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ON DOUBLE COVERS OF GRAPHS

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In [1] D. A. Waller has proposed the following problem:

Characterization problem for covering graphs.

A graph D with 2n vertices is a double cover graph if it admits a vertex-labelling such that

(i) *each integer* $r \in \{1, \ldots, n\}$ *occurs exactly twice (as* r *and* r'*),*
(ii) *adjacencies occur in pairs, in the form* ($r \sim s$ *and* $r' \sim s'$) *or* ($r \sim s'$ *and* $r' \sim s$).

Such a labelling of D determines a quotient graph G and a 2:1 *projection morphism* $p: D \to G$, *given by* $p(r) = p(r') = r$.

Problem 1. *Characterize double cover graphs.*

Problem 2. *Characterize graphs uniquely expressible as a double cover.*

From the definition of a double cover graph quoted above it is not clear, whether it is possible that r might be adjacent to both s and s' or that r might be adjacent to r'. But the following definition from paper [2] by M. Farzan clearly excludes these cases:

Given a map $f: E(G) \to Z_2$, *the graph* $D = dc(G, f)$ *is a double cover of G when* $V(D) = V(G) \times Z_2$ *and* $[(u, x), (v, y)] \in E(D)$ *if and only if* $[u, v] \in E(G)$ *and* $f([u, v]) = xy$.

Here Z_2 denotes a group of the order 2.

If we therefore consider undirected graphs without loops and multiple edges, in the double cover of a graph the subgraph induced by the set $\{r, s, r', s'\}$, where $r \neq s$, contains either only edges rs and $r's'$, or only edges rs' and $r's$, or no edges.

We shall prove some theorems concerning double covers of graphs. We consider finite undirected graphs without loops and multiple edges.

Theorem 1. *Let H be a finite undirected graph. The following two assertions are equivalent:*
(a) *There exists an automorphism* α *of H such that* $\alpha(\alpha(x)) = x$ *and* $d(x, \alpha(x)) \geqq 3$ *holds for each vertex x of H.*
(b) *H is a double cover graph.*

Remark. The symbol $d(x, y)$ denotes the distance of x and y in H.

Proof. (a) $\Rightarrow$ (b). The sets $\{x, \alpha(x)\}$ for all $x \in V(H)$, where $V(H)$ denotes the vertex set of H, form a partition of $V(H)$ in which each class has exactly two elements. We choose an element from each class and denote the set of the chosen elementy by W. Further we put $W' = V(H) - W$. We denote the elements of W by $1, \ldots, n$, where n is the cardinality of W. For each element $r \in W$ the image $\alpha(r) \in W'$; we denote it by r'. Now let r, s be two elements of W. The vertices r and r' are not adjacent, because by the assumption $d(r, r') \geqq 3$. If r and s are adjacent, so are their images $r' = \alpha(r)$, $s' = \alpha(s)$. The vertices r and s' are not adjacent, because otherwise there would exist a path of the length 2 connecting r and r' with the inner vertex s'; analogously r' and s are not adjacent. If r and s' are adjacent, so are their images $r' = \alpha(r)$, $s = \alpha(s')$ and neither r and s, nor r' and s' are adjacent. Therefore H is a double cover graph.

(b) $\Rightarrow$ (a). Let H be a double cover graph. Define α so that $\alpha(r) = r'$, $\alpha(r') = r$ for each $r \in \{1, \ldots, n\}$. Then obviously $\alpha(\alpha(x)) = x$ for each $x \in V(H)$. The adjacency between r and s is equivalent to the adjacency of r' and s' and the adjacency between r and s' is equivalent to the adjacency of r' and s, therefore α is an automorphism of H. Now let $r \in \{1, \ldots, n\}$. As always $r \neq r'$, we have $d(r, r') \geqq 1$. If $d(r, r') = 1$, then r and r' would be adjacent, which was excluded. If $d(r, r') = 2$, then there would exist a vertex of H adjacent to both r and r'. If this vertex were s for some $s \in \{1, \ldots, n\} - \{r\}$, then there would be an adjacency between r and s and between r' and s, which was excluded; analogously if the mentioned vertex would be s'. Therefore $d(r, r') \geqq 3$ for each $f \in \{1, \ldots, n\}$, i.e. $d(x, \alpha(x)) \geqq 3$ for each $x \in V(H)$.

