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ONE GENERALIZATION OF QUASIFIELD

JAROLÍM BUREŠ—JOSEF KLOUDA

Among other affine planes the translation planes are characterized by the transitivity of their translation group (to any two points of the plane there exists exactly one translation, carrying the first into the second one). In paper [1] by J. André translation planes are described via so-called congruences. These are coverings of the group by its proper subgroups such that any two distinct ones of them generate the whole group.

In this article another algebraic description of translation planes by means of generalized quasifields is introduced A generalized quasifield differs from the quasifield in such a way that the solution of the equation $a.x=b$ and the conditions $1.a=a$ and $0.a=0$ are not supposed.

In §§ 1, 2 we introduce the basic notations and generalize the notation of the quasifield. In §§ 3 and 4 are concerned with coordinate transformations and quasifields associated to the same affine plane. The method due to G. Menichetti [3] is generalized here. In the last § 5 we investigate the geometric properties of algebraic systems introduced in the preceding paragraphs. This work was prepared under the direction of V. Havel.

§ 1. Translation planes

Definition. *A partition in an additive non-trivial group G is defined as a set $\mathscr{K}$ of non-trivial subgroups of G such that to every non-zero $x \in A$. A partition $\mathscr{K}$ in a group G is called congruence if*

$$\forall A, B \in \mathscr{K}; \quad A \neq B \quad A+B=G$$

In [1] pp. 165 it is proved that the existence of a congruence $\mathscr{K}$ in G implies commutativity of G and

$$\forall A, B \in \mathscr{K}; \quad A \neq B \quad A \oplus B=G$$

Let $\mathscr{K}$ be a congruence in G. By a $\mathscr{K}$-endomorphism of G there is meant an endomorphism φ of G satisfying $\varphi(A) \subseteq A$ for every $A \in \mathscr{K}$. The set of all

$\mathscr{K}$-endomorphisms with the sum and product defined in the usual way is said to be a kernel $K(\mathscr{K})$ of a congruence $\mathscr{K}$. In [1], pp. 167 it is proved:

Proposition 1. *Kernel $K(\mathscr{K})$ of a congruence $\mathscr{K}$ is a skewfield.*

If $\mathscr{K}$ is a congruence in G we can consider G as a vector space over $K(\mathscr{K})$. Then every $A \in \mathscr{K}$ is a vector subspace of the vector space G.

Let W be a vector space over a skewfield F, W^+ its additive group and $\mathscr{K}$ a congruence in W^+ such that every $A \in \mathscr{K}$ is a vector subspace of W. For each $f \in F$ define a map α_f: $W \to W$ by $\alpha_f(x) = f.x$. Obviously $\alpha_f \in K(\mathscr{K})$.

Definition. *Let W be a vector space over a skewfield F and $\mathscr{K}$ a congruence in W^+ such that every $A \in \mathscr{K}$ is a vector subspace of W. If $K(\mathscr{K}) = \{\alpha_f, f \in F\}$, we shall say that $\mathscr{K}$ is a congruence in W.*

Let $\mathscr{K}$ be a congruence in a vector space W. Let us construct a plane $\alpha(W, \mathscr{K})$ in the following way: The points of $\alpha(W, \mathscr{K})$ are just the elements of W and the lines of $\alpha(W, \mathscr{K})$ are just subsets of W of the form $U + a$, $U \in \mathscr{K}$, $a \in W$.

Proposition 2. *$\alpha(W, \mathscr{K})$ is a translation plane.* For proofs, see [1] pp. 163.

§ 2. Generalized quasifield

Definition. *A right generalized quasifield or GVW-system is an algebraic structure consisting of a set Q of at least two elements provided with two operations, addition and multiplication satisfying the following axioms:*

(VW 1) $(Q, +)$ *is a group, its neutral element will be denoted by* 0.

(VW 2) $\forall\, a, b, c \in Q \;\; a.(b + c) = a.b + a.c$

(VW 3) *There exists in Q the right unit element $1 \in Q$ satisfying $\forall\, a \in Q \;\; a.1 = a$.*

(VW 4) $\forall\, a, b \in Q; \quad a = 0 \quad \exists! \quad x \in Q \quad x.a = b$

(VW 5) $\forall\, a, b, c \in Q; \quad a \neq b \quad \exists! \quad x \in Q \quad -a.x + b.x = c$

This in a GVW-system neither a left distributivity nor associativity of multiplication is supposed. One does not require that $0.a = 0$, $1.a = a$ for $a \in Q$. Neither is the equation $a.x = b$ supposed to have a solution for any given $a, b \in Q$, $a \neq 0$. Notice that (VW 2) implies $\forall\, a \in Q \;\; a.0 = 0$ and $\forall\, a, b \in Q \;\; -(a.b) = a.(-b)$

Proposition 3. *The addition of any GVW-system is commutative.*

Proof. Given a GVW-system Q, construct a translation plane $\alpha(Q)$ as follows: Points of $\alpha(Q)$ are just the ordered pairs (x, y) $Q \times Q$. Lines of $\alpha(Q)$ are just all the subsets of points (x, y) satisfying either $y = a.x + b$ (for a, b fixed), or $x = a$ (for $a \in Q$ fixed). Translations of $\alpha(Q)$ are precisely all the mappings $(x, y) \mapsto (x + a \;\; y + b)$, $a, b \in Q$. The group $T(\alpha(Q))$ of all translations of $\alpha(Q)$ is isomorphic to the direct sum $Q^+ + Q^+$ of the additive group Q^+ of the GVW-system Q. As $T(\alpha(Q))$ is abelian, Q^+ is also abelian

Definition. *Let G be a non-trivial additive group and $\mathscr{L}$ a system of its endomorphisms satisfying:*

(1) $\forall\, a, b \in G;\quad a \neq 0 \quad \exists!\ \ X^a_b \in \mathscr{L} \quad X^a_b a = b$

(2) $\forall\, X, Y \in \mathscr{L};\quad X \neq Y \quad (X - Y)(G) = G$

Then the ordered pair $(G, \mathscr{L})$ will be called an L-system.

Let $G, \mathscr{L})$ be an L-system, $a \in G$; $a \neq 0$. For $\forall x, y \in G$ define $x\,.\,y = X^a_x y$ $(X^a_x \in \mathscr{L})$. The group G provided with this multiplication is a GVW-system $Q_a(G, \mathscr{L})$, the unity element of which the element a is. Conversely, if Q is a GVW-system, we can consider the set of endomorphisms $\mathscr{L}_Q = \{\varphi_a\,;\ \varphi_a(x) = a\,.\,x,\ a \in Q\}$.

Denote by Q^+ the additive group of the GVW-system Q. Then $(Q^+, \mathscr{L}_Q)$ is an L-system. Obviously $Q_1(Q^+, \mathscr{L}_Q) = Q$

Proposition 4. *If (G, L) is a L-system, then G is an abelian group.*

Proof. Being additive group of the L-system $Q_a(G, \mathscr{L})$ is abelian.

Definition. *Let $(G, \mathscr{L})$be an L-system. An $\mathscr{L}$-endomorphism of G is an endomorphism ϱ of G commuting with every $X \in \mathscr{L}$, i.e. $\forall\, a \in G\ \forall\, X \in \mathscr{L}\ \varrho X a = X \varrho a$.*

Definition. *The set $K(G, \mathscr{L})$ of just all $\mathscr{L}$-endomorphisms of G is called kernel of $\mathscr{L}$-system $(G, \mathscr{L})$.*

Proposition 5. *Kernel $K(G, \mathscr{L})$ is a skewfield.*

Proof. Sum and product of two $\mathscr{L}$-endomorphisms is defined by $(\varrho + \sigma)(a) = \varrho(a) + \sigma(a)$, $(\varrho\sigma)(a) = \varrho(\sigma(a))$.

