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THE ORTHOGONALITY IN AFFINE PARALLEL STRUCTURE

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ABSTRACT. The paper deals with orthogonality of lines in an incidence structure. As a closure condition of orthogonality, the reduced pentagonal condition is used. For finding the algebraic expression of orthogonality in parallel structure, we use the Reidemeister condition except the reduced pentagonal condition.

Introduction

In this paper the orthogonality of lines in an affine parallel structure constructed over a non-planar right nearfield is investigated.

In section 1 we construct an affine parallel structure $\mathcal{A}$ over a non-planar right nearfield and its extension $\overline{\mathcal{A}}$, adding improper points and an improper line. It is proved that the Reidemeister condition (more general as in [5]) is fulfilled in this structure $\overline{\mathcal{A}}$ (see the Theorem 5 and its proof).

Section 2 includes the definition of the orthogonality of lines and its properties in the structure $\overline{\mathcal{A}}$. As a closure condition for the orthogonality of lines a reduced pentagonal condition is introduced. By this condition as well as by the Reidemeister condition from the section 1 the property (RP) of the structure $\overline{\mathcal{A}}$ is derived – the Theorem 6 on "orthogonal rectangles".

An algebraic expression of the orthogonality of lines in $\overline{\mathcal{A}}$ is derived in the section 3. Some properties of the "directions" of orthogonal lines as well as the properties of a certain mapping of the coordinate nearfield onto itself are proved. It is shown that this mapping is an automorphism. Using this fact the condition for the directions of the orthogonal lines is expressed in Theorem 11, which is the main result of this paper.

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An example of a non-planar right quasifield of the first type is given in the section 4. By a modification of the operation of the multiplication we obtain a non-planar right nearfield.

1. The parallel structure over a non-planar right nearfield

We recall the definition of the non-planar right nearfield (see [3], [4]).

DEFINITION 1. *A right nearfield is defined as a triplet $\mathcal{S} = (S, +, \cdot)$, where S is an at least two-element set equipped with two binary operations "$+$" and "$\cdot$" such that*

(S1) $(S,+)$ *is an abelian group with a null element* $\underline{0}$;
(S2) $(S^{\#},\cdot)$ *is a group with a neutral element* $\underline{1}$, *where* $S^{\#} = S \setminus \{\underline{0}\}$;
(S3) $\underline{0} \cdot a = \underline{0}$ *for all* $a \in S$;
(S4) $a \cdot (b+c) = a \cdot b + a \cdot c$ *for all* $a,b,c \in S$ *(right distributive law).*

A right nearfield is planar whenever

(S5) $\#\{x \in S \mid x = a \cdot x + b\} = 1$ *for all* $a, b \in S$, $a \neq 1$.

A right nearfield is called non-planar when it is not planar.

We recall some known properties of the right nearfield without proof.

LEMMA 1. *For any right nearfield $\mathcal{S} = (S, +, \cdot)$ the following holds:*

(1) $a \cdot (b-c) = a \cdot b - a \cdot c$ *for all* $a,b,c \in S$,
(2) $a \cdot \underline{0} = \underline{0}$ *for all* $a \in S$,
(3) $a \cdot (-b) = (-a) \cdot b = -a \cdot b$ *for all* $a, b \in S$.

We shall use the following notions:

DEFINITION 2.

(a) *The kernel* $\mathrm{Ker}(\mathcal{S})$ *of a given right nearfield* $\mathcal{S} = (S,+,\cdot)$ *is the set of all elements* $a \in S$ *such that*

$$(x+y) \cdot a = x \cdot a + y \cdot a \qquad \text{for all} \quad x, y \in S.$$

(b) *The centre* $\mathrm{Z}(\mathcal{S})$ *of* $\mathcal{S} = (S,+,\cdot)$ *is the set of all elements* $a \in S$ *such that*

$$x \cdot a = a \cdot x \qquad \text{for all} \quad x \in S.$$

The following properties of $\mathrm{Ker}(\mathcal{S})$ and $\mathrm{Z}(\mathcal{S})$ follow from Definition 1 and 2.

LEMMA 2.

(a) $\mathrm{Ker}(\mathcal{S})$ *is always a skewfield,*
(b) $\mathrm{Z}(\mathcal{S})$ *is always a field.*

Since any finite right nearfield is planar, by $\mathcal{S} = (S, +, \cdot)$ we will mean an infinite non-planar right nearfield, further ($\#S$ is an infinite cardinal number).

In what follows we omit the notation $a \cdot b$ and we shall use briefly ab.

We are going to construct an affine parallel structure $\mathcal{A} = (P, \mathcal{L})$ over a given nearfield $\mathcal{S} = (S, +, \cdot)$:

Ordered pairs $(x, y) \in S^2$ are points from $\mathcal{A}$, therefore $P = S^2$; lines of the first type are the sets

$$\{(x, y) \in S^2 \mid x = c\}, \qquad c \in S;$$

lines of the second type are the sets

$$\{(x, y) \in S^2 \mid y = d\}, \qquad d \in S;$$

lines of the third type are the sets

$$\{(x, y) \in S^2 \mid y = ax + b\}, \qquad a \neq \underline{0}, \quad a, b \in S.$$

Thus

$$\mathcal{L} = \{\{(x, y) \in S^2 \mid x = c\} \mid c \in S\} \cup \{\{(x, y) \in S^2 \mid y = d\} \mid d \in S\} \\ \cup \{\{(x, y) \in S^2 \mid y = ax + b\} \mid a \neq \underline{0}, \ a, b \in S\}.$$

A line g, h, k of the first, second and the third type will be denoted briefly as $g = (x = c)$, $c \in S$, $h = (y = d)$, $d \in S$, $k = (y = ax + b)$, $a \neq \underline{0}$, $a, b \in S$, respectively.

The incidence of points from P and lines from $\mathcal{L}$ in the structure $\mathcal{A}$ is considered in the sense "to be an element".

A relation " $\|$ " of parallelity of lines from $\mathcal{L}$ is determined as follows: any line of the first type is parallel to any line of the first type and to no other line from $\mathcal{L}$. Similarly, any line of the second type is parallel to any line of the second type and to no others. Two lines $(y = a_1 x + b_1)$ and $(y = a_2 x + b_2)$ of the third type are parallel if and only if they have the same direction, i.e.

$$(y = a_1 x + b_1) \parallel (y = a_2 x + b_2) \iff a_1 = a_2.$$

It is easy to check the following:

(a) Any line of the first type intersects any line of the second as well as the third type (we also say they have a common point).

(b) Any line of the second type intersects any line of the first as well as the third type.

(c) Two lines of the third type may or may not have a common point. Since (S5) does not hold for $\mathcal{S}$, there are non-parallel lines of the third type having no common point in $\mathcal{L}$. Moreover, the following lemma holds.

LEMMA 3. *If a point $(a,b) \in P$ does not lie on a line $(y = kx + q) \in \mathcal{L}$, $k \neq \underline{0}$, then there is a line containing the point (a,b) and having no common point with the line $(y = kx + q)$.*

Proof. See the proof of Theorem 3.2 in [3]. □

The common point $B \in P$ of two different lines $g, h \in \mathcal{L}$ (the intersection point of g, h) will be denoted by $B = g \cap h$.

We have just constructed the structure $\mathcal{A} = (P, \mathcal{L})$ with the incidence of points and lines as well as with the parallelity of lines. It is easy to see that the axioms of an affine parallel structure (introduced by J. Andre in [1]) are fulfilled in $\mathcal{A}$.

The following is fulfilled in the affine parallel structure $\mathcal{A} = (P, \mathcal{L})$: If $A, B \in P$ are two mutually different points, then there is exactly one line $g \in \mathcal{L}$ connecting A, B. This line g will be denoted as $g = AB$ and called a *join line* of points A, B.

If $A = (a,b)$, $B = (c,d)$, then

$$\begin{aligned} g &= (x = a) && \text{whenever } a = c,\ b \neq d; \\ g &= (y = b) && \text{whenever } b = d,\ a \neq c; \\ g &= (y = kx + q) && \text{whenever } a \neq c,\ b \neq d, \text{ where } k = (d-b)(c-a)^{-1}, \\ &&& \qquad\qquad\qquad\qquad q = -ka + b. \end{aligned} \tag{4}$$

Remark 1. The constructed affine parallel structure $\mathcal{A} = (P, \mathcal{L})$ is a central translation structure in the sense of J. Andre [2].

Translations of $\mathcal{A}$ are mappings.