Now we shall study graphs uniquely expressible as double covers. Two expressions of a given graph as a double cover will be considered as different if the corresponding partitions into two-element classes $\{r, r'\}$ for $r \in \{1, \ldots, n\}$ are distinct. Evidently two such expressions are different if and only if different automorphisms α from Theorem 1 correspond to them. A graph H will be called uniquely expressible as a double cover if it is a double cover graph and any two expressions of H as a double cover determine the same partition of the vertex set of H into two-element classes $\{r, r'\}$ for $r \in \{1, \ldots, n\}$.

The following result follows immediately from Theorem 1.

Corollary 1. *Let H be a finite undirected graph. The following two assertions are equivalent:*

(c) *There exists exactly one automorphism α of H such that $\alpha(\alpha(x)) = x$ and $d(x, \alpha(x)) \geqq 3$ holds for each vertex x of H.*

(d) *H is uniquely expressible as a double cover graph.*

Exemples of such graphs are the circuit of the length 6 and the graph of the 3-dimensional cube. Both these graphs have the diameter 3 and in each of them there exists to each vertex exactly one vertex which has the distance 3 from it and is its image in an involutory automorphism.

Corollary 2. *Let H be a graph consisting of two isomorphic connected components. The graph H is uniquely expressible as a double cover graph if and only if none of its connected components has a non-identical automorphism.*

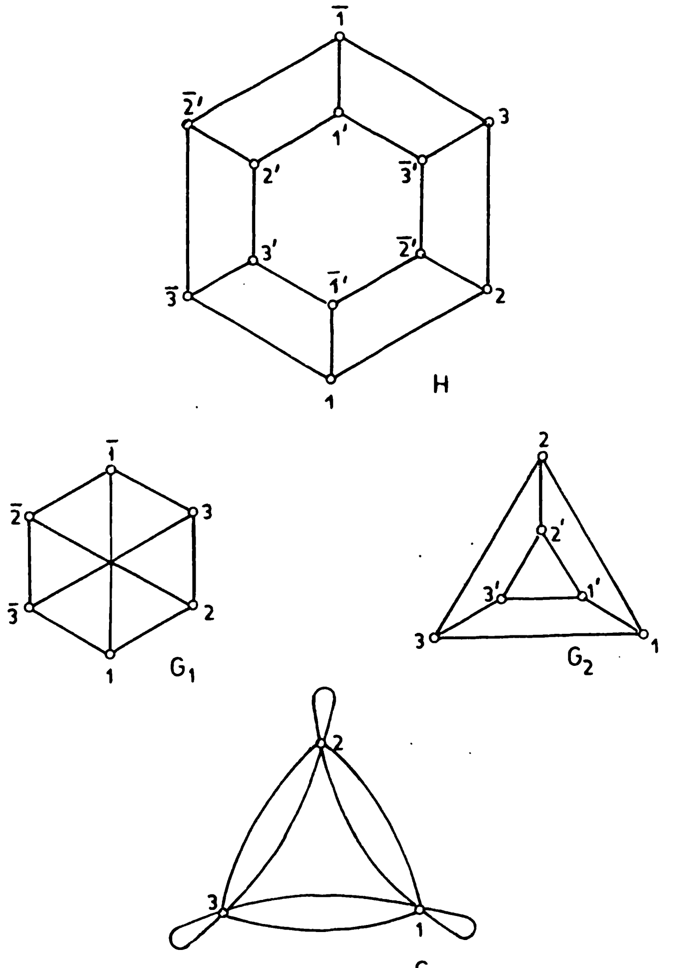

Fig. 1

Now we shall consider graphs which can be expressed as double covers by means of two distinct automorphisms α, β, which are commutative. We shall use the concept of a pseudograph. A pseudograph is a graph in which loops and multiple edges are admitted.