Obviously $K(G, \mathscr{L})$ is an associative ring with unity element. It suffices to prove that every endomorphism $\varrho \in K(G, \mathscr{L})$ $\varrho \neq 0$ is automorphism of G. In fact, if $\varrho \in K(G, \mathscr{L})$ is an automorphism it is $\varrho(X(\varrho^{-1}x)) = Xx$ which means $X\varrho^{-1}x = \varrho^{-1}Xx$.

Thus we consider $\varrho \in K(G, \mathscr{L})$ for which there exists $g \in G$, $g \neq 0$ such that $\varrho(g) = 0$. Then for $\forall h \in G$ it is $\varrho h \varrho X^g_h(g) = X^g_h(0) = 0$ i.e. $\varrho = 0$. For $\varrho \neq 0$, $\varrho(g) = 0$ implies $g = 0$. Let us show now that for $\varrho \neq 0$ it is $\varrho(G) = G$. Let $H, k \in G$, $k \neq 0$. Then $\varrho(X^{\varrho k}_h k) = X^{\varrho k}_h(\varrho k) = h$. So we have shown that every $\varrho \in K(G, \mathscr{L})$; $\varrho \neq 0$ is an automorphism of G, finishing the proof.

In this sense we can take G as a vector space over the field $K(G, \mathscr{L})$. $\mathscr{L}$ is then the set of endomorphisms of this vector space.

Definition. *The kernel $K(Q)$ of a GVW-system Q is the set*

$$\{x \in Q;\ \ (a + b)x = a\,.\,x + b\,.\,x,\ \ (a\,.\,b)\,.\,x = a(b\,.\,x)\ \ \forall a, b \in Q\}$$

Proposition 6. *The kernel $K(Q)$ of a GVW-system Q is a skewfield.*

Proof. Obviously $0,1 \in K(Q)$. et x, $K($) Th n for all $a, b \in Q$:

$$(a+b)\ (x+y)\ \ (a\ \ b).\ \ \ (a+b).y$$
$$= a.x+b.x+a\ \ '+b.y\ \ \ .(\ \ +y)+b.(x+y)$$

$$(a.b)(x+y)=(a.b)y+(a.b)x-a(b.\ \ )+a.(b.y)=a.(b.(x+y))$$

$$(a+b)(x\ \ y)=((\ \ \ b).x)\ y\ \ (a.x+b.x)y-$$
$$=(a\ \ x)y+(b.x)y=a\ (\ \ \ )+b\ (x.y)$$

$$(ab).(x\ \ y)\ \ ((\ b)x)y\ \ a.((bx)y)-a.(b(xy))$$

Let $x \neq 0$ x, y $K(Q)$. Th n th r e ısts ex tly one) el m nt $z \in Q$ such that $z.x-y$.

$$((a+b)z)x-(a+b)\ \ '\ \ a.y\ \ b.\ \ -a(\ .x)+b\ (\ .x)=$$
$$-(a.z).x\ \ \ \ )\ \ \ (\ \ z\ \ \ b\ z)\ x$$

and according to (VW) g

$$a+b).z\ \ \ .z+\ \ \ \ \forall\ \ ,b \in Q$$

Furthemore

$$((a.b)\ z).x\ \ (a.\ \ \ v\ \ a\ (b.\ )\ \ )\ \ (a\ (b.z)).x$$

and by (VW 4) $(a.b).z-a\ (b\ \ \ )$ $a, b \in Q$ Thu if $x \in K(Q)$, $x \neq 0$, then $x\ ^1 \in K(Q)$, $x\ ^1.x-1$ nd the p f ˙ f ˙ d

Propositi n 7. $K(Q_a(G, \mathscr{L}$ r c $(G\ \mathscr{L})$.

Proof. For $\forall r \in K(Q\ (G, \mathscr{L})$ $r\ \rightarrow$ an e domorphism of G. Since $(\varphi r)(X_x^a y)=(x.y(.\ =x\ \ y.\ )$ $y\ r)$ ((y) for all x $y \in G$, φr belongs to $G(G, \mathscr{L})$. For $\forall s \in K(G$) ef a As for er $x \in G$ there is $x.(s.a)-X_x^a(s.a)-s\ X$) for r x $\in G$

$$(x+y).(s.a)=\ \ (x\ \ \ \ )\ \ s.\ \ \ \ s\ \ -\ \ (s.a)+y\ (s.a)$$
$$(x.y)\ (s\ \ a)\ \ \ \ .\ \ \ y\ \ \ \ \ X\ \ \ \ X_x\ \ \ \ y-X_x^a(y\ (\ \ a)-$$
$$-x\ (y\ (\ .\ \ \ \ \ \ \ \ t\ \ \ \ \ \psi s\ \ K(Q\ (G, \mathscr{L}))$$

If $s \in K(Q, \mathscr{L})$, e h φ $s($) ($=s(x)$ for every $x \in G$, getting thus $\varphi\psi s = s$, which implie that i ın˙ iv F h rmore for $\in K(Q_a(G, \mathscr{L}))$ there is $\psi\varphi r = \varphi r(a) = a.r$. But b caus a ı a unı y element of the fıeld $K(Q_a(G, \mathscr{L}))$, there is $a.r=r$ or $\psi\varphi r$ r whi h h s that ψ ıs a mapping of $K(G, \mathscr{L})$ onto $K(Q_a(G, \mathscr{L})$. Now to complet th p o f t remains to verify that there is $\psi(s_1+s_2)=\psi s_1+\psi s_2$ $(s_1\ s\ )-$ s $(\ _2$

Defin˙tion. *Let* $(G,$ (') *b t o* *ms* φ *a homomorphi m of* G *into* G', ψ *a mapping of* $\mathscr{L}$ *i t* $\mathscr{L}$ *I* $\forall X$ $\forall$ G $\varphi(Xx)-(\psi X)(\varphi x)$, *we call the* *ordered pair* (φ, ψ) *a homomorphism of an L-system* $(G, \mathscr{L})$ *into an L-system* $(G', \mathscr{L}')$. *If* (φ, ψ) *is a homomorphism of* $(G, \mathscr{L})$ *into* $(G', \mathscr{L}')$, φ *is an isomorphism of L-systems* $(G, \mathscr{L})$ $(G', \mathscr{L}')$.

If (φ, ψ) is an isomorphism of the L-system $(G, \mathscr{L})$ and $(G', \mathscr{L}')$, we have $\psi X = \varphi X \varphi^{-1}$ for every $X \in \mathscr{L}$. Conversely, if $\varphi G \to G'$ is an isomorphism such that $\varphi \mathscr{L} \varphi^{-1} = \mathscr{L}'$, let us set $\psi X = \varphi X \varphi^{-1}$. Then (φ, ψ) is an isomorphism of the L-systems $(G, \mathscr{L})$ and $(G', \mathscr{L}')$. Thus we can speak about an isomorphism φ of the L-systems $(G, \mathscr{L})$ $(G', \mathscr{L}')$ if it is an isomorphism of G onto G' such that $\mathscr{L}' = \{\varphi X \varphi^{-1}, X \in \mathscr{L}\}$.