$\tau(a,b)\colon P \to P$, $(x,y) \mapsto (x + a,\ y + b)$, for all $(a,b) \in S^2$. The image of a line $(x = c)$, $c \in S$, $(y = kx + q)$, $k, q \in S$ under the translation $\tau(a,b)$ is the line $(x = c)\tau(a,b) = (x = c + a)$, $(y = kx + q)\tau(a,b) = (y = kx - ka + q + b)$, respectively. Any line $(y = kx + q)$, where $k = ba^{-1}$, $q \in S$ is fixed (as gross) under translation $\tau(a,b)$, $a \neq \underline{0}$. The translation $\tau(\underline{0}, b)$, fixes (as gross) any line $(x = c)$, $c \in S$.

If we denote

$$T = \{\tau(a,b) \mid (a,b) \in S^2\},$$

then $(T, \cdot)$ is a translation group of all translations of $\mathcal{A}$. The group $(T, \cdot)$ operates strictly simply transitively in the set P of points of $\mathcal{A}$.

In order to make formulations more simple as well as to express closure conditions we will extend the structure $\mathcal{A} = (P, \mathcal{L})$ adding improper points and an improper line. The improper point $\hat{\mathfrak{g}}$ of a line $g \in \mathcal{L}$ is the set

$$\hat{\mathfrak{g}} = \{g' \in \mathcal{L} \mid g' \parallel g\}.$$

The improper line h_∞ is the set of improper points of all lines from $\mathcal{L}$. A common improper point of all lines of the first and second type will be denoted as (∞), $(\underline{0})$, respectively. A common improper point of all parallel lines of third type $(y = k_0x + q)$, $q \in S$ ($k_0 \in S$ is fixed) will be denoted as (k_0). Hence

$$h_\infty = \{(k) \mid k \in S\} \cup \{(\infty)\}.$$

The structure $\overline{\mathcal{A}} = (P \cup h_\infty, \mathcal{L} \cup \{h_\infty\})$ is a projective parallel structure with prominent improper line h_∞ in the sense of J. Andre. Any proper line from $\overline{\mathcal{A}}$ is a line from $\mathcal{L}$ extended by the addition of its improper point.

We denote as $\overline{A_1A_2\dots A_n}$ $(n > 2)$ the fact that mutually distinct points $A_1, A_2, \dots, A_n \in P \cup h_\infty$ are collinear i.e. there is a line $g \in \mathcal{L} \cup \{h_\infty\}$ containing all ones.

We shall use the following lemma in the sequel.

Lemma 4. *Let A, B, C, D be mutually distinct proper points of the structure $\overline{\mathcal{A}}$ constructed over a non-planar nearfield $\mathcal{S} = (S, +, \cdot)$. Let $\overline{AB(\underline{0})} \wedge \overline{CD(\underline{0})} \wedge \overline{AD(\infty)} \wedge \overline{BC(\infty)}$. If $\overline{AC(k)}$ for some $k \in S^\#$, then $\overline{BD(-k)}$.*

Proof. We have $A = (a, b)$, $B = (c, b)$, $C = (c, d)$, $D = (a, d)$ from the assumptions of Lemma for some $a, b, c, d \in S$, $a \neq c$, $b \neq d$. Using the third relation of (4) and the properties of the nearfield $\mathcal{S}$ the proof is finished. □

The Reidemeister condition (of a certain type) in an affine parallel structure (extended by the addition of improper points and an improper line) constructed over a non-planar left quasifield was introduced and studied in [5]. Fulfilling that condition in the mentioned incidence structure implies that the coordinate quasifield is a non-planar left nearfield (see the Theorem 3 in [5]). Since the coordinate algebra of the structure $\overline{\mathcal{A}} = (P \cup h_\infty, \mathcal{L} \cup \{h_\infty\})$ constructed in this paper is the non-planar right nearfield $\mathcal{S} = (S, +, \cdot)$, the mentioned Reidemeister condition holds. (The proof follows from the fact that the coordinate algebra is a non-planar right nearfield and the lines of the third type are of the form $y = kx + q$, $k \neq \underline{0}$, $k, q \in S$).

We will prove the validity of a more general closure condition than that of Reidemeister in [5].

Definition 3. *Let $\overline{\mathcal{A}} = (P \cup h_\infty, \mathcal{L} \cup \{h_\infty\})$. We say that the Reidemeister condition of the type $(\underline{0}, \infty, k, l, m, n)$, $k, l, m, n \in S^\#$, $k \neq l \neq m$ is satisfied in $\overline{\mathcal{A}}$, constructed over a non-planar right nearfield $\mathcal{S} = (S, +, \cdot)$ whenever $\overline{\mathcal{A}}$ has the property*

(R) *Let $R \in P$ be a point and $A, B, C, D, A', B', C', D' \in P \setminus \{R\}$. If any eleven of the twelve relations from $\overline{AB(\underline{0})}$, $\overline{CD(\underline{0})}$, $\overline{A'B'(\underline{0})}$, $\overline{C'D'(\underline{0})}$,*

$\overline{AD(\infty)}$, $\overline{BC(\infty)}$, $\overline{A'D'(\infty)}$, $\overline{B'C'(\infty)}$, $\overline{RAA'(k)}$, $\overline{RBB'(1)}$, $\overline{RCC'(m)}$, $\overline{RDD'(n)}$ *hold, then the remaining twelfth relation holds as well (see Fig. 1).*

If the Reidemeister condition of the type $(\underline{0}, \infty, k, l, m, n)$ *holds for any* $k, l, m \in S^{\#}$, $k \neq l \neq m$, *then we say the Reidemeister condition holds in* $\overline{\mathcal{A}}$.

THEOREM 5. *In the parallel structure* $\overline{\mathcal{A}} = \big(P \cup h_\infty,\, \mathcal{L} \cup \{h_\infty\}\big)$ *constructed over a non-planar right nearfield* $\mathcal{S} = (S, +, \cdot)$ *the Reidemeister condition is satisfied.*

Proof. Since $\mathcal{A} = (P, \mathcal{L})$ is a central translation structure and the incidence relation between the points and lines as well as the relation of parallelity of lines (see Remark 1) are preserved in any translation structure, it is sufficient to prove Theorem 5 for the point $R = (\underline{0}, \underline{0})$.

Let $k, l, m \in S^{\#}$, $k \neq l \neq m$. Suppose, for example, the first eleven relations hold from (R) for given points $A, B, C, D, A', B', C', D' \in P \setminus \{R\}$. If A, A' are arbitrary proper points of line $R(k)$, $A \neq R \neq A'$ then the points B, B' lying on $R(l)$, C, C' lying on $R(m)$ and the points D, D' satisfying the first eleven relations from (R) are determined uniquely. It is sufficient to show $D' \in RD$ and to determine the direction n of the line RD.

Let a, a' be the first coordinate of the points A, A', respectively. From the incidence $A, A' \in R(k) = (y = kx)$ we have the second coordinates of A, A', thus $A = (a, ka)$, $A' = (a', ka')$, respectively.

Since AB, $A'B'$ are the lines of the second type and the points B, B' are lying on the line $R(l) = (y = lx)$, the coordinates of B, B' are

$$B = (l^{-1} \cdot ka,\, ka), \qquad B' = (l^{-1} \cdot ka',\, ka').$$

Since BC, $B'C'$ are the lines of the first type and the points B, B' are lying on the line $R(m) = (y = mx)$ we have

$$C = (l^{-1} \cdot ka,\, ml^{-1} \cdot ka), \qquad C' = (l^{-1} \cdot ka',\, ml^{-1} \cdot ka').$$

Since AD, $A'D'$ are the lines of the first type and CD, $C'D'$ are the lines of the second type the coordinates of D, D' are

$$D = (a,\, ml^{-1} \cdot ka), \qquad D' = (a',\, ml^{-1} \cdot ka').$$

Let n be the direction of RD, i.e. $RD = (y = nx)$. The equality $ml^{-1} \cdot ka = na$ results $n = ml^{-1} \cdot k$, thus

$$RD = (y = ml^{-1} \cdot kx).$$

The coordinates of D' satisfy the equation of the line RD, consequently $D' \in RD$ and thus the twelfth relation $\overline{RDD'(n)}$ from (R) holds and $n = ml^{-1} \cdot k$ is uniquely determined by the elements $k, l, m \in S^{\#}$, $k \neq l \neq m$ now. We can prove the theorem when other eleven relations hold in a similar way. □

COROLLARY. *If the relations $\overline{AB(\underline{0})}$, $\overline{CD(\underline{0})}$, $\overline{AD(\infty)}$, $\overline{BC(\infty)}$ hold for any mutually distinct points $A, B, C, D \in P$ and a point $R \in P \setminus \{A, B, C, D\}$ does not lie on any line AB, BC, CD, AD, then the directions k, l, m, n of the lines RA, RB, RC, RD satisfy the equality $n = ml^{-1} \cdot k$.*

P r o o f. Follows from Theorem 5 immediately. □

2. The reduced pentagonal condition

Define a relation "$\perp$" (orthogonality of lines) in the set $\mathcal{L}$ of lines of the affine parallel structure $\mathcal{A} = (P, \mathcal{L})$ constructed over a non-planar right nearfield $\mathcal{S} = (S, +, \cdot)$ in the following way (requiring only the validity of so-called trivial axioms of orthogonality).

