1} \in D_{d_1}^-$. Hence $\mathscr{D}^- = \{D_1^-, \ldots, D_{d_1}^-\}$ is an inside-semidomatic partition of $T$ and, as the outdegree of $z$ is $d_1 - 1$, we have $d^-(T) = d_1$. We have proved the assertion for the case $d_1 \leqq d_2$.

If $d_1 > d_2$, $d_2 < n/2$, we construct a tournament $\hat{T}$ such that $d^-(\hat{T}) = d_2$, $d^+(\hat{T}) = d_1$. By reversing the orientation of all edges of $\hat{T}$ we obtain the required tournament $T$.

If $d_1 = d_2 = n/2$, we take $W = Z = \emptyset$. Then we may put $D_i^+ = D_i^- = \{u_i, v_i\}$ for $i = 1, \ldots, n/2$; these sets form a partition of $V(T)$ which is simultaneously inside-semidomatic and outside-semidomatic, hence $d^+(T) \geqq n/2$, $d^-(T) \geqq n/2$. From Proposition 4 we obtain the equalities.

The following theorem is an analogon of a theorem in [2] concerning the domatic number.

**Theorem 2.** *Let $G$ be a directed graph. Then the following two assertions are equivalent:*
(i) *$G$ contains a factor $G_0$ which is bipartite and has no sink.*
(ii) *$d^-(G) \geqq 2$.*

Proof. Suppose that $G$ contains the described factor $G_0$. It is a bipartite graph, hence there exists a partition $\{D_1, D_2\}$ of $V(G_0) = V(G)$ such that each edge of $G_0$ joins two vertices of distinct classes of this partition. As $G_0$ has no sink, each vertex of $D_2 = V(G) - D_1$ is an initial vertex of an edge of $G_0$ and the terminal edge of this edge is in $D_1$; hence $D_1$ is inside-semidominating in $G_0$ and analogously so is $D_2$. Thus $\{D_1, D_2\}$ is an inside-semidomatic partition of $G_0$ and also of $G$ and $d^-(G) \geqq 2$.

Now suppose that $d^-(G) \geqq 2$. Then there exists an inside-semidomatic partition

$\{D_1, D_2\}$ of $G$. By $G_0$ we denote the factor of $G$ whose edge set is the set of edges of $G$ joining vertices of $D_1$ with vertices of $D_2$; this is a bipartite graph. Suppose that $G_0$ has a sink $u$; without loss of generality let $u \in D_2$. Then there exists no edge from $u$ to a vertex of $D_1$ and $D_1$ is not inside-semidominating, which is a contradiction.

**Theorem 2'.** *Let $G$ be a directed graph. Then the following two assertions are equivalent:*
*(i') $G$ contains a factor $G_0'$ which is bipartite and contains no source.*
*(ii') $d^+(G) \geqq 2$.*

This theorem is dual to Theorem 2.

**Corollary 2.** *If every cycle of a directed graph $G$ has an odd length, then $d^-(G) = d^+(G) = 1$.*

A question may be asked, whether $d^-(G) \geqq 2$ and $d^+(G) \geqq 2$ imply $d(G) \geqq 2$. We shall show that this is not true. Let $V(G) = \{u_1, u_2, u_3, u_4, u_5\}$ and let the edges of $G$ be $\overrightarrow{u_1u_2}$, $\overrightarrow{u_2u_3}$, $\overrightarrow{u_3u_4}$, $\overrightarrow{u_3u_5}$, $\overrightarrow{u_4u_5}$, $\overrightarrow{u_5u_1}$. The reader may verify himself that $d^-(G) = d^+(G) = 2$ and $d(G) = 1$.

## REFERENCES


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## ПОЛУДОМАТИЧЕСКИЕ ЧИСЛА ОРИЕНТИРОВАННЫХ ГРАФОВ


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### Резюме


Подмножество $D$ множества $V(G)$ вершин ориентированного графа $G$ называется внутренне полудоминантным (или внешне доминантным), если для всякой вершины $x \in V(G) - D$ существует вершина $y \in D$ такая, что дуга $\overrightarrow{xy}$ (или $\overrightarrow{yx}$, соответственно) принадлежит графу $G$. Максимальное число классов разбиения множества $V(G)$, все классы которого являются внутренне полудоминантными (или внешне доминантными) множествами в $G$, называется внутренне полудоматическим (или внешне полудоматическим) числом графа $G$ и обозначается через $d^-(G)$ (или $d^+(G)$ соответственно), Исследуются свойства чисел $d^-(G)$ и $d^+(G)$.