If two L-systems $(G, \mathscr{L})$ and $(G', \mathscr{L}'à$ are isomorphic, then the fields $K(G, \mathscr{L})$ and $K(G', \mathscr{L}')$ are also isomorphic. Namely if φ is an isomorphism of $(G, \mathscr{L})$ onto $(G', \mathscr{L}')$ it suffices assign to any $\lambda \in K(G, \mathscr{L})$ an element $\varphi \lambda \varphi^{-1} \in K(G', \mathscr{L}')$.

Proposition 8. *If* φ *is an isomorphism of an L-system* $(G, \mathscr{L})$ *onto an L-sy tem* $(G', \mathscr{L})$, *then* φ *is an isomorphism of the GVW-system* $Q_a(G, \mathscr{L})$ *onto* $Q_{\varphi(a)}$ $(G, \mathscr{L}')$. *If the GVW-systems* $Q_a(G, \mathscr{L})$, $Q_b(G', \mathscr{L})$ *are isomorphic, then the L-systems* $(G, \mathscr{L})$, $(G', \mathscr{L}')$ *are also isomorphic.*

Proof. Let φ be an isomorphism of the L-systems $(G, \mathscr{L})$ onto $(G', \mathscr{L}')$. Then $\varphi X_x^a \varphi^{-1}(\varphi(a)) = \varphi X_x^a a = \varphi(x)$. Hence for all $x, y \in G$ we get

$$\varphi(x\,.\,y) = \varphi(X_x^a y) = \varphi X_x^a)\varphi^{-1}(\varphi y) = X_{\varphi x}^{\varphi a}$$
$$(\varphi y) = \varphi(x)\varphi(y)\,.$$

For the proof of the converse part let us suppose that is an isomorphism of the GVW-system, $Q_a(G, \mathscr{L})$ onto the GVW-system $Q_b(G', \mathscr{L}')$. Then φ is an isomorphism of G onto G'. For any $X_x^a \in \mathscr{L}$ and any $y \in G$ there holds

$$\varphi X_x^a \varphi^{-1} \quad (\varphi y) = \varphi(x\,.\,y) = \varphi(x)\varphi(y) = X_{\varphi(x)}^{\varphi(a)}(y)$$

and therefore $\varphi \mathscr{L} \varphi^{-1} = \mathscr{L}'$.

Definition. *A right generalized quasifield Q is called a right VW-system (right Veblen-Weddeburn system, right quasifield) if there is*

(VW 6) $\forall a \in Q \qquad 0\,.\,a = 0$

(VW 7) $\forall a \in Q \qquad 1\,.\,a = a\,.$

A GVW-system $Q_a(G, \mathscr{L})$, where $(G, \mathscr{L})$ is an L-system is a VW-system if and only if $\mathscr{L}$ contains the zero endomorphism 0 and the identity endomorphism I. Namely, if we have $0, \mathrm{I} \in \mathscr{L}$, then $0\,.\,x = X_0^a x = 0(x) = 0$, $a\,.\,x = X_a^a x = \mathrm{I}(x) = x$ for all $x \in G$.

Example. $R \oplus R$ is a vector space over the field R of real numbers. For $\forall \lambda, a, b, \in R$, $a^2 + b^2 = 1$ we define $(\lambda, a, b)(x_1, x_2) = (\lambda((2a^2 - 1)x_1 + 2abx_2),$ $\lambda(2abx_1 + (2b^2 - 1)x_2))$.

For a ve tor $(y_1$) e d no e $n(y_1, y_2) - y_1^2 + y$ A simple calculation shows that tl ere is

$$N(x \quad x_2) \quad N((\ a, b)(x_1, x_2)).$$

Now we are going to prove that the mapping $(\lambda, a, b) - (\lambda', a, b)$ is injective for $(\lambda \ a, b) \neq (\lambda \ , a', b')$. T co responding mappings are distinct). If there is $\lambda \neq \lambda'$, we ha e

$$N((\quad)(\ _1, x \ )) - \lambda^2 N(x_1, x_2 \ ,$$
$$N((\quad , b')(x_1, \ _2) - \lambda'^2 N(x_1, x_2).$$
$$N(x_1, x_2 \quad \lambda' \ N(x_1, x_2)$$

if and only if $N(x \ x \ ) \ 0, x_1 - 0 \ x_2 \ 0$

Let us have iov $\lambda - $ 0. T n

$$[(\lambda \ a \ b \ - \lambda \quad ' \quad ')](\ _1 \quad) \qquad ((a^2 \quad a'^2)x_1 + (ab - a'b')x$$
$$c \quad ' \ ) \ _1 + (b^2 - b' \ ) \ ).$$

Because of

$$\begin{vmatrix} a & & a & -a\,b' \\ & a'b & b & b \end{vmatrix} - \begin{vmatrix} a & b \\ a', & b' \end{vmatrix} \neq 0$$

we get from $[(\lambda \ a, \ ) - (\lambda \ , a', b \ )](x_1 \ x_2)$ $(0,0)$ that there is $x_1 \quad x = 0$. We show that tl e et of all mappi igs (λ, a, b) is transitive on $R \oplus R$, i.e. that to every $(x_1 \ x_2) \neq (0 \ 0)$ $(y_1. \ y \ )$ $R \oplus R$ there exi ts a mapping $(\lambda \ a, b)$ such that $(\lambda, a, b)(x_1, x_2) - (y_1 \ y$

If $(y_1, y_2) - (0,0$, we t k λ- (. hus et $(y \ , y_2) \neq (0 \ 0)$, that is to say $N(y \ , y_2)$ -0, $N(x_1 \ x \ ) \neq$) n there exi t $\mu \in R$, $\mu \neq 0$ such that $N(y_1, y \ ) - \mu N(x_1 \ x_2)$. Choosing $\lambda \quad \sqrt{\mu}$ ve g t $N(\lambda^{-1} y_1, \lambda^{-1} y_2) = N(x_1, x_2)$. Now there exists ($b) \in R \times R$ such that

$$\begin{vmatrix} \lambda x \ + y \ (2 \ ) \ , & (\lambda x \ + y_2)((2\lambda)^{-1} \\ & b \end{vmatrix} = 0$$

$$+ b^2 \quad 1$$

(λ, a, b) is the desired apping Let us d note by $\mathscr{L}$ the set of all mappings (λ, a, b) Th $((R + \ R)^+ \quad)$ is an L-s t m.

§ 3. Line r coordinates

Let $\alpha(W, \mathscr{K})$ b a r slation pl ne $A \quad \mathscr{J}$, S a subspace of W, isomorphic to A and such that $A \oplus S \quad W, p \in V$ I $\cdot A \quad S$ be an isomorphism. If $a \in W$, then th r exists exa t y o e ( ) uc tl at $a - p - y \quad \Phi x$ The vectors x, y are linear coordinates of the point a in the coordinate system (p, A, S, Φ). To any $B \in \mathscr{K}$, $B \neq A$ there exists an endomorphism X_B of A such that

$$B = \{X_B x + \Phi x;\ x \in A\}$$

We denote by $\mathscr{L}(A, S)$ the system of all endomorphisms X_B, $B \neq A$.