(01) $g \perp h \implies h \perp g$ for all $g, h \in \mathcal{L}$,

(02) $\big((g \perp h) \ \&\ (h \parallel k)\big) \implies g \perp k$ for all $g, h, k \in \mathcal{L}$.

(03) For any line $g \in \mathcal{L}$ and for any point $B \in g$ there is exactly one line $h \in \mathcal{L} \setminus \{g\}$ such that $B \in h$ and $h \perp g$.

R e m a r k 2.

(a) If $g \perp h$, then g, h is called *mutually orthogonal.*

(b) From axioms of parallel structure and from the axioms (02), (03) we have for a given point $B \in P$ and a given line $g \in \mathcal{L}$ that there exists just one line $h \in \mathcal{L} \setminus \{g\}$ such that $B \in h$ and $h \perp g$.

(c) According to the axiom (03) no line $g \in \mathcal{L}$ of the structure $\mathcal{A}$ is isotropic (i.e. $g \perp g$ does not hold).

(d) In the structure $\mathcal{A}$ with an orthogonality of lines

$$\big((g \perp h) \ \&\ (g \perp k)\big) \implies h \parallel k$$

is valid for all mutually distinct lines $g, h, k \in \mathcal{L}$.

If an orthogonality is defined in the affine parallel structure $\mathcal{A} = (P, \mathcal{L})$ over a non-planar right nearfield $\mathcal{S} = (S, +, \cdot)$, then we will always suppose the following:

1) Any line $(x = c)$, $c \in S$ of the first type is orthogonal to any line $(y = d)$, $d \in S$ of the second type and vice versa.
2) A perpendicular to a given line $(y = x)$ of the third type is a line $(y = ex)$, $e \in S^{\#} \setminus \{1\}$ of the third type.

3) A perpendicular to a given line $(y = ax)$, $a \in S^{\#}$ of the third type will be denoted as $(y = a'x)$, $a' \in S^{\#} \setminus \{a\}$; this is also a line of the third type.

As a closure condition for the orthogonality of lines in $\mathcal{A}$ over a non-planar right nearfield $\mathcal{S} = (S, +, \cdot)$ a special case of pentagonal condition dealing with an intersection point of the altitudes of a pentagon will be used (see [7]). Since any two non-parallel lines of the third type need not have an intersection point in our structure, some additional requirements concerning the sides of the pentagon must be fulfilled. Thus we shall obtain the following reduced pentagonal condition.

Definition 4. *We say that a structure $\overline{\mathcal{A}} = (P \cup h_\infty,\ \mathcal{L} \cup \{h_\infty\})$ constructed over a non-planar right nearfield $\mathcal{S} = (S, +, \cdot)$ satisfies a reduced pentagonal condition whenever $\overline{\mathcal{A}}$ has the following property:*

(P) *For any pentagonal $ABCDE$ from $\overline{\mathcal{A}}$ satisfying the relations $\overline{BC(e)}$, $\overline{CD(\underline{1})}$, $\overline{DE(\infty)}$, $\overline{EA(\underline{0})}$ and in which AB is a line of the third type the following holds: If four altitudes of $ABCDE$ (i.e. four perpendiculars going from the vertices to the opposite sides) are going through a point Q, then the fifth altitude goes through Q too (see Fig. 2).*

From Fig. 2 it is evident how we can construct the perpendicular h to a given line g of the third type in the case when two couples of mutually orthogonal lines are known (a line of the first type and a line of the second type; a line $(y = x + q)$ and a line $(y = ex + q')$) and the reduced pentagonal condition holds (i.e. $\overline{\mathcal{A}}$ satisfies the property (P)).

If in $\overline{\mathcal{A}}$ the reduced pentagonal condition holds, then if the Reidemeister condition is satisfied too (by Theorem 5 it holds in $\overline{\mathcal{A}}$) we can derive another closure condition of an orthogonality of lines in $\overline{\mathcal{A}}$ useful for an algebraic expression of an orthogonality in $\mathcal{A}$.

Definition 5. *We shall say a structure $\overline{\mathcal{A}} = (P \cup h_\infty,\ \mathcal{L} \cup \{h_\infty\})$ constructed over a non-planar right nearfield $\mathcal{S} = (S, +, \cdot)$ has the property* (RP) *whenever following holds:*

(RP) *Let $R, K, L, M, N \in P$ be mutually distinct points for which $\overline{KL(\underline{0})}$, $\overline{MN(\underline{0})}$, $\overline{KN(\infty)}$, $\overline{LM(\infty)}$ fulfil and the point R does not lie on any of lines KL, LM, MN, NK. Similarly let R', K', L', M', $N' \in P$ be another mutually distinct points for which $\overline{K'L'(\infty)}$, $\overline{M'N'(\infty)}$, $\overline{K'N'(\underline{0})}$, $\overline{L'M'(\underline{0})}$ fulfil and the point R' does not lie on any of lines $K'L'$, $L'M'$, $M'N'$, $N'K'$. If any three of the four relations*

$RK \perp R'K'$, $RL \perp R'L'$, $RM \perp R'M'$, $RN \perp R'N'$ are satisfied then the fourth is satisfied too. (See Fig. 3).

THEOREM 6. *If a structure $\overline{\mathcal{A}} = \big(P \cup h_\infty, \mathcal{L} \cup \{h_\infty\}\big)$ constructed over a non-planar right nearfield $\mathcal{S} = (S, +, \cdot)$ has the property* (P) *then it has the property* (RP).

P r o o f. Since $\overline{\mathcal{A}}$ is a central translation structure preserving the incidence of points and lines as well as the parallelity of lines (see Remark 1) we can suppose without loss of generality that $R = (\underline{0}, \underline{1})$, $R' = (-e^{-1}, \underline{0})$ has the property $(y = x) \perp (y = ex)$ where $e \in S^{\#} \setminus \{1\}$.

Let K_0, L_0, M_0, N_0 be points satisfying the Reidemeister condition with the points R, K, L, M, N where $K_0 = (1, y_0)$. Similarly let K_0', L_0', M_0', N_0' be points satisfying the assumption of (R) with points R', K', L', M', N' where $K_0' = (1 - e^{-1}, y_0')$.

Suppose for example the first three relations from (R) hold. Then $RK_0 \perp R'K_0'$, $RL_0 \perp R'L_0'$, $RM_0 \perp R'M_0'$ also hold. To prove the fourth relation $RN \perp R'N'$ from (RP) it is sufficient to show that $RN_0 \perp R'N_0'$. Let k, l, m, n be directions of the lines RK_0, RL_0, RM_0, RN_0, respectively. By Theorem 5, $n = m\ell^{-1} \cdot k$. Using the reduced pentagonal condition (which is satisfied in $\overline{\mathcal{A}}$ by the assumption of the theorem) we construct the perpendiculars to the lines RK_0, RL_0, RM_0, RN_0 going through the point R' in such a way that we choose a common point A of all pentagonals $AB_1C_1D_1E$, $AB_2C_2D_2E$, $AB_3C_3D_3E$, $AB_4C_4D_4E$ in the point $R = (\underline{0}, \underline{1})$ and a common intersection point Q of altitudes of those pentagonals in the point $R' = (-e^{-1}, \underline{0})$ (see Fig. 4). It is easy to show that $E = (\underline{1} - e^{-1}, \underline{1})$ and also

$$\begin{aligned}
B_1 &= (-k^{-1}, \underline{0}), & C_1 &= (-e^{-1}, -\underline{1} + ek^{-1}), & D_1 &= (\underline{1} - e^{-1}, ek^{-1}), \\
B_2 &= (-1^{-1}, \underline{0}), & C_2 &= (-e^{-1}, -\underline{1} + el^{-1}), & D_2 &= (\underline{1} - e^{-1}, el^{-1}), \\
B_3 &= (-m^{-1}, \underline{0}), & C_3 &= (-e^{-1}, -\underline{1} + em^{-1}), & D_3 &= (\underline{1} - e^{-1}, em^{-1}), \\
B_4 &= (-n^{-1}, \underline{0}), & C_4 &= (-e^{-1}, -\underline{1} + en^{-1}), & D_4 &= (\underline{1} - e^{-1}, en^{-1}).
\end{aligned}$$

