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ON d-IDEALS IN d-ALGEBRAS

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ABSTRACT. We introduce the notions of d-subalgebra, d-ideal, $d^\sharp$-ideal and d^*-ideal in d-algebras, and investigate relations among them. Furthermore, we are able to define the idea of a quotient d-algebra and to prove a fundamental theorem of d-morphisms for d-algebras as a consequence.

1. Introduction

Y. Imai and K. Iséki [II] and K. Iséki [Is1] introduced two classes of abstract algebras: namely, BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [HL1], [HL2] Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim [NK] introduced the notion of d-algebras which is another generalization of BCK-algebras, and investigated relations between d-algebras and BCK-algebras. In this paper we discuss the ideal theory in d-algebras. We introduce the notions of d-subalgebra, d-ideal, $d^\sharp$-ideal and d^*-ideal, and investigate relations among them. Furthermore, we are able to define the idea of a quotient d-algebra and to prove a fundamental theorem of d-morphisms for d-algebras as a consequence.

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2. Preliminaries

A *d-algebra* is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(I) $x * x = 0$,
(II) $0 * x = 0$,
(III) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all x, y in X.

A *BCK-algebra* is a d-algebra $(X, *, 0)$ satisfying the following additional axioms:

(IV) $((x * y) * (x * z)) * (z * y) = 0$,
(V) $(x * (x * y)) * y = 0$

for all x, y, z in X.

In a BCK-algebra $(X, *, 0)$ the following hold:

(1) $(x * y) * x = 0$,
(2) $((x * z) * (y * z)) * (x * y) = 0$

for arbitrary $x, y, z \in X$.

A non-empty subset I of a BCK-algebra X is called a *BCK-ideal* of X if

(i) $0 \in I$,
(ii) $x \in I$ and $y * x \in I$ imply $y \in I$,

for all $x, y \in X$.

PROPOSITION 2.1. *Let X be a d-algebra. If $x \neq y$ and $x * y = 0$, then $y * x \neq 0$.*

Proof. By (III), it is straightforward. □

3. d-ideals

DEFINITION 3.1. Let $(X, *, 0)$ be a d-algebra and $\emptyset \neq I \subseteq X$. I is called a *d-subalgebra* of X if $x * y \in I$ whenever $x \in I$ and $y \in I$. I is called a *BCK-ideal* of X if it satisfies:

(D_0) $0 \in I$,
(D_1) $x * y \in I$ and $y \in I$ imply $x \in I$.

I is called a *d-ideal* of X if it satisfies (D_1) and

(D_2) $x \in I$ and $y \in X$ imply $x * y \in I$, i.e., $I * X \subseteq I$.

EXAMPLE 3.2. Let $X := \{0, a, b, c, d\}$ be a d-algebra which is not a BCK-algebra with the Cayley table as follows:

$*$	0	a	b	c	d
0	0	0	0	0	0
a	a	0	a	0	a
b	b	b	0	c	0
c	c	c	b	0	c
d	c	c	a	a	0

Then $I := \{0, a\}$ is a d-ideal of X.

EXAMPLE 3.3. Let $X := \{0, a, b, c\}$ be a d-algebra which is not a BCK-algebra with the Cayley table as follows:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	b	b	0	0
c	c	c	a	0

Then $J := \{0, a, c\}$ satisfies (D_2), but not (D_1) since $b * c = 0 \in J$ and $c \in J$, but $b \notin J$, i.e., J is a d-subalgebra, but not a BCK-ideal of X.

In a d-algebra, a BCK-ideal need not be a d-subalgebra, and also a d-subalgebra need not be a BCK-ideal as shown in the following example.

EXAMPLE 3.4. Let $X := \{0, a, b, c\}$ be a d-algebra which is not a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	c	0	0
c	c	c	c	0

Then $I := \{0, a, b\}$ is a BCK-ideal which is not a d-subalgebra of X, while $J := \{0, c\}$ is a d-subalgebra which is not a BCK-ideal of X.

Clearly, $\{0\}$ is a d-subalgebra of every d-algebra X and every d-ideal of X is a d-subalgebra, but the converse need not be true.

EXAMPLE 3.5. Let $X := \{0, a, b, c\}$ be a d-algebra which is not a BCK-algebra with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	b
b	b	b	0	0
c	c	c	c	0

Then $I := \{0, a\}$ is a d-subalgebra of X, but not a d-ideal of X, since $a * c = b \notin I$.

LEMMA 3.6. *If I is a d-ideal of a d-algebra X, then $0 \in I$.*

Proof. Since $I \neq \emptyset$, there exists x in I and hence $0 = x * x \in I$ by (D_2). □

Note that every d-ideal of a d-algebra is a BCK-ideal, but the converse need not be true. In Example 3.5, $I := \{0, a\}$ is a BCK-ideal of X, but not a d-ideal of X.

PROPOSITION 3.7. *Let I be a d-ideal of a d-algebra X. If $x \in I$ and $y * x = 0$, then $y \in I$.*

Proof. Assume that $x \in I$ and $y * x = 0$. By Lemma 3.6 and (D_1), we have $y \in I$. This completes the proof. □

DEFINITION 3.8. Let X be a d-algebra. A d-ideal I of X is called a *$d^\sharp$-ideal* of X if, for arbitrary $x, y, z \in X$,

(D_3) $x * z \in I$ whenever $x * y \in I$ and $y * z \in I$.