Proposition 9. *$(A, \mathscr{L}(A, S))$ is an L-system.*

Proof. Let $x \in A$, $x \neq 0$, $y \in A$. There exists exactly one element $B \in \mathscr{K}$ such that $y + \Phi x \in B \neq A$. That means that there exists exactly one element $X \in \mathscr{L}(A, S)$ such that $Xx = y$. Further let $X, y \in \mathscr{L}(A, S)$, $X \neq Y$, $z \in A$. Because

$$\{Xx + \Phi x;\ x \in A\} \oplus \{Yy + \Phi y;\ y \in A\} = W,$$
$$Xx + \Phi x + Yy + \Phi y = z$$

we get

$$\Phi x + \Phi y = 0, \quad x = -y, \quad (X - Y)(x) = Xx - Yx = z$$
$$\text{i.e.} \quad (X - Y)(A) = A$$

Let us consider a line $A + b$ and let (x_b, y_b) be coordinates of the point b. Coordinates of any point $y + b$, $y \in A$ are then $(x_b, y_b + y)$ and the line $A + b$ is described by the equation $x = x_b$. Conversely, the set consisting of all points (x, y) satisfying $x = c$, $c \in A$ is the line $A + \Phi c$.

Let us have now a line $B + b$, $B \neq A$. The coordinates of any point $X_B z + \Phi z + y_b + \Phi x_b$, $z \in A$ are $(X_B z + y_b,\ z + x_b)$. Hence the equation of $B + b$ is $y = X_B x - X_B x_b + y_b$. Conversely, the set of (x, y) satisfying $y = X_B x + c$ is the line $B + c$. Choosing in A a unit element $a = 0$, we get a GVW-system $Q_a(A, \mathscr{L}(A, S))$ and then any line of α is described either by the equation $y = b\,.\,x + c$ or by $x = c$.

The translation plane $\alpha(Q_a(A, \mathscr{L}(A, S))$ is isomorphic to the plane α. The described isomorphism is a mapping sending every (x, y) to $y + \Phi x$.

Proposition 10. *Let α(W, K) be a translation plane (p, A, S, Φ) a coordinate system. Then $K(\alpha)$ is isomorphic to $K(A, \mathscr{L}(A, S))$.*

Proof. Let $s \in K(\alpha)$. s is an endomorphism of W^+ leaving every element of the congruence invariant. We define $\varphi s = s/A$ (restriction of s to A) Φ is an isomorphism of A with S i.e. $s\Phi(x) = \Phi(sx)$ i.e. $s\Phi(x) = \Phi\varphi s x$ for every $x \in A$.) Further let $X \in \mathscr{L}(A, S)$. Then $\{Xx + \Phi x;\ x \in A\} \in \mathscr{K}$ and thus for $s \in K(\alpha)$ we have s $(Xx + \Phi x) = s\ Xx + \Phi sx \in \{Xx + \Phi x;\ x \in A\}$, which means that there exists $y \in A$ such that $s\ Xx + \Phi sx = Xy + \Phi y$. From the uniqueness of this expression it follows $\Phi sx = \Phi y$ and because Φ is an isomorphism, we have $y = sx$, $s\ Xx = Xsx$, i.e. $(\varphi s)Xx = X(\varphi s)x$. We have shown that $\varphi s \in K(A, \mathscr{L}(A, S))$.

Let $r \in K(A, \mathscr{L}(A, S))$. We define $\psi(r)$ to be an endomorphism of W^+ such that: $(\psi r)(x + y) = rx + \Phi r \Phi^{-1} y$ where $x \in A$, $y \in S$. Since $(\psi r)(A) \subset A$,

$\psi r(\{Xx+\Phi x\,;\, x \quad\})\ \ \{r \quad x+ \quad rx\,,\, x\in A\}-\{Xrx+\Phi rx\,;\, x\in A\}\subset Xy+\Phi y\,;$ $y\in A\}$, $X\in\mathscr{L}(A, S)$. We have $\psi r\in K(\alpha)$. It can be easily seen that

$$\varphi\psi \quad \mathrm{id}_{K(A \ \mathscr{L}\ A\ S))}\,, \qquad \psi\varphi=\mathrm{id}_{K(\alpha)}$$

$$\psi(r_1+r_2)=\psi(r_1)+\psi(r_2), \qquad \psi(r_1, r_2)=\psi(r_1)\psi(r_2)$$

and from this our a sertion immediately follows

For an L-system $(G, \mathscr{L})$ $K(Q_a(G, \mathscr{L}))$ is always isomorphic to $K(G, \mathscr{L})$. Therefore $K(\alpha)$ i isomorphic with $K(Q_a(A, \mathscr{L}(A, S))$. But generally for a GVW-system Q, $K(Q)$ ne d not be isomorphic with $K(\alpha(Q))$.

Example. Q $\{0, 1, 2\ 3\}$. An dditive roup of the GVW-system is defined in t e followin w y:

	1	2	3
1	0	3	2
2	3	0	1
	2	1	0

Multi l'c ti n in Q id f e fo lo (e g $2.1-2$, $0.2=2$)

	0	1	2	3
0	0	0	2	2
1	0	1	1	0
2	0	2	3	1
		3	0	3

$K(Q)-\{0,1\}$ be 1 3).$2-3$, but $1.2+3.2=1$ $(1+3).3=1$, but 1 $3+3.3=3$.

The t p i t o $\alpha($) i the t $Q\times Q$. A group of just all translations of the tran lat n pl ne α $Q)$ s morph c to the group $Q^+\oplus Q^+$ A congruence in $Q^+\oplus Q^+$ relat d t the lan $\alpha(Q)$ consi t o the groups

$$\{(0\ 0), (1\ 0), (3,2), 2\ 2)\}, \quad \{(0\ 0), (1,1), (3,0), (2,1)\}$$
$$\{(0\ 0), (1,2)\ \ 2,3)\ (\ \ 1)\}, \quad \{(0,0)\ (3,3), (2,0), (1,3)\}$$
$$\{(0,0), (0,1), (0,2\ \ (0,3)\}$$

$K(\alpha(Q))$ contai the zero ndo orphi m, the identity endomorphism as well as the endomorphism $(0,1)\mapsto(0,\llcorner)\to(0,3)$ $(1,3)\to(3,3)\to(2,0)$ and $(0,1)\to(0,3)\to$ $(0,2)$ As $|K(Q)|$ 2, $|K(\alpha(Q))|-4$, $K(Q)$ and $K(\alpha(Q))$ cannot be isomorphic. Ch sing $A\in\mathscr{K}$ here $\mathscr{K}$ i co ruence in G and S a subgroup of G isomorphic with A ch that $A\oplus S$ G e n a in g t (A, S) such that $(A,\mathscr{L}(A, S))$ is an L-sy tem. Then th ern l $K(A, \mathscr{L}(A, S))$ is i omorphic to the field of all $\mathscr{K}$-endomorphism λ uch that $\lambda\Phi x-\Phi\lambda x$ for all $x\in A$, where Φ is a given i omorph m of Then the follo i g can be easily proved.

Propos't' 11. *T e k rn l $K(Q)$ is isomorphic with a subfield of $K(\alpha(Q))$.*

§ 4. Quasifield

Definition. *We call an L-system* $(G, \mathscr{L})$ *a QL-system if contains the zero and the identity endomorphisms.*

If $(G, \mathscr{L})$ is a QL-system then $Q_a(G, \mathscr{L})$ is a quasifield. If Q is a quasifield, then $(Q^+, \mathscr{L}_Q)$ is a QL-system. Let $\alpha(W, \mathscr{K})$ be a translation plane. $A, B, C \in \mathscr{K}$, $A \neq B \neq C \neq A$. Then there exists an isomorphism Φ of A onto B such that $\{x + \Phi x; x \in A\} = C$. The coordinate system (p, A, B, Φ) will be denoted in this case by (p, A, B, C) and instead of $\mathscr{L}(A, S)$ we write $\mathscr{C}(A, B, C)$. It is easy to see that $(A^+, \mathscr{C}(A, B, C))$ is a QL-system.