Theorem 2. *Let H be a finite undirected graph, let α, β be two distinct automorphisms of H such that $\alpha(\alpha(x)) = \beta(\beta(x)) = x$, $d(x, \alpha(x)) \geqq 3$,*

$d(x, \beta(x)) \geqq 3$, $\alpha(\beta(x)) = \beta(\alpha(x))$ for each vertex x of H. Let G_1 (or G_2) be the quotient graph for the expression of H as a double cover by means of α (or β respectively). Then for $i \in \{1, 2\}$ the graph G_i has an automorphism γ_i such that $\gamma_i(\gamma_i(x)) = x$ for each vertex x of G_i and this automorphism has the following property: The mappings γ_1, γ_2 have the same number of fixed vertices and if we identify x with $\gamma_i(x)$ for each vertex x of G_i so that all edges remain, we obtain isomorphic pseudographs for both $i = 1$ and $i = 2$.

Proof. As α, β are commutative in the group Aut H of all automorphisms of H, the set $\{\varepsilon, \alpha, \beta, \alpha\beta\}$, where ε is the identical mapping of the vertex set of H, is a subgroup of Aut H; denote it by $\mathfrak{G}$. Each orbit of $\mathfrak{G}$ has either two or four elements, because $\alpha(x) \neq x$, $\beta(x) \neq x$ for each $x \in V(H)$. From each orbit of $\mathfrak{G}$ we choose one element and the set of the chosen elements will be denoted by W. Further we put $W' = \{\alpha(x) \mid x \in W\}$, $\bar{W} = \{\beta(x) \mid x \in W\}$, $\bar{W}' = \{\alpha\beta(x) \mid x \in W\}$. At least one orbit of $\mathfrak{G}$ has four elements; otherwise there would be $\alpha = \beta$. Let p be the number of orbits of $\mathfrak{G}$ with four elements, let q be the number of all orbits of $\mathfrak{G}$. We label the vertices of W by $1, \ldots, q$ so that the elements which were chosen from the orbits with four elements might have the labels $1, \ldots, p$. For each $r \in \{1, \ldots, q)$ denote $r' = \alpha(r)$, $\bar{r} = \beta(r)$, $\bar{r}' = \alpha\beta(r)$. Evidently $r' \in W'$, $\bar{r} \in \bar{W}$, $\bar{r}' \in \bar{W}'$. For $1 \leqq r \leqq p$ the elements $r, r', \bar{r}, \bar{r}'$ are pairwise different. For $p + 1 \leqq r \leqq q$ we have $r = \bar{r}' \neq r' = \bar{r}$. Now the quotient graph G_1 has the vertex set $\{1, \ldots, q, \bar{1}, \ldots, \bar{p}\}$ and two vertices r, s of this set are adjacent in it if and only if in H either the pairs $\{r, s\}$, $\{r', s'\}$ or the pairs $\{r, s'\}$, $\{r', s\}$ are adjacent. The quotient graph G_2 has the vertex set $\{1, \ldots, q, 1', \ldots, p'\}$ and two vertices r, s of this set are adjacent in it if and only