According to the assumption we have $K_0' = D_1 = (\underline{1} - e^{-1}, ek^{-1})$ and $L_0' = D_2 = (\underline{1} - e^{-1}, el^{-1})$. The point M_0' must lie on the line QD_3 and $\overline{L_0'M_0'(\underline{0})}$ holds, hence $M_0' = (ml^{-1} - e^{-1}, el^{-1})$. For the coordinates of the point N_0' we have

$$N_0' = (ml^{-1} - e^{-1}, ek^{-1}).$$

It is sufficient to show that the point N_0' lies on the line QD_4 with the equation

$$QD_4 = \big(y = ek^{-1}lm^{-1}(x + e^{-1})\big).$$

Since $y = ek^{-1}lm^{-1}(ml^{-1} - e^{-1} + e^{-1}) = ek^{-1}lm^{-1}ml^{-1} = ek^{-1}$ holds, the point N_0' lies on QD_4, thus $R'N_0' \perp RN_0$, $R'N' \perp RN$. We can prove the theorem in a similar way when the other three of the four relations (RP) are fulfilled. □

3. The algebraic expression of the orthogonality in $\overline{\mathcal{A}}$

By the orthogonality of lines $((01), (02), (03)$ and $(x = c) \perp (y = d)$ satisfied for any $c, d \in S$) of a parallel structure $\overline{\mathcal{A}} = \big(P \cup h_\infty,\ \mathcal{L} \cup \{h_\infty\}\big)$ constructed over a non-planar right nearfield $\mathcal{S} = (S, +, \cdot)$ a map from the set $S^{\#}$ of non-zero elements of S into itself is given. Namely

$$a \mapsto a' \iff (y = ax) \perp (y = a'x) . \tag{5}$$

From (5) as well as the axioms and the properties of mutually orthogonal lines follows

Lemma 7. *The map $a \mapsto a'$ given by* (5)

(a) *is bijective,*
(b) *is involutorial i.e. $a'' = (a')' = a$,*
(c) *has no fixed element, i.e. for no element $a \in S^{\#}$ $a' = a$ holds.*

Since a perpendicular to the line $(y = x)$ of the third type was denoted as $(y = ex)$, $e \in S^{\#} \setminus \{\underline{1}\}$ we can write

$$\underline{1}' = e , \qquad e' = \underline{1} . \tag{6}$$

In the following we always suppose that the reduced pentagonal condition (P) as well as the condition (RP) (as it was just proved) are satisfied in $\overline{\mathcal{A}}$ constructed over a non-planar right nearfield $\mathcal{S} = (S, +, \cdot)$. Other nontrivial properties of the map $a \mapsto a'$ are involved in the following lemma.

Lemma 8. *For all $a, b \in S^{\#}$ we have*

(7) $(-a)' = -a'$,
(8) $(a^{-1})' = e \cdot (a')^{-1} \cdot e$,
(9) $(a \cdot b)' = b' \cdot e^{-1} \cdot a'$.

P r o o f.

a) Let $a \in S^{\#}$ be an arbitrary element. We will find the perpendiculars to the lines $(y = ax + \underline{1})$, $(y = -ax + \underline{1})$ by the reduced pentagonal condition (P), where $A = (\underline{0}, \underline{1})$ and $E = (\underline{1} - e^{-1}, \underline{1})$ are common vertices of the pentagons $ABCDE$, $AB'C'D'E$ and $Q = (-e^{-1}, \underline{0})$ is an intersection point of their altitudes (see Fig. 5). When $AB = (y = ax + \underline{1})$ has a direction a, then the line OF has the direction $-a$ by Lemma 4, where $O = (\underline{0}, \underline{0})$, $F = A(\underline{0}) \cap B(\infty)$. Hence $AB' \parallel OF$ and $AB' = (y = -ax + \underline{1})$. The coordinates of the further vertices of the pentagons are:

$$B = (-a^{-1}, \underline{0}), \qquad C = (-e^{-1}, -1 + ea^{-1}), \qquad D = (\underline{1} - e^{-1}, ea^{-1}),$$
$$B' = (a^{-1}, \underline{0}), \qquad C' = (-e^{-1}, -1 - ea^{-1}), \qquad D' = (\underline{1} - e^{-1}, -ea^{-1}).$$

If $G = (-e^{-1}, -ea^{-1})$, $H = (\underline{1} - e^{-1}, \underline{0})$, then the image of the point Q, D under translation $\tau(0, -ea^{-1})$, is G, H, respectively. Moreover $GH \parallel QD$. Thus GH has the direction a'. QD' has the direction $-a'$ by Lemma 4. Using the construction and the condition (P) we have the direction of QD' is $(-a)'$. Hence $(-a)' = -a'$.

b) We use Theorem 6 and the method of its proof to prove (8). If $a \in S^{\#}$ is an arbitrary element, then we can determine the points (designation as in the proof of Theorem 6):

$$K_0 = (\underline{1}, \underline{1} + \underline{1}), \qquad K_0' = (\underline{1} - e^{-1}, e),$$
$$L_0 = (a^{-1}, \underline{1} + \underline{1}), \qquad L_0' = (\underline{1} - e^{-1}, ea^{-1}),$$
$$M_0 = (a^{-1}, a^{-1} + \underline{1}), \qquad M_0' = (a^{-1} - e^{-1}, ea^{-1}),$$
$$N_0 = (\underline{1}, a^{-1} + \underline{1}), \qquad N_0' = (a^{-1} - e^{-1}, e).$$

The direction of AK_0, AL_0, AM_0, AN_0, QK_0', QL_0', QM_0', QN_0' is $k = \underline{1}$, $l = a$, $m = \underline{1}$, $n = a^{-1}$, $k' = e$, $l' = a'$, $m' = e$, $n' = e(a')^{-1} \cdot e$, respectively (for the latter we have used the corollary of Theorem 5). By Theorem 6 $QN_0' \perp AN_0$ hence $n' = (a^{-1})'$. Consequently we have $(a^{-1})' = e \cdot (a')^{-1} \cdot e$.

c) In this case we also use Theorem 6 and the method of its proof. Let $a, b \in S^{\#}$ be arbitrary elements. We will determine the points (designation as in the proof of Theorem 6):

$$K_0 = (\underline{1}, \underline{1} + \underline{1}), \qquad K_0' = (\underline{1} - e^{-1}, e),$$
$$L_0 = (b^{-1}, \underline{1} + \underline{1}), \qquad L_0' = (\underline{1} - e^{-1}, b'),$$
$$M_0 = (b^{-1}, a + \underline{1}), \qquad M_0' = ((a')^{-1}e - e^{-1}, b'),$$
$$N_0 = (\underline{1}, a + \underline{1}), \qquad N_0' = ((a')^{-1} \cdot e - e^{-1}, e).$$

The direction of AK_0, AL_0, AM_0, AN_0, QK_0', QL_0', QM_0', QN_0' is $k = \underline{1}$, $l = b$, $m = ab$, $n = a$, $k' = e$, $l' = b'$, $m' = b'e^{-1}a'$, $n' = a'$, respectively (for the expression of m' we have used corollary of Theorem 5). By Theorem 6 $QM_0' \perp AM_0$ hence $m' = (ab)'$. Consequently we have $(ab)' = b' \cdot e^{-1} \cdot a'$. □

LEMMA 9. *For any* $a, b \in S^{\#}$ *we have*

$$(a+b)' = \left[(a')^{-1} + (b')^{-1}\right]^{-1} \tag{10}$$

if and only if $e \in \operatorname{Ker}(\mathcal{S})$, *where* e *is from* (6).