Proposition 12. *Let Q be a quasifield Then the kernel $K(Q)$ is isomorphic to the kernel $K(\alpha(Q))$.*

Proof. Let us choose $A = \{(x, 0); x \in Q\}$, $B = \{(0, x); x \in Q\}$, $C = \{(x, x); x \in Q\}$. The sets of (x, y) satisfying $x = 0$, resp. $y = 0$, resp. $x = y$ are lines and therefore A, B, C are elements of the congruence $\mathscr{K}$ corresponding to the translation plane $\alpha(Q)$. An easy argument shows that the QL-system $(Q^+, \mathscr{L}_Q)$ is isomorphic to the QL-system $(A, \mathscr{C}(A, B, C))$. As $K(\alpha)$ is isomorphic to $K(A, \mathscr{C}(A, B, C))$, $K(\alpha)$ is isomorphic to $K(Q^+, \mathscr{L}_Q)$. But $K(Q^+, \mathscr{L}_Q)$ is isomorphic to $K(Q)$ and the proposition is proved.

To any given coordinate system (p, A, B, C) and unit element $a \in A$; $a \neq 0$ there can be associated a unique quasifield $Q = Q_a(A, \mathscr{C}(A, B, C))$. But after the passage to another unit element or another coordinate system, the new quasifield need not be isomorphic with the quasifield Q.

That is to say we can have various non-isomorphic quasifields associated to a given translation plane.

Proposition 13. *Let $\alpha(W, \mathscr{K})$ be a translation plane. Two quasifields $Q_a(A, \mathscr{C}(A, B, C))$ and $Q_b(A, \mathscr{C}(A, B, C))$ are isomorphic if and only if there exists an affinity $\mathscr{A}$ of the plane $\alpha(W, \mathscr{K})$ such that $\Phi_{\mathscr{A}}(A) = A$, $\Phi_{\mathscr{A}}(B) = B$, $\Phi_{\mathscr{A}}(C) = c$; $\Phi_{\mathscr{A}}(a) = b$ $(\Phi_{\mathscr{A}}(c - d) = \mathscr{A}(c) - \mathscr{A}(d))$.*

Proof. Let $Q_a(A, \mathscr{C}(A, B, C))$ be isomorphic to $Q_b(A', \mathscr{C}(A', B', C'))$. Then there exists an isomorphism φ of the QL-system $(A, \mathscr{C}(A, B, C))$ onto the QL-system $(A', \mathscr{C}(A', B', C'))$ such that $\varphi(a) = b$. Let Φ be an isomorphism of A onto B such that $\{x + \Phi x; x \in A\} = C$ and a Φ' isomorphism of A' onto B' such that $\{x + \Phi' x; x \in A'\} = C$. We define an automorphism of W^+ setting $\psi(x + \Phi y) = \varphi x + \Phi' \varphi y$ Obviously $\psi(A) = A'$, $\psi(B) = B'$, $\psi(C) = C'$. Let $X \in \mathscr{C}(A, B, C))$ then $\psi\{Xx + \Phi x; x \in A\}$ $\{(\varphi X \varphi^{-1})(x) + \Phi' \varphi x; x \in A\} \in \mathscr{K}$.

Thus we have shown that ψ is an automorphism of W^+ preserving the congruence $\mathscr{K}$. It is not difficult to verify that ψ is a semilinear automorphism of W and $\psi(a) = b$.

Let φ be a semilinear automorphism of W preserving the congruence $\mathscr{K}$, and let

$\varphi(A)=A'$, $\varphi(B)=B'$, $\varphi(C)=\bar{C}$, $\varphi(a)=b$. Let $\psi=\varphi/_A$. Then ψ is an isomorphism of A^+ with A'^+.

As $\varphi(x+\Phi x)=\varphi x+\varphi\Phi x=\psi x+\varphi\Phi x\in C'$, there exists $y\in A'$ such that $\psi x+\varphi\Phi x=y+\Phi' y$. Moreover as $A'\oplus B'=W$, we have $y=\psi x$, $\varphi\Phi x=\Phi'\psi x$. For $X\in\mathscr{C}(A,B,C)$ we get $\varphi\{Xx+\Phi x,\ x\in A\}=\{(\psi X\psi^{-1})\psi x+\varphi\Phi x,\ x\in A\}=\{(\psi X\psi^{-1})y+\Phi' y,\ \ y\in A'\}$. Thus $\psi X\psi^{-1}\in\mathscr{C}(A',B',C')$. Conversely if $Y\in\mathscr{C}(A',B',C')$, then $\varphi^{-1}(\{Yy+\Phi' y,\ \ y\in A'\})=\{(\psi^{-1}Y\psi)(\psi^{-1}y)\Phi\psi^{-1}y,\ y\in A'\}=\{\psi^{-1}Y\psi x+\Phi x, x\in A\}\in\mathscr{K}$, which implies $\psi^{-1}Y\psi\in\mathscr{C}(A,B,C)$. Thus ψ is an isomorphism of the QL-system $(A,\mathscr{C}(A,B,C))$ onto the QL-system $(A',\mathscr{C}(A',B',C'))$ and therefore the quasifields $Q_a(A,\mathscr{C}(A,B,C))$ and $Q_{\psi a=\varphi a=b}(A',\mathscr{C}(A',B',C'))$ are isomorphic. The proof is finished.

For the sake of brevity introduce the symbol ∞. If $(G,\mathscr{L})$ is a QL-system, $\mathscr{L}^*=\mathscr{L}\cup\{\infty\}$, we shall call the pair $(G,\mathscr{L}^*)$ a QL*-system.

Definition. *Let $(G,\mathscr{L}^*)$ be a QL*-system, $S,T,U\in\mathscr{L}$, $S\neq T\neq U\neq S$. Then*

a) $(\infty,T,U)X=(X-T)(U-T)^{-1}$ *for all* $X\in\mathscr{L}$
$(\infty,T,U)\infty=\infty$

b) $(S,\infty,U)X=(X-S)(U-S)$ *for all* $X\in\mathscr{L}$, $X\neq S$
$(S,\infty,U)S=\infty$
$(S,\infty,U)\infty=0$

c) $(S,T,\infty)X=(X-S)(X-T)$ *for all* $X\in\mathscr{L}$, $X\neq S$
$(S,T,\infty)S=\infty$
$(S,T,\infty)\infty=\mathrm{I}$

d) $(S,T,U)X=(X-S)^{-1}(X-T)(U-T)^{-1}(U-S)$ *for all* $X\in\mathscr{L}$, $X\neq S$
$(S,T,U)S=\infty$
$(S,T,U)\infty=(U-T)^{-1}(U-S)$

I — *identical endomorphism.*

Definition. *A QL*-system $(G,\mathscr{L}^*)$ is said to be associated to a QL*-system $(G',\mathscr{L}'^*)$ if there are given φ — an isomorphism of G onto G' and $S,T,U\in\mathscr{L}^*$, $S\neq T\neq U\neq S$ such that $\mathscr{L}'^*=\{\varphi(S,T,U)X\varphi^{-1},\ X\in\mathscr{L}^*\}$, where $\varphi\infty\varphi^{-1}=\infty$.*

Definition. *We say that a quasifield Q is associated to a translation plane α if α is isomorphic to the plane $\alpha(Q)$.*

Proposition 14. *Two quasifields Q, Q' are associated to a translation plane α if and only if the QL*-system $(Q^+,\mathscr{L}_Q^*)$ is associated to the QL*-system $(Q'^+,\mathscr{L}_{Q'}^*)$.*