if in H either the pairs $\{r, s\}$, $\{\bar{r}, \bar{s}\}$, or the pairs $\{r, \bar{s}\}$, $\{\bar{r}, s\}$ are adjacent. Now we define a pseudograph G_0. The vertex set of G_0 will be the set $\{1, \ldots, q\}$. If $r \in \{1, \ldots, q\}$, then in H the vertex r is adjacent neither to r' nor to $\bar{r}$, but it may be adjacent to $\bar{r}'$ (if $\bar{r}' \neq r$). In this case there is the edge $r\bar{r}$ in G_1 and the edge rr' in G_2. In G_0 at r there will be a loop. Now let r, s be two distinct elements of $\{1, \ldots, p\}$. If in H the vertex r is adjacent to s and to none of the vertices $s', \bar{s}, \bar{s}'$, also the pairs $\{r', s'\}$, $\{\bar{r}, \bar{s}\}$, $\{\bar{r}', \bar{s}'\}$ are adjacent in H. In G_1 the pairs $\{r, s\}$, $\{\bar{r}, \bar{s}\}$ and in G_2 the pairs $\{r, s\}$, $\{r', s'\}$ are adjacent. In G_0 there will be one edge joining r and s. The cases where r is adjacent to exactly one of the vertices $s', \bar{s}, \bar{s}'$ are analogous. If r is adjacent to s, then it cannot be adjacent to s' or to $\bar{s}$, but it may be adjacent to $\bar{s}'$. If this occurs and $s \neq \bar{s}'$, then in H also the pairs $\{r', s'\}$, $\{r', \bar{s}\}$, $\{\bar{r}, \bar{s}\}$, $\{\bar{r}, s'\}$, $\{\bar{r}', \bar{s}'\}$, $\{\bar{r}', s\}$ are adjacent. In G_1 the pairs $\{r, s\}$, $\{r, \bar{s}\}$, $\{\bar{r}, \bar{s}\}$, $\{\bar{r}, s\}$, in G_2 the pairs $\{r, s\}$, $\{r, s'\}$, $\{r', s\}$, $\{r', s'\}$ are adjacent. In G_0 there will be two edges joining r and s. New let $r \in \{p + 1, \ldots, q\}$, $s \in \{1, \ldots, p\}$, i.e. $r = \bar{r}'$. If r is adjacent to s, then it is adjacent to $\bar{s}'$ and it cannot be adjacent to s' and $\bar{s}$. Then in G_1 the pairs $\{r, s\}$, $\{r, \bar{s}\}$ and in G_2 the pairs $\{r, s\}$, $\{r, s'\}$ are adjacent. In G_0 there will be two edges joining r and s. Analogously if r is adjacent to s or s'. Finally, if $\{r, s\} \subseteq \{p + 1, \ldots, q\}$ and r is adjacent to s in H, then the pair $\{r, s\}$ is adjacent in both G_1, G_2 and it will be joined by one edge in G_0; analogously if r is adjacent to s' or $\bar{s}$. Now define γ_1 so that $\gamma_1(r)=\bar{r}$, $\gamma_1(\bar{r})=r$ for $r \in \{1, ..., p\}$ and $\gamma_1(r)=r$ for $r \in \{p+1, q\}$. The mapping γ_2 will be defined so that $\gamma_2(r)=r'$, $\gamma_2(r')=r$ for $r \in \{1, ..., p\}$, $\gamma_2(r)=r$ for $r \in \{p+1, ..., q\}$. We see that the pseudograph G_0 is obtained from both G_1 and G_2 in the described way.