Proof. Let $a, b \in S^{\#}$. By the reduced pentagonal condition (P) the perpendiculars to lines $(y = a^{-1}x + \underline{1})$, $(y = b^{-1}x + \underline{1})$, $y = (a+b)^{-1}x + \underline{1})$ are determined, where $A = (\underline{0}, \underline{1})$, $E = (\underline{1} - e^{-1}, \underline{1})$ are the common vertices of the pentagons $AB_1C_1D_1E$, $AB_2C_2D_2E$, $EB_3C_3D_3E$ and $Q = (-e^{-1}, \underline{0})$ is a common intersection point of their altitudes. The coordinates of the further vertices of the considered pentagons are

$$\begin{aligned} &B_1 = (-a, \underline{0}), && C_1 = (-e^{-1}, -\underline{1} + ea), && D_1 = (\underline{1} - e^{-1}, ea), \\ &B_2 = (-b, \underline{0}), && C_2 = (-e^{-1}, -\underline{1} + eb), && D_2 = (\underline{1} - e^{-1}, eb), \\ &B_3 = (-(a+b), \underline{0}), && C_3 = \big(-e^{-1}, -\underline{1} + e(a+b)\big), && D_3 = \big(\underline{1} - e^{-1}, e(a+b)\big). \end{aligned}$$

By (P) QD_1, QD_2, QD_3 is perpendicular to $AB_1 = (y = a^{-1}x + \underline{1})$, $AB_2 = (y = b^{-1}x + \underline{1})$, $AB_3 = \big(y = (a+b)^{-1}x + \underline{1}\big)$, respectively. Their directions are

$$(a^{-1})' = ea, \qquad (b^{-1})' = eb, \qquad \left[(a+b)^{-1}\right]' = e(a+b).$$

By the axiom (S4) $e(a+b) = ea + eb$. Hence

$$\left[(a+b)^{-1}\right]' = (a^{-1})' + (b^{-1})'.$$

Using (8) we have

$$e\left[(a+b)'\right]^{-1} \cdot e = e(a')^{-1} \cdot e + e(b')^{-1} \cdot e$$

from which

$$\left[(a+b)'\right]^{-1} = \left[(a')^{-1} \cdot e + (b')^{-1} \cdot e\right] \cdot e^{-1}.$$

The right side of the last equation is equal to $(a')^{-1} + (b')^{-1}$ if and only if $e^{-1} \in \operatorname{Ker}(\mathcal{S})$, i.e. $e \in \operatorname{Ker}(\mathcal{S})$. Hence we have

$$(a+b)' = \left[(a')^{-1} + (b')^{-1}\right]^{-1}.$$

□

Using the mapping $a \mapsto a'$ of the set $S^{\#}$ of non-zero elements of $\mathcal{S}$ (defined by (5)) and its properties (6), (7), (8), (9), (10) we define the new mapping $^{-}: S \to S$ as follows:

Let $\overline{\mathcal{A}} = (P \cup h_\infty,\ \mathcal{L} \cup \{h_\infty\})$ be a structure constructed over the non-planar right nearfield $\mathcal{S} = (S, +, \cdot)$ and satisfying the reduced pentagonal condition (P). Let $A = (\underline{0}, \underline{1})$, $Q = (-e^{-1}, \underline{0})$, $F = (\underline{0}, \underline{1})$ be points from P, where $e \in S^{\#} \setminus \{1\}$ satisfies (6). An arbitrary line from $\mathcal{L}$ going through F intersects the line $(y = \underline{0})$ in the point $X = (b, \underline{0})$, $b \in S$. The line (determined by (P)) going through Q and perpendicular to the line FX intersects the line $(x = \underline{0})$ in the point $Y = (\underline{0}, \overline{b})$ (see Fig. 6). A mapping $X \mapsto Y$ of the set of the points X of the line $(y = \underline{0})$ onto the set of the points Y of the line $(x = \underline{0})$ is a bijective one in which to the point $(\underline{1}, \underline{0})$, $(\underline{0}, \underline{0})$ lying on $(y = \underline{0})$, there corresponds the point $(\underline{0}, \underline{1})$, $(\underline{0}, \underline{0})$, lying on $(x = \underline{0})$, respectively. This mapping determines the bijective one

$$^{-}: S \to S, \qquad b \mapsto \overline{b} \tag{11}$$

in which $\underline{1} \mapsto \overline{\underline{1}} = \underline{1}$ and $\underline{0} \mapsto \overline{\underline{0}} = \underline{0}$.

The equation of the line FX, where $X = (b, \underline{0})$, $b \neq \underline{0}$ is

$$FX = (y = b^{-1}x - 1)$$

and the equation of the perpendicular QY $(QY \perp FX)$ is

$$QY = \big(y = (b^{-1})' \cdot x + \overline{b}\big).$$

The coordinates of Q satisfy the equation of the line QY, hence

$$\underline{0} = (b^{-1})' \cdot (-e^{-1}) + \overline{b}.$$

Using (8) we obtain

$$\begin{aligned} &\overline{b} = e \cdot (b')^{-1} \qquad \text{for all} \quad b \in S^{\#}, \\ &\overline{\underline{0}} = \underline{0}. \end{aligned} \tag{12}$$

For the mapping $^{-}: S \to S$, $b \mapsto \overline{b}$ determined by (12) the following lemma is satisfied:

LEMMA 10. *For any $e \in S^{\#} \setminus \{1\}$ satisfying* (6) *the mapping* $^{-}: S \to S$, $b \mapsto \overline{b}$ *determined by* (12) *is an automorphism of the multiplicative group* $(S^{\#}, \cdot)$ *of the nearfield* $\mathcal{S}$.

If $e \in \mathrm{Ker}(\mathcal{S})$ and (6) *is fulfilled, then the mapping "$^{-}$" is an automorphism of the nearfield $\mathcal{S} = (S, +, \cdot)$.*

If $e \in \mathrm{Z}(\mathcal{S})$ and (6) *is fulfilled, then the mapping "$^{-}$" is an identical automorphism of the nearfield $\mathcal{S} = (S, +, \cdot)$.*

Proof. From the definition of the mapping $^{-}: S \to S$, $b \mapsto \overline{b}$ we can see that it is bijective and $\overline{\underline{0}} = \underline{0}$, $\overline{\underline{1}} = \underline{1}$. To prove the first assertion we will show that for any two elements $a, b \in S^{\#}$ we have

$$\overline{a^{-1}} = (\overline{a})^{-1}, \qquad \overline{a \cdot b} = \overline{a} \cdot \overline{b}.$$

$$\begin{aligned} \overline{a^{-1}} &= e \cdot \left[(a^{-1})'\right]^{-1} = e \cdot \left[e \cdot (a')^{-1} \cdot e\right]^{-1} = e \cdot e^{-1} \cdot (a') \cdot e^{-1} \\ &= (a') \cdot e^{-1} = \left[e \cdot (a')^{-1}\right]^{-1} = (\overline{a})^{-1}. \end{aligned}$$

We have used (8) and the properties of the nearfield. Using (9) and the properties of the nearfield we obtain

$$\begin{aligned} \overline{a \cdot b} &= e \cdot \left[(a \cdot b)'\right]^{-1} = e \cdot (b' \cdot e^{-1} \cdot a')^{-1} = e \cdot \left[(a')^{-1} \cdot e \cdot (b')^{-1}\right] \\ &= \left[e \cdot (a')^{-1}\right] \cdot \left[e \cdot (b')^{-1}\right] = \overline{a} \cdot \overline{b}. \end{aligned}$$

To prove the second assertion it is sufficient to show additional equations, namely for any $a, b \in S^{\#}$

$$\overline{-a} = -\overline{a}, \qquad \overline{a + b} = \overline{a} + \overline{b}.$$

Using (7) and the properties of the nearfield we have:

$$\overline{-a} = e \cdot \left[(-a)'\right]^{-1} = e \cdot (-a')^{-1} = -e(a')^{-1} = -\overline{a}.$$

Using Lemma 9 and the fact that $e \in \mathrm{Ker}(\mathcal{S})$, we have

$$\begin{aligned} \overline{a + b} &= e \cdot \left[(a + b)'\right]^{-1} = e \cdot \left\{\left[(a')^{-1} + (b')^{-1}\right]^{-1}\right\}^{-1} \\ &= e \cdot (a')^{-1} + e \cdot (b')^{-1} = \overline{a} + \overline{b}. \end{aligned}$$

To prove the third assertion we use the expression of direction a' of the perpendicular line QD to the line $AB = (y = ax + \underline{1})$, $a \neq \underline{0}$ in the condition (P).

It is easy to show that $a' = ea^{-1}$. Hence for any $a \in S^{\#}$

$$\overline{a} = e \cdot (a')^{-1} = e \cdot (e \cdot a^{-1})^{-1} = e \cdot a \cdot e^{-1} .$$

If $e \in Z(S)$, then

$$\overline{a} = e \cdot a \cdot e^{-1} = a \cdot e \cdot e^{-1} = a ,$$

thus $a \mapsto \overline{a}$ is an identical mapping. □

For the image of $e \in S^{\#}$ (see (6)) under the mapping "$^{-}$" we have by (12)

$$\overline{e} = e \cdot (e')^{-1} = e \cdot \underline{1}^{-1} = e \cdot \underline{1} = e .$$

Hence e is fixed with respect to the automorphism "$^{-}$". Using (12) we can express the direction b' of the perpendicular line to $(y = bx)$, $b \neq \underline{0}$. We obtain

$$b' = (e^{-1} \cdot \overline{b})^{-1} , \qquad b \neq \underline{0} . \tag{13}$$

Since a perpendicular line to $(y = c)$, $c \in S$ is any line $(x = d)$, $d \in S$ with non-defined direction, it is suitable to express the mutual perpendicular (proper) lines from $\mathcal{L}$ in the following way

$$(y = bx + c) \perp \big(x = (e^{-1} \cdot \overline{b}) \cdot y + d\big) , \qquad b, c, d \in S . \tag{14}$$

By this relation we can also express a perpendicular line to a line of the second type with zero-direction because $\overline{\underline{0}} = \underline{0}$.