Proof. Let us suppose that $(Q^+,\mathscr{L}_Q)$ is isomorphic to $(A,\mathscr{C}(A,B,C))$ and let Φ be an isomorphism of A onto B such that $\{x+\Phi x, x\in A\}=C$. We assume first that the quasifields are associated to the same affine plane.

a) $(Q'^+,\mathscr{L}_{Q'})$ is isomorphic to $(A,\mathscr{C}(A,B',C'))$. Let $B'=\{Tx+\Phi x,\ x\in A\}$, $C'=\{Ux+\Phi x,\ x\in A\}$. For the mappings Φ': $A\to B'$ $\Phi' x=T(U-T)^{-1}x+$

$\Phi(U-T)^{-1}x$ there holds $\{x+\Phi'x,\ x\in A\}=\{x+T(U-T)^{-1}x+\Phi(U-T)^{-1}x,$ $x\in A\}=\{(U-T)y+Ty+\Phi y,\quad y\in A\}=C'$. If $X\in\mathscr{C}(A,B,C)$, then $\{(X-T)(U-T)^{-1}x+\Phi'x,\ \ x\in A\}=\{(x-T)y+Ty+\Phi y,\ \ y\in A\}=\{Xy+\Phi y,$ $y\in A\}$.

b) $(Q'^{+},\mathscr{L}_{Q'})$ is isomorphic to $(A,\mathscr{C}(A',A,C'))$. Let $A'=\{Sx+\Phi x,\ x\in A\}$, $C'=\{Ux+\Phi x,\ x\in A\}$. The mapping $\Phi'(Sx+\Phi x)=(U-S)x$ is an isomorphism of A onto A for which there holds $\{x+\Phi'x,\ x\in A'\}=\{Sx+\Phi x+Ux-Sx,$ $x\in A\}=C'$. For $X\in\mathscr{C}(A,B,C)$, $X\neq S$ there is $\{S(X-S)^{-1}(U-S)z+\Phi(X-S)^{-1}(U-S)z+\Phi'(Sz+\Phi z),\quad Sz+\Phi,\quad z\in A'\}=\{S(X-S)^{-1}y+\Phi(X-S)^{-1}y+y,\quad y\in A\}=\{Sx+\Phi x+(X-S)x,\quad x\in A\}=\{Xx+\Phi x,\ x\in A\}$.

c) $(Q'^{+},\mathscr{L}_{Q'})$ is isomorphic to $(A,\mathscr{C}(A',B',A))$. Let $A'=\{Sx+\Phi x,\ x\in A\}$, $B'=\{Tx+\Phi x,\ x\in A\}$. If we define $\Phi'(Sx+\Phi x)=T(-x)+\Phi(-x)$ we get an isomorphism of A' onto B' such that $\{Sx+\Phi x+\Phi'(Sx+\Phi x),\ x\in A\}=\{Sx+\Phi x+T(-x)+\Phi(-x),\ x\in A\}=\{(S-T)x,\ x\in A\}=A$.

Now for $X\in\mathscr{C}(A,B,C)$, $X\neq S$ let us set $D(X)=\{S(X-S)^{-1}(X-T)z+\Phi(X-S)^{-1}(X-T)z+\Phi'(Sz+\Phi z),\quad z\in A\}=\{S(X-S)^{-1}(X-T)z+\Phi(X-S)^{-1}(X-T)z-Tz-\Phi z,\ z\in A\}$. Further for $X\neq T$ we have $D(X)=\{Sy+\Phi y-T(X-T)^{-1}(X-S)y-\Phi(X-T)^{-1}(X-S)y,\ y\in A\}$. But because of $(S-X)y=(X-T)(X-T)^{-1}(S-X)y$ there is $X(y+(X-T)^{-1}(S-X)y)=Sy+T(X-T)^{-1}(S-X)y$ and thus $D(X)=\{X(y+(X-T)^{-1}(S-X)y)+\Phi(y+(X-T)^{-1}(S-X)y),\ y\in A\}$.

If now $x\in A$, there exist $a,\ b\in A$ such that $Xx+\Phi x=Sa+\Phi a+Tb+\Phi b$. From this it follows that $\Phi(x)=\Phi(a+b)$, $x=a+b$, $X(a+b)=Sa+Tb$, that is $(S-X)a=(X-T)b$, $b=(X-T)^{-1}(S-X)a$. To any $x\in A$ we can find $a\in A$ such that $x=a+(X-T)^{-1}(S-X)a$. Therefore $D(X)=\{Xx+\Phi x,\ x\in A\}$. If $X=T$, one can directly verify that $D(T)=\{Tx+\Phi x,\ x\in A\}$.

d) $(Q'^{+},\mathscr{L}_{Q'})$ is isomorphic to $(A',\mathscr{C}(A',B',C'))$, where $A'\neq A$, $B'\neq A$, $C'\neq A$. Let $A'=\{Sx+\Phi x,\ x\in A\}$, $B'=\{Tx+\Phi x,\ x\in A\}$, $C'=\{Ux+\Phi x,$ $x\in A\}$. The mapping $\Phi'(Sx+\Phi x)=T(U-T)^{-1}(S-U)x+\Phi(U-T)^{-1}(S-U)x$ is an isomorphism of A' onto B' for there holds $\{y+\Phi'y,\ y\in A'\}=\{Sx+\Phi x+T(U-T)^{-1}(S-U)x+\Phi(U-T)^{-1}(S-U)x,\quad x\in A\}=\{Ux+\Phi x,$ $x\in A\}=C'$. If $D=\{Sx+\Phi x+T(U-T)^{-1}(S-U)x+\Phi(U-T)^{-1}(S-U)x,$ $x\in A\}$, then $D=\{U(x+(U-T)^{-1}(S-U)x)+\Phi(x+(U-T)^{-1}(S-U)x),$ $x\in A\}$, because $Sx+T(U-T)^{-1}(S-U)x=Ux+U(U-T)^{-1}(S-U)x$ for every $x\in A$. If $y\in A$, then there exists $a,\ b\in A$ such that $Uy+\Phi y=Sa+\Phi a+Tb+\Phi b$. This implies $\Phi y=\Phi a+\Phi b$, $y=a+b$, $U(a+b)=Sa+Tb$, $(S-U)a=(U-T)b$, $b=(U-T)^{-1}(S-U)a$. Thus to every $y\in A$ there exists $x\in A$ such that $y=x+(U-T)^{-1}(S-U)x$ and therefore $D=\{Uy+\Phi y,\ y\in A\}=C'$.

For $X\in\mathscr{C}(A,B,C)$, $X\neq S$ we set $D(X)=\{S(X-S)^{-1}(X-T)(U-T)^{-1}(U-S)z+\Phi(X-S)^{-1}(X-T)(U-T)^{-1}(U-S)z+\Phi'(Sz+\Phi z),\qquad z\in A\}=$

$-\{S(X-S)^{-1}(X-T)y+\Phi(X-S)^{-1}(X-T)y-Ty-\Phi y,\ y\in A\}$. Because of $(X-T)y=(X-S)(X-S)^{-1}(X-T)y$, there is $S(X-S)^{-1}(X-T)y-Ty=$ $=X((X-S)^{-1}(X-T)y-y)$ and thus $D(X)=\{X((X-S)^{-1}(X-T)y-y)+$ $+\Phi((X-S)^{-1}(X-T)y-y),\ y\in A\}$. If $x\in A$, there exist $a, y\in A$ such that $Xx+$ $+\Phi x=Sa+\Phi a-Ty-\Phi y$ From this it follows that $\Phi x=\Phi a-\Phi y$, $x=a-y$, $Xa-Xy=Sa-Ty$, $(X-S)a=(X-T)y$, $x=(X-S)^{-1}(X-T)y-y$, $D(X)=$ $\{Xx+\Phi x,\ x\in A\}$.