The author of the problem presents an example of a graph which can be expressed as a double cover graph in two ways such that the quotient graphs are not isomorphic. It is the graph of the 6-sided prism. The quotient graphs are the graph of the 3-sided prism and the graph $K_{3,3}$. The corresponding automorphisms are commutative. In Fig. 1 we see the labelling of this graph H by 1, 2, 3, 1′, 2′, 3′, $\bar{1}$, $\bar{2}$, $\bar{3}$, $\bar{1}'$, $\bar{2}'$, $\bar{3}'$, the quotient graphs G_1, G_2 and the pseudograph G_0.

Theorem 3. *Let G_1, G_2 be two finite undirected graphs, which are expressible as double covers of the same graph G. Then there exists a graph D which can be expressed as a double cover of G_1 and simultaneously as a double cover of G_2.*

Proof. Let the number of vertices of G be n. Let the vertices of G_1 be labelled by r and r' for $r \in \{1, ..., n\}$ and let the vertices of G_2 be labelled by r and $\bar{r}$ for $r \in \{1, ..., n\}$ so that both these labellings might fulfil the conditions from the definition of a double cover graph. Let the vertices of D be $1, ..., n, 1', ..., n'$, $\bar{1}, ..., \bar{n}, \bar{1}', ..., \bar{n}'$. The vertices r, s for $\{r, s\} \subseteq \{1, ..., n\}$ are adjacent in D if and only if they are adjacent in both G_1 and G_2; then also the pairs $\{r', s'\}$, $\{\bar{r}, \bar{s}\}$, $\{\bar{r}', \bar{s}'\}$ are adjacent in D. Two vertices r, s' are adjacent if and only if r, s' are adjacent in G_1 and r, s are adjacent in G_2; then also the pairs $\{r', s\}$, $\{\bar{r}, \bar{s}'\}$, $\{\bar{r}', \bar{s}\}$ are adjacent in D. The vertices $r, \bar{s}$ are adjacent in D if and only if the vertices r, s are adjacent in G_1 and the vertices $r, \bar{s}$ are adjacent in G_2; then also the pairs $\{r', \bar{s}'\}$, $\{\bar{r}, s\}$, $\{\bar{r}', s'\}$ are adjacent in D. The vertices $r, \bar{s}'$ are adjacent in D if and only if the vertices r, s' are adjacent in G_1 and $r, \bar{s}$ are adjacent in G_2; then also the pairs $\{r', \bar{s}\}$, $\{\bar{r}, s'\}$, $\{\bar{r}', s\}$ are adjacent. No other edges than the described ones are in D. The graph D is a double cover of both G_1 and G_2.

The last theorem will show that the situation described in Theorem 2 is not too rare. Before stating it, we shall prove a lemma.

Lemma. *Let H be an undirected graph, let α, β be two automorphisms of H such that $\alpha(\alpha(x)) = \beta(\beta(x)) = x$, $d(x, \alpha(x)) \geqq 3$, $d(x, \beta(x)) \geqslant 3$ for each vertex x of H. Then the mapping $\varphi = \alpha\beta\alpha$ has also these properties.*

Proof. In the group Aut H we have $\alpha^2=\varepsilon$, $\beta^2=\varepsilon$, where ε is the identical mapping of H. Hence

$$\varphi^2 = \alpha\beta\alpha^2\beta\alpha = \alpha\beta^2\alpha = \alpha^2 = \varepsilon .$$

As $d(x, \beta(x)) \geqq 3$ for each vertex x of H, this holds also for the vertex $\alpha(x)$, where

x is an arbitrary vertex of H and we have $d(\alpha(x), \beta\alpha(x) \geqq 3$ for each vertex x of H. As α is an automorphism of H, it preserves the distance and we have

$$d(x, \alpha\beta\alpha(x)) = d(\alpha^2(x), \alpha\beta\alpha(x)) = d(\alpha(x), \quad \beta\alpha(x)) \geqq 3 .$$

Theorem 4. *Let H be a finite undirected graph, let α, γ be two automorphisms of H such that $\alpha(\alpha(x)) = \gamma(\gamma(x)) = x$, $d(x, \alpha(x)) \geqq 3$, $d(x, \gamma(x)) \geqq 3$ for each vertex x of H. Let the order of $\alpha\gamma$ in* Aut H *be even. Then there exists an automorphism β of H such that $\beta(\beta(x)) = x$, $d(x, \beta(x)) \geqq 3$, $\alpha(\beta(x)) = \beta(\alpha(x))$ for each vertex x of H.*

Proof. Let the order of $\alpha\gamma$ in Aut H be $2k$, where k is a positive integer. Put $\beta = \gamma(\alpha\gamma)^{k-1}$. Then

$$\alpha\beta = \alpha\gamma(\alpha\gamma)^{k-1} = (\alpha\gamma)^k = (\alpha\gamma)^{-k} = (\gamma\alpha)^k = \gamma(\alpha\gamma)^{k-1} = \beta\alpha ,$$

hence $\beta(\alpha(x)) = \alpha(\beta(x))$ for each vertex x of H. Further properties can be proved by induction from Lemma.

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О ДВОЙНЫХ ПОКРЫТИЯХ ГРАФОВ

Богдан Зелинка

Резюме

Граф D с $2n$ вершинами является графом двойного покрытия, если он допускает помечение вершин такое, что каждое число $r \in \{1, ..., n\}$ появляется точно два раза (как r и r') и смежности появляются парами во форме ($r \sim s$ и $r' \sim s'$) или ($r \sim s'$, и $r' \sim s$). Это определение ввел Д. А. Уоллер. В статье доказаны некоторые теоремы о графах двойного покрытия.