By Lemma 7 the mapping $a \mapsto a'$ determined by (5) is involutorial, i.e. $a'' = a$ for all $a \in S$. If $a \neq \underline{0}$, then $a' = (e^{-1}, \overline{a})^{-1}$. That means

$$a = a'' = (a')' = \big[(e^{-1} \cdot \overline{a})^{-1}\big]' = \Big[e^{-1} \cdot \overline{(e^{-1} \cdot \overline{a})^{-1}}\Big]^{-1} = \overline{(e^{-1} \cdot \overline{a})} \cdot e ,$$

hence we have

$$\overline{e^{-1} \cdot \overline{a}} = a \cdot e^{-1} \qquad \text{for all} \quad a \in S . \tag{15}$$

Summarizing all the facts above we obtain the following theorem which, determines an algebraic expression of the orthogonality in $\overline{\mathcal{A}}$.

THEOREM 11. *Suppose that in the parallel structure* $\overline{\mathcal{A}} = \big(P \cup h_\infty, \mathcal{L} \cup \{h_\infty\}\big)$ *constructed over the non-planar right nearfield* $S = (S, +, \cdot)$ *the orthogonality of the proper lines satisfying the axioms* (01), (02), (03) *is defined and the reduced pentagonal condition* (P) *holds. Then any line* $(y = ax + b)$, $a, b \in S$ *is perpendicular to any line* $\big(x = (e^{-1} \cdot \overline{a})y + c\big)$, $c \in S$, *where* $e \in S^{\#} \setminus \{\underline{1}\}$ *is determined by* (6) *and the mapping* $a \mapsto \overline{a}$ *is an automorphism of the multiplicative group* $(S^{\#}, \cdot)$ *of the nearfield* S *for which the following holds:*

$$\overline{\underline{0}} = \underline{0} , \quad \overline{\underline{1}} = \underline{1} , \quad \overline{e} = e , \quad \overline{e^{-1} \cdot \overline{a}} = a \cdot e^{-1} .$$

If $e \in \mathrm{Ker}(S)$, *then the mapping* $a \mapsto \overline{a}$ *is an automorphism of the nearfield* S.

If $e \in Z(S)$, *then the mapping* $a \mapsto \overline{a}$ *is the identical automorphism of the nearfield* S.

4. Examples of the infinite non-planar right quasifield and nearfield

Let $(C,+,\cdot)$ be the field of complex numbers $z = x + y\mathrm{i}$, where x, y are any real numbers and i the imaginary unit ($\mathrm{i}^2 = -1$). Zero and unit elements will be denoted by $\underline{0}$ and $\underline{1}$, respectively. If $z = x + y\mathrm{i}$, then the conjugate complex number of z will be denoted by $z^{\#} = x - y\mathrm{i}$. The mapping $z \mapsto z^{\#}$ is evidently a non-identical automorphism of $(C,+,\cdot)$.

Let t be an indeterminate over $(C,+,\cdot)$. Denote $C[t]$ the ring of polynomials of the indeterminate t with the coefficients from $(C,+,\cdot)$. If $a(t) = a_0t^n + a_1t^{n-1} + \cdots + a_{n-1}t + a_n$, ($n$ is positive integer, $a_0 \neq 0$) is a polynomial from $C[t]$ then its degree will be denoted by $n = \sigma a$ and by $a^{\#}(t) = a_0^{\#}t^n + a_1^{\#}t^{n-1} + \cdots + a_{n-1}^{\#}t + a_n^{\#}$ we will denote a polynomial with the conjugate coefficients $a_j = x_j - y_j\mathrm{i}$, $j \in \{0, 1, \dots, n\}$. It is clear that $\sigma a^{\#} = \sigma a$.

Let $C(t)$ be a quotient field of $C(t)$. If $u(t) \in C(t)$, then

$$u(t) = \frac{a(t)}{b(t)}, \qquad b(t) \neq 0$$

for some $a(t), b(t) \in C[t]$. The operation of addition (multiplication) in $C(t)$ will be denoted by "+" ("$\cdot$").

Let $u^{\#}(t) = \dfrac{a^{\#}(t)}{b^{\#}(t)}$, where $u(t) = \dfrac{a(t)}{b(t)} \in C(t)$. The mapping $u(t) \mapsto u^{\#}(t)$ is a non-identical automorphism of the field $\big(C(t),+,\cdot\big)$ and $u^{\#\#}(t) = (u^{\#})^{\#}(t)$ $= u(t)$ holds. Define a new binary operation "$\circ$" as follows:

For all $u(t) = \dfrac{a(t)}{b(t)}$, $v(t) = \dfrac{c(t)}{d(t)}$ from $C(t)$ we state

$$u(t) \circ v(t) = \frac{a^{\#}(t)}{b^{\#}(t)} \cdot \frac{c(t + \sigma a - \sigma b)}{d(t + \sigma a - \sigma b)}. \tag{16}$$

THEOREM 12. $\big(C(t),+,\circ\big)$ *is a non-planar right quasifield of the first type without unity.*

Remark 3. A right quasifield is an ordered triplet $\mathcal{G} = (Q,+,\cdot)$, where Q is a non-empty (at least two element) set with two binary operations "$+$", "$\cdot$" satisfying

(q1) $(Q,+)$ is an abelian group with a zero element $\underline{0} \in Q$.
(q2) $(Q^{\#},\cdot)$ is a groupoid, where $Q^{\#} = Q \setminus \{\underline{0}\}$.
(q3) For all element $a \in Q$: $\underline{0} \cdot a = \underline{0}$.
(q4) $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in Q$ (the right distributive low).
(q5) $\#\{(x,y) \in Q^2 \mid xa + y = b \;\&\; xc + y = d\} = 1$
for any $a, b, c, d \in Q$, $a \neq c$.

A right quasifield $\mathcal{G}$ is termed a planar (non-planar), whenever the condition of planarity holds (does not hold):

(q6) $\#\{x \in Q \mid ax = bx + c\} = 1$ for all $a, b, c \in Q$, $a \neq b$.

A right quasifield is called a right quasifield of the first (second) type if $(Q^{\#}, \cdot)$ is a quasigroup (is not a quasigroup).

Proof of Theorem 12. The axioms (q1), (q2), (q3) hold evidently. The groupoid $(C^{\#}(t), \cdot)$, where $C^{\#}(t) = C(t) \setminus \{\underline{0}\}$ is certainly non-commutative. We will prove (q4). Let $u(t) = \dfrac{a(t)}{b(t)}$, $v(t) = \dfrac{c(t)}{d(t)}$, $w(t) = \dfrac{e(t)}{f(t)} \in C(t)$. Put $r = \sigma a - \sigma b$. We have

$$\begin{aligned} u(t) \circ [v(t) + w(t)] &= \frac{a(t)}{b(t)} \circ \frac{c(t)f(t) + d(t)e(t)}{d(t)f(t)} \\ &= \frac{a^{\#}(t)}{b^{\#}(t)} \cdot \frac{c(t+r)f(t+r) + d(t+r)e(t+r)}{d(t+r) \cdot f(t+r)} \\ &= \frac{a^{\#}(t)}{b^{\#}(t)} \cdot \left[\frac{c(t+r)}{d(t+r)} + \frac{e(t+r)}{f(t+r)}\right] = u(t) \circ v(t) + u(t) \circ w(t), \end{aligned}$$

hence (q4) holds.