Let us suppose that $(Q^+, \mathscr{L}_Q)$ is associated to $(Q'^+, \mathscr{L}^*_{Q'})$. Then there exist $S, T, U\in\mathscr{L}^*_Q$ and an isomorphism Φ of the group Q onto the group Q'^+ such that $\mathscr{L}^*_Q\{\varphi(S, T, U)X\varphi^{-1},\ X\in\mathscr{L}^*_Q\}$. If

a) $S=\infty$, let us set $A'=A$, $B'=\{Tx+\Phi x,\ x\in A\}$, $C'=\{Ux+\Phi x,\ x\in A\}$,

b) $T=\infty$, let us set $A'=\{Sx+\Phi x,\ x\in A\}$, $B'=A$, $C'=\{Ux+\Phi x,\ x\in A\}$,

c) $U=\infty$, let us set $A'=\{Sx+\Phi x,\ x\in A\}$, $B'=\{Tx+\Phi x,\ x\in A\}$, $C'=A$,

d) $S\neq\infty$, $T\neq\infty$, $U\neq\infty$ let us set $A'=\{Sx+\Phi x,\ x\in A\}$, $B'=\{Tx+\Phi x,$ $x\in A\}$, $C'=\{Ux+\Phi x,\ x\in A\}$, then $(Q'^+, \mathscr{L}_Q)$ is isomorphic to $(A', \mathscr{L}(A', B', C'))$. This finishes the proof.

Proposition 15. *Let there be given a translation plane $\alpha(W, \mathscr{K})$ with a coordinate system (p, A, B, C) and let $(x_{p'}, y_{p'})$ be coordinates of the point p' with respect to the coordinate system. Further let $S, T, U\in\mathscr{L}(A, B, C)$, $S\neq T\neq U\neq S$, $A'=\{Sx+\Phi x,\ x\in A\}$, $B'=\{Tx+\Phi x,\ x\in A\}$, $C'=\{Ux+\Phi x,$ $x\in A\}$. $\varphi(\ )$ $S+\Phi x$ is an isomorphism of A onto A. A point z with coordinates (x, y) with respect to the coordinate system (p, A, B, C) has coordinates (x', y') with respect to the coordinate system*

a) (p', A, B', C') *and there is* $\quad x=(U-T)^{-1}x'+x_p$
$\quad y=T(U-T)^{-1}x'+y'+y_p$

b) (p', A', A, C') *and there is* $\quad x=\varphi^{-1}y'+x_p$
$\quad y=(U-S)\varphi^{-1}x'+S\varphi^{-1}y'+y_p$

c) (p', A', B', A) *and there is* $\quad x=\varphi^{-1}x'+\varphi^{-1}y'+x_p$
$\quad y=T\varphi^{-1}x'+S\varphi^{-1}y'+y_p$

d) (p', A', B', C') *and there is*
$\quad x=(U-T)^{-1}(S-U)\varphi^{-1}x'+\varphi^{-1}y'+x_p$
$\quad y=T(U-T)^{-1}(S-U)\varphi^{-1}x'+S\varphi^{-1}y'+y_p$

Proof.

a) $z-p'=y'+\Phi'x'=y'+T(U-T)^{-1}x'+\Phi(U-T)^{-1}x'$. Further we have $z-$ $p=y+\Phi x=(z-p')+(p'-p)=y'+T(U-T)^{-1}x'+\Phi(U-T)^{-1}x'+y_p+\Phi x_p$,

b) $z-p'=y'+\Phi'x'=S\varphi^{-1}y'+\Phi\varphi^{-1}y'+\Phi'(S\varphi^{-1}x'+\Phi\varphi^{-1}x')=S\varphi^{-1}y'+$ $\Phi\varphi^{-1}y'+(U-S)\varphi^{-1}x'$. Further we have $z-p=y+\Phi x=(z-p')+(p'-p)=$ $S\varphi^{-1}y'+\Phi\varphi^{-1}y'+(U-S)\varphi^{-1}x'+y_p+\Phi x_p$.

c) $z-p'=y'+\Phi'x'=S\varphi^{-1}y'+\Phi\varphi^{-1}y'+\Phi'(S\varphi^{-1}x'+\Phi\varphi^{-1}x')=S\varphi^{-1}y'+$

$\Phi\varphi^{-1}y' - T\varphi^{-1}x' - \Phi\varphi^{-1}x'$; thus we get $z-p = y+\Phi x = (z-p')+(p'-p) = S\varphi^{-1}y' + T\varphi^{-1}x' + y_{p'} + \Phi\varphi^{-1}y' + \Phi\varphi^{-1}x' + \Phi x_{p'}$,

d) $z-p' = y'+\Phi'x' = S\varphi^{-1}y' + \Phi\varphi^{-1}y' + \Phi(S\varphi^{-1}x' + \Phi\varphi^{-1}x') = S\varphi^{-1}y' + \Phi\varphi^{-1}y' + T(U-T)(S-U)\varphi^{-1}x' + \Phi(U-T)^{-1}(S-U)\varphi^{-1}x'$, and from this

$$z-p = y+\Phi x = (z-p')+(p'-p)$$
$$= S\varphi^{-1}y' + T(U-T)^{-1}(S-U)\varphi^{-1}x' + y_{p'} + \Phi\varphi^{-1}y' + \Phi(U-T)^{-1}(S-U)\varphi^{-1}x' + \Phi x_{p'}.$$

§ 5. Geometrical properties of L-systems

Lemma 1. *Let $(G, \mathscr{L})$ be a L-system, $U, V \in \mathscr{L}$, $U \neq V$, $\mathscr{L}'(U) = \{X-U, X \in \mathscr{L}\}$, $\mathscr{L}''(U, V) = \{(X-U)(V-U)^{-1}, X \in \mathscr{L}\}$.*

a) *$(G, \mathscr{L}'(U))$ is an L-system such that $\mathscr{L}'(U)$ contains the zero endomorphism.*

b) *$(G, \mathscr{L}''(U, V))$ is a QL-system.*

Proof. The set $\{\{(Xx, x), x \in G\}, X \in \mathscr{L}\} \cup \{(x, 0), x \in G\}$ is a congruence in the group $G \oplus G$. We denote $A = \{(x, 0), x \in G\}$.

a) Let us denote $S = \{(Ux, x), x \in G\}$. The mapping Φ: $A \to S$ defined by $\Phi(x, 0) = (Ux, x)$ is an isomorphism. For every $X \in \mathscr{L}$ there is $\{((X-U)x, 0) + \Phi(x, 0), x \in G\} = \{((X-U)x + Ux, x), x \in G\} = \{(Xx, x), x \in G\}$.

b) Let us denote $B = \{(Ux, x), x \in G\}$, $C = \{(Vx, x), x \in G\}$. Because of $\{(x, 0) + (U(V-U)^{-1}x, (V-U)^{-1}x), x \in G\} = \{(x + U(V-U)^{-1}x, (V-U)^{-1}x, x \in G\} = \{((V-U)y + Uy, y), y \in G\} = \{(Vy, y), y \in G\}$, the mapping $\Phi(x, 0) = (U(V-U)^{-1}x, (V-U)^{-1}x)$ is an isomorphism of A onto B such that $\{(x, 0) + \Phi(x, 0), x \in G\} = C$. For $X \in \mathscr{L}$ there holds $\{((X-U)(V-U)^{-1}x, 0) + \Phi(x, 0), x \in G\} = \{((X-U)(V-U)^{-1}x + U(V-U)^{-1}x, (V-U)^{-1}x), x \in G\} = \{((X-U)y + Uy, y), y \in G\} = \{(Xy, y), y \in G\}$.

An L-system $(G, \mathscr{L})$ and an element $a \in G \dot{-} \{0\}$ determine a GVW-system $Q_a(G, \mathscr{L})$ and an affine translation plane $\alpha(Q_a(G, \mathscr{L}))$. If $b \in G \dot{-} \{0\}$, the planes $\alpha(Q_a(G, \mathscr{L}))$ and $\alpha(Q_b(G, \mathscr{L}))$ are isomorphic. We say that an L-system $(G, \mathscr{L})$ is associated to a plane α if there exists an $a \in G \dot{-} \{0\}$ such that α is isomorphic to $\alpha(Q_a(G, \mathscr{L}))$ and we write $\alpha = \alpha(G, \mathscr{L})$.

From the construction performed in the proof of lemma 1 it follows:

Proposition 16. *If $(G, \mathscr{L})$ is an L-system, then for every $U, V \in \mathscr{L}$; $U \neq V$ the systems $(G, \mathscr{L})$, $(G, \mathscr{L}'(U))$, $(G, \mathscr{L}''(U, V))$ are associated to the same affine plane.*

For the sake of brevity we introduce the following mappings defined for all $U, V, X, Y \in \varepsilon(G)$ (the set of all endomorphisms of the group G) for which the expressions make sense:

1. $\Phi_U: \varepsilon(G) \times \varepsilon(G) \to \varepsilon(G)$

   $\Phi_U(X, Y) = X + Y - U$
2. $\Psi_{UV}: \varepsilon(G) \to \varepsilon(G)$

$\Psi_{UV}(X)=(V-U)(X-U)^{-1}(V-U)+U$

3 $\chi_{UV}: \varepsilon(G)\times\varepsilon(G)\to\varepsilon(G)$

$\chi_{UV}(X, Y)=(X-U)(V-U)^{-1}(Y-U)+U$

4. $\tau_{UV}: \varepsilon(G)\times\varepsilon(G)\to\varepsilon(G)$

$\tau_{UV}(X, Y)=(X-U)(V-U)^{-1}(Y-U)-(Y-U)(V-U)^{-1}(X-U)$.

Let us suppose moreover that G does not contain elements of the order 2 (i.e. $x+x=0$ implies $x=0$).

Lemma 2. *Let $(G, \mathscr{L})$ be a L-system. If there exists $U\in\mathscr{L}$ such that $(\mathscr{L}'(U), +)$ is a group, then for every V, U_1 with $V\neq U_1$ $(\mathscr{L}'(U_1), +)$ and $(\mathscr{L}''(U_1, V), +)$ are* roups.

Proof. If $\mathscr{L}'(U)$ is a group, then

a) for every X, $Y\in\mathscr{L}$ there is $(X-U)+(Y-U)\in\mathscr{L}'(U)$, i.e. $X+Y-U\in\mathscr{L}$ and

b) the inverse element to $X-U$ in $\mathscr{L}'(U)$ is an element $X'-U$ such that $(X-U)+(X'-U)=0$, i.e. $X+X'=2U$ and vice versa, $(\mathscr{L}'(U_1), +)$ is a group if and only if

(i) $\Phi_{U_1}(\mathscr{L})\subset\mathscr{L}$

(ii) $\forall X\in\mathscr{L}\ \exists x'\in\mathscr{L}\qquad X+X'=2U_1$.

Let $U_1\in\mathscr{L}$. To $X\in\mathscr{L}$ there exists $X'\in\mathscr{L}$ with $X+X'=2U$ and setting $X_1'=X'+2U_1-2U$ we have $X+X_1'=2U_1$, $X_1'=((X'+U_1-U)+U_1)-U_1$, $A=X'+U_1-U$, $A+U_1-U\in\mathscr{L}$. To $U_1\in\mathscr{L}$ there exists $U_1'\in\mathscr{L}$ with $U_1+U_1'=2U$ and for X, $Y\in\mathscr{L}$ there holds $X+Y-U_1=((X+Y-U)+U_1')-U\in\mathscr{L}$ for $A-X+Y-U\in\mathscr{L}$, $A+U_1'-U\in\mathscr{L}$. The second part follows immediately from the equality $(X-U_1)(V-U_1)^{-1}+(Y-U_1)(V-U_1)^{-1}=(X+Y-2U_1)\cdot(V-U_1)^{-1}$ and from the foregoing part.

Proposition 17. 1. *A QL-system $(G, \mathscr{L}''(U, V))$ satisfies the condition $X\in\mathscr{L}''(U, V)\Rightarrow X^{-1}\in\mathscr{L}''(U, V)$ if and only if the QL-system $(G, \mathscr{L})$ has the property $\Psi_{UV}(\mathscr{L})\subset\mathscr{L}$.*

2. The QL-system $(G, \mathscr{L}''(U, V))$ is a skewfield with respect to the addition and composition of homomorphisms $((\mathscr{L}''(U, V), +, \circ)$ is a skewfield) if and only if for some U_1 $(\mathscr{L}'(U_1), +)$ is a group, $\chi_{UV}(\mathscr{L}\times\mathscr{L})\subset\mathscr{L}$ and $\Psi_{UV}(\mathscr{L})\subset\mathscr{L}$.

3 Under the assumption of 2, $\mathscr{L}''(U, V)$ is a field if and only if $\tau_{UV}(\mathscr{L}\times\mathscr{L})=0$, i e. $\mathscr{L}\times\mathscr{L}\subset\operatorname{Ker}\tau_{UV}$.

Proof. 1. $X=(A-U)(V-U)^{-1}$, $X^{-1}=(V-U)(A-U)^{-1}$, $(V-U)(A-U)^{-1}=(Z-U)(V-U)^{-1}\in\mathscr{L}''(U, V)$ if and only if $Z-(V-U)(A-U)^{-1}(V-U)+U\in\mathscr{L}$.

2. $X=(A-U)(V-U)^{-1}$, $Y=(B-U)(V-U)^{-1}$ are elements $\mathscr{L}''(U, V)$ $X\circ Y=(A-U)(V-U)^{-1}(B-U)(V-U)^{-1}=(Z-U)(V-U)^{-1}\in\mathscr{L}''(U, V)$ if and only if $Z=(A-U)(V-U)^{-1}(B-U)+U\in\mathscr{L}$. The identity isomorphism $I=(V-U)(V-U)^{-1}\in\mathscr{L}''(U, V)$ for every U, $V\in\mathscr{L}$, $X^{-1}\in\mathscr{L}''(U, V)$ by 1.

3. $X \circ Y = Y \circ X$ if and only if in the notation of 2 there is $(A-U)(V-U)^{-1}(B-U)(V-U)^{-1} = (B-U)(V-U)^{-1}(A-U)(V-U)^{-1} = \tau_{UV}(A, B) = 0$.

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ОДНО ОБОБЩЕНИЕ СУСТЕМЫ ВЕБЛЕН—ВЕДЕРБЕРНА

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Резюме

В статье находится алгебраическое описание плоскостей трансляций при помощи обобщения системы Веблен—Ведерберна. При помощи линейных преобразований определяется соотношение между системами Веблен—Ведерберна принадлежащими к такой же плоскости трансляций.