Now we are going to prove that the left distributive law $[u(t)+v(t)] \circ w(t) = u(t) \circ w(t) + v(t) \circ w(t)$ does not hold in $(C(t), +, \circ)$. Let $r = \sigma a - \sigma b$, $s = \sigma c - \sigma d$. For the right side we have

$$u(t) \circ w(t) + v(t) \circ w(t) = \frac{a^{\#}(t)}{b^{\#}(t)} \cdot \frac{e(t+r)}{f(t+r)} + \frac{c^{\#}(t)}{d^{\#}(t)} \cdot \frac{e(t+s)}{f(t+s)}.$$

On the other hand for the left side we have

$$\begin{aligned} &[u(t) + v(t)] \circ w(t) \\ &= \frac{a(t) \cdot d(t) + b(t) \cdot c(t)}{b(t) \cdot d(t)} \circ \frac{e(t)}{f(t)} = \frac{a^{\#}(t) \cdot d^{\#}(t) + b^{\#}(t) \cdot c^{\#}(t)}{b^{\#}(t) \cdot d^{\#}(t)} \cdot \frac{e(t+m-n)}{f(t+m-n)} \\ &= \frac{a^{\#}(t)}{b^{\#}(t)} \cdot \frac{e(t+m-n)}{f(t+m-n)} + \frac{c^{\#}(t)}{d^{\#}(t)} \cdot \frac{e(t+m-n)}{f(t+m-n)}, \end{aligned}$$

where $m = \sigma(ad + bc) = \max(\sigma a + \sigma d, \sigma b + \sigma c)$, $n = \sigma b + \sigma d$. Consider the three cases:

$$m = \begin{cases} \sigma a + \sigma d \\ \sigma b + \sigma c \\ \sigma a + \sigma d = \sigma b + \sigma c. \end{cases}$$

We can see that the left side of the left distributive law is equal to the right side only for the third case. Hence the left distributive law does not hold.

Now we will prove that $(C^{\#}(t), \circ)$ is a quasigroup. For any $u(t) = \dfrac{a(t)}{b(t)}$ and $v(t) = \dfrac{c(t)}{d(t)}$ from $C^{\#}(t)$ the equation $u(t) \circ x(t) = v(t)$ has the following unique solution

$$x(t) = \frac{b^{\#}(t-r) \cdot c(t-r)}{a^{\#}(t-r) \cdot d(t-r)}, \qquad \text{where} \quad r = \sigma a - \sigma b .$$

It follows from the computation

$$\begin{aligned} u(t) \circ x(t) &= \frac{a(t)}{b(t)} \circ \frac{b^{\#}(t-r) \cdot c(t-r)}{a^{\#}(t-r) \cdot d(t-r)} \\ &= \frac{a^{\#}(t)}{b^{\#}(t)} \cdot \frac{b^{\#}(t-r+r) \cdot c(t-r+r)}{a^{\#}(t-r+r) \cdot d(t-r+r)} = \frac{c(t)}{d(t)} = v(t) . \end{aligned}$$

Similarly for any $u(t)$, $v(t) \in C^{\#}(t)$ the equation $y(t) \circ u(t) = v(t)$ has the unique solution

$$y(t) = \frac{b^{\#}(t-r+s) \cdot c^{\#}(t)}{a^{\#}(t-r+s) \cdot d^{\#}(t)}, \qquad \text{where} \quad r = \sigma a - \sigma b, \quad s = \sigma c - \sigma d .$$

It follows from the computation that

$$\begin{aligned} y(t) \circ u(t) &= \frac{b^{\#}(t-r+s) \cdot c^{\#}(t)}{a^{\#}(t-r+s) \cdot d^{\#}(t)} \circ \frac{a(t)}{b(t)} = \frac{b^{\#\#}(t-r+s) \cdot c^{\#\#}(t)}{a^{\#\#}(t-r+s) \cdot d^{\#\#}(t)} \cdot \\ &\cdot \frac{a(t-r+s)}{b(t-r+s)} = \frac{b(t-r+s) \cdot c(t)}{a(t-r+s) \cdot d(t)} \cdot \frac{a(t-r+s)}{b(t-r+s)} = \frac{c(t)}{d(t)} = v(t) . \end{aligned}$$

We have used the property $a^{\#\#}(t) = a(t)$, which is satisfied for all $a(t) \in C(t)$.

The quasigroup $(C^{\#}(t), \circ)$ is not a loop since it has no unity. For $\underline{1}$ the equation $\underline{1} \circ u(t) = u(t)$ holds but $u(t) \circ \underline{1} = u^{\#}(t) = u(t)$. Hence $\underline{1}$ is a left unity.

Since "$\circ$" does not satisfy the associative law the quasigroup $(C^{\#}(t), \circ)$ is not a semigroup. In order to prove that the associative law does not hold it is sufficient to compute the following.

$$\begin{aligned} u(t) \circ \big[v(t) \circ w(t)\big] &= \frac{a(t)}{b(t)} \circ \left[\frac{c^{\#}(t)}{d^{\#}(t)} \cdot \frac{e(t+s)}{f(t+s)}\right] \\ &= \frac{a^{\#}(t)}{b^{\#}(t)} \cdot \frac{c^{\#}(t+r) \cdot e(t+s+r)}{d^{\#}(t+r) \cdot f(t+s+r)} , \end{aligned}$$

where $r = \sigma a - \sigma b$, $s = \sigma c - \sigma d$.

On the other hand we have

$$\begin{aligned}&\left[u(t) \circ v(t)\right] \circ w(t)\\ &= \left[\frac{a^{\#}(t)}{b^{\#}(t)} \cdot \frac{c(t+r)}{d(t+r)}\right] \circ \frac{e(t)}{f(t)} = \frac{a^{\#\#}(t) \cdot c^{\#}(t+r)}{b^{\#\#}(t) \cdot d^{\#}(t+r)} \cdot \frac{e(t+r+s)}{f(t+r+s)}\\ &= \frac{a(t) \cdot c^{\#}(t+r) \cdot e(t+r+s)}{b(t) \cdot d^{\#}(t+r) \cdot f(t+r+s)}.\end{aligned}$$

Hence $u(t) \circ \left[v(t) \circ w(t)\right] \neq \left[u(t) \circ v(t)\right] \circ w(t)$.

Since $\left(C^{\#}(t), \circ\right)$ is a quasigroup it is easy to prove the validity of (q5), i.e. for any $r(t)$, $s(t)$, $u(t)$, $v(t)$ from $C(t)$, $r(t) \neq u(t)$ the following system of equations

$$\begin{aligned}x(t) \circ r(t) + y(t) &= s(t)\\ x(t) \circ u(t) + y(t) &= v(t)\end{aligned}$$

has the unique solution $\left(x(t),\, y(t)\right)$.

From the first equation we can express $y(t)$:

$$y(t) = -x(t) \circ r(t) + s(t) \tag{17}$$

and substituting it in the second one we obtain

$$x(t) \circ \left[u(t) - r(t)\right] = v(t) - s(t).$$

Since $\left(C^{\#}(t), \circ\right)$ is a quasigroup the last equation has the unique solution. From (17) we can find the unique $y(t)$.

Now we will prove that the quasifield $\left(C(t), +, \circ\right)$ is not planar. That means there are $u(t)$, $v(t)$, $w(t) \in C(t)$, $u(t) \neq v(t)$ for which

$$u(t) \circ x(t) = v(t) \circ x(t) + w(t)$$

has no solution. Let $u(t) = \underline{1}$, $v(t) = -t$, $w(t) = \underline{1}$. We will prove that

$$x(t) = -t \circ x(t) + \underline{1} \tag{18}$$

has no solution. Suppose $x(t) = \dfrac{a(t)}{b(t)}$ ($b(t) \neq \underline{0}$, $a(t)$ and $b(t)$ are coprime elements) is a solution of (18). That means

$$\frac{a(t)}{b(t)} = -\frac{t}{\underline{1}} \circ \frac{a(t)}{b(t)} + \underline{1}.$$

Using (16) we have

$$t \cdot a(t+1) \cdot b(t) + a(t) \cdot b(t+1) = b(t) \cdot b(t+1)\,. \tag{19}$$

Put $m = \sigma a(t) = \sigma a(t+1)$, $n = \sigma b(t) = \sigma b(t+1)$.

From (19) we have

$$1 + m + n = n + n \implies m = n - 1\,. \tag{20}$$

From (19) we have

$$t \cdot a(t+1) \cdot b(t) = \big[b(t) - a(t)\big] \cdot b(t+1)\,.$$

Since $b(t)$ divides without remainder the left side of the last equation it must divide also the right side. Since $a(t)$ and $b(t)$ are coprime elements $b(t)$ must divide the polynomial $b(t+1)$. It is possible only when $n = 0$. By (20) $m = -1$, which is a contradiction. Consequently (q6) does not hold in $\big(C(t), +, \circ\big)$.

Summarizing we obtain that $\big(C(t), +, \circ\big)$ is a non-planar right quasifield of the first type. Its multiplicative group $\big(C^{\#}(t), \circ\big)$ is non-associative, having no unit. □

The element $\underline{1}$ from the constructed right non-planar quasifield $\big(C(t), +, \circ\big)$ is only a left unit with respect to "$\circ$". Using the construction 1 from [4] we can change the operation "$\circ$" so that the element $\underline{1}$ would be a right as well as a left unit. In order to do it we define a mapping

$$\varphi\colon C(t) \to C(t)\,, \qquad x(t) \mapsto x(t) \circ \underline{1}$$

and the new operation "$\bullet$" of the multiplication in $C(t)$ is defined

$$u(t) \bullet v(t) = \varphi^{-1}\big[u(t)\big] \circ v(t)\,, \qquad \text{for all} \quad u(t),\ v(t) \in C(t)\,.$$

In order to prove that $\underline{1}$ is a left and right unit with respect to the new operation "$\bullet$", we will express "$\bullet$" by the original operation "$\circ$" in the field $\big(C(t), +, \circ\big)$.

Since $\varphi\big[x(t)\big] = x(t) \circ \underline{1} = x^{\#}(t)$ and the automorphism $x(t) \mapsto x^{\#}(t)$ is involutorial, $\varphi^{-1}\big[x(t)\big] = x^{\#}(t)$. Hence for any $u(t) = \dfrac{a(t)}{b(t)}$, $v(t) = \dfrac{c(t)}{d(t)}$ from $C(t)$ we have

$$\begin{aligned} u(t) \bullet v(t) = u^{\#}(t) \circ v(t) &= \frac{a^{\#\#}(t)}{b^{\#\#}(t)} \cdot \frac{c(t + \sigma a - \sigma b)}{d(t + \sigma a - \sigma b)} \\ &= \frac{a(t)}{b(t)} \cdot \frac{c(t + \sigma a - \sigma b)}{d(t + \sigma a - \sigma b)} = u(t) \cdot v(t + \sigma a - \sigma b)\,. \end{aligned}$$

Changing "$\circ$" to "$\bullet$" the properties of $\big(C(t), +, \circ\big)$ are preserved, except those of the non-existence of unit and the non-associativeness of "$\circ$". More precisely

THEOREM 13. $\big(C(t), +, \bullet\big)$ *is a non-planar right nearfield.*

Proof. See the proof of Theorem 5.1 in [3]. The Theorem 13 is a special case of Theorem 5.1 from [3] in such a sense that in our case F is the field of complex numbers and the indexing set $\mathcal{J}$ is a singleton. □

THEOREM 14. *The centre of the nearfield* $\big(C(t), +, \bullet\big)$ *is the field* $(C, +, \cdot)$ *of complex numbers.*

Proof. See the proof of Theorem 5.2 in [3]. The operations "$\circ$" and "$\bullet$" coincide on C. □

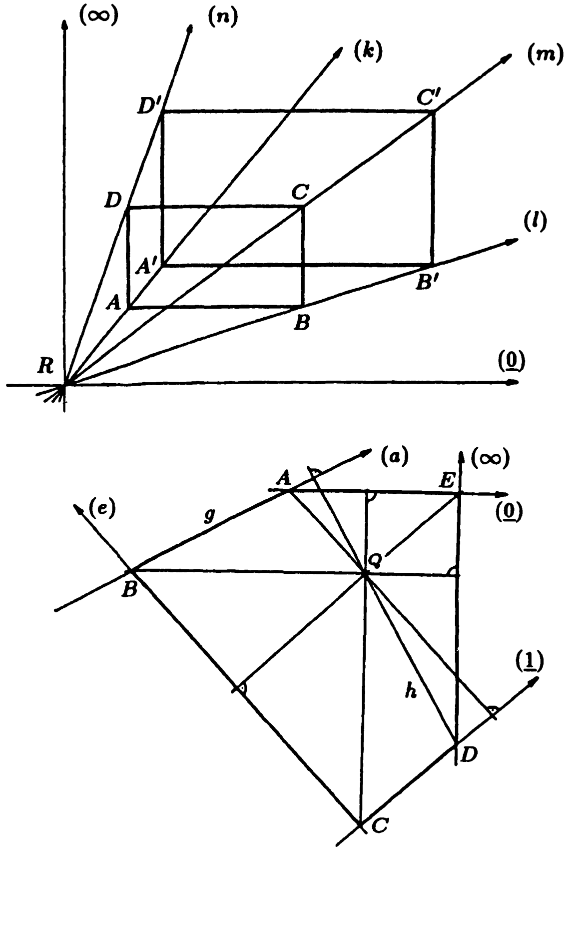

Figure 1 and Figure 2.

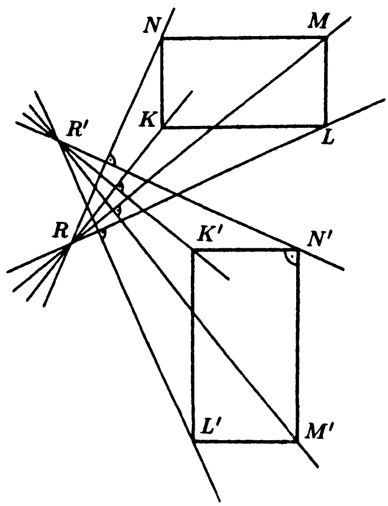

Figure 3.

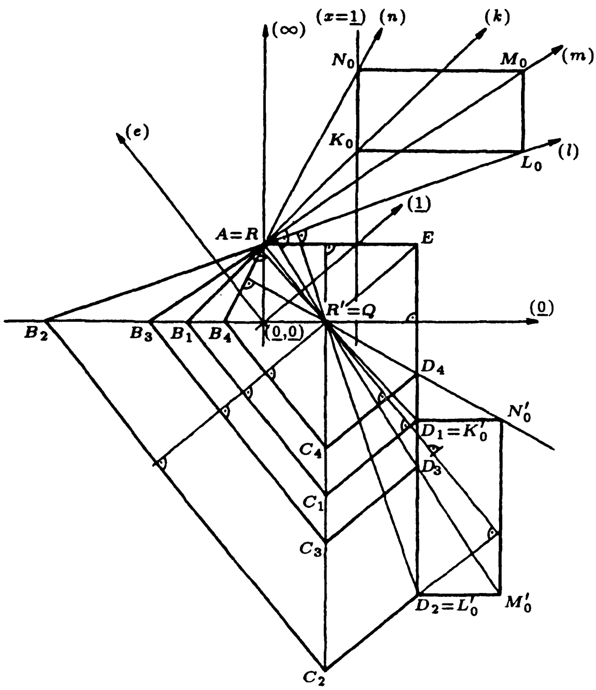

Figure 4.

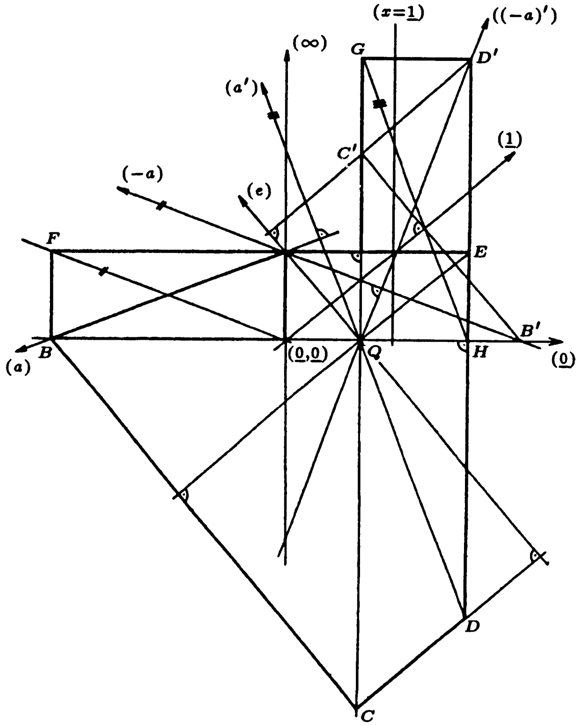

Figure 5.

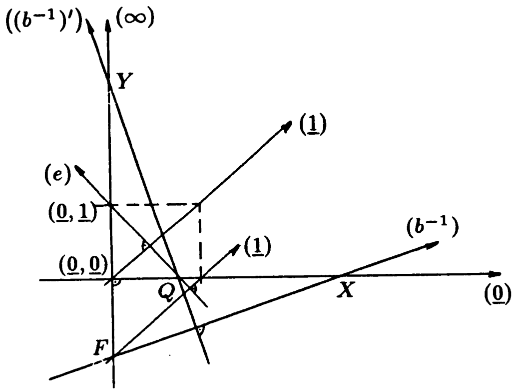

Figure 6.

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