EXAMPLE 3.9. Let X be a d-algebra as in Example 3.5. Then $K := \{0, a, b\}$ is a $d^\sharp$-ideal of X.

Obviously, every $d^\sharp$-ideal is a d-ideal, but the converse need not be true.

EXAMPLE 3.10. Let X be a d-algebra as in Example 3.2. Then $L := \{0, a\}$ is a d-ideal which is not a $d^\sharp$-ideal of X, since $b * d = 0 \in L$, $d * c = a \in L$, but $b * c = c \notin L$.

Note that we can see that $d^\sharp$-ideal $\subsetneq$ d-ideal $\subsetneq$ d-subalgebra and $d^\sharp$-ideal $\subsetneq$ d-ideal $\subsetneq$ BCK-ideal in d-algebras.

In a d-algebra X, the identity $(x * y) * x = 0$ does not hold in general. For instance, in Example 3.5, we know that $(a * c) * a = b * a = b \neq 0$.

DEFINITION 3.11. A d-algebra X is called a *d^*-algebra* if it satisfies the identity $(x * y) * x = 0$ for all $x, y \in X$.

Clearly, a BCK-algebra is a d^*-algebra, but the converse need not be true.

EXAMPLE 3.12. Let $X := \{0, 1, 2, \dots\}$ and let the binary operation $*$ be defined as follows:

$$x * y := \begin{cases} 0 & \text{if } x \leq y, \\ 1 & \text{otherwise.} \end{cases}$$

Then $(X, *, 0)$ is a d-algebra which is not a BCK-algebra (see [NK; Example 2.8]). We can easily see that $(X, *, 0)$ is a d^*-algebra.

THEOREM 3.13. *In a d^*-algebra, every BCK-ideal is a d-ideal.*

Proof. Let I be a BCK-ideal of a d^*-algebra X and let $x \in I$, $y \in X$. Since $(x * y) * x = 0$ for all $x, y \in X$, it follows from Proposition 3.7 that $x * y \in I$. Hence I is a d-ideal of X. □

The following corollary is obvious.

COROLLARY 3.14. *In a d^*-algebra, every BCK-ideal is a d-subalgebra.*

DEFINITION 3.15. If a $d^\sharp$-ideal I of a d-algebra X satisfies

(D_4) $x * y \in I$ and $y * x \in I$ imply $(x * z) * (y * z) \in I$ and $(z * x) * (z * y) \in I$

for all $x, y, z \in X$, then we say that I is a *d^*-ideal* of X.

In Example 3.3, the set $I := \{0, a\}$ is a d^*-ideal of X. Obviously, every d^*-ideal in a d-algebra is a $d^\sharp$-ideal, but the converse does not hold in general.

EXAMPLE 3.16. Let $X := \{0, a, b, c\}$ be a set with the following Cayley table:

$*$	0	a	b	c
0	0	0	0	0
a	a	0	0	a
b	c	b	0	c
c	c	b	b	0

Then $(X, *, 0)$ is a d-algebra, but not a BCK-algebra. We can see that $I := \{0, a\}$ is a $d^\sharp$-ideal, but not d^*-ideal, since $0 * a = 0 \in I$ and $a * 0 = a \in I$, but $(c * 0) * (c * a) = c * b = b \notin I$.

LEMMA 3.17. (Iséki et al. [IT1]) *Let I be a BCK-ideal of a BCK-algebra X. If $x \in I$ and $y * x = 0$ then $y \in I$.*

THEOREM 3.18. *If $(X, *, 0)$ is a BCK-algebra, then every BCK-ideal of X is a d^*-ideal of X.*

Proof. Let I be a BCK-ideal of X and let $x \in I$ and $y \in X$. Since $(x * y) * x = 0$ by (1), it follows from Lemma 3.17 that $x * y \in I$, proving (D_2).

Assume that $x * y \in I$ and $y * z \in I$ for all $x, y, z \in I$. Then $\big((x * z) * (y * z)\big) * (x * y) = 0$ by (2), and hence $(x * z) * (y * z) \in I$. Since $y * z \in I$ and since I is a BCK-ideal of X, it follows that $x * z \in I$. This proves (D_3).

Let $x * y$, $y * x \in I$ for all $x, y \in X$. Then, by (IV) and (2), we have

$$\big((z * x) * (z * y)\big) * (y * x) = 0 \qquad \text{and} \qquad \big((x * z) * (y * z)\big) * (x * y) = 0,$$

respectively. It follows from Lemma 3.17 that $(z * x) * (z * y) \in I$ and $(x * z) * (y * z) \in I$, proving (D_4). This completes the proof. □

Remark 3.19.

(i) In a d^*-algebra, the concept of d-ideal, d-subalgebra and BCK-ideal coincide.

(ii) In a BCK-algebra, the concept of d-ideal, $d^\sharp$-ideal, d^*-ideal and BCK-ideal coincide.

4. Quotient d-algebras

Let $(X; *, 0_X)$ and $(Y; *, 0_Y)$ be d-algebras. A mapping $f\colon X \to Y$ is called a *d-morphism* ([NK]) if $f(x * y) = f(x) * f(y)$ for all $x, y \in X$. Note that $f(0_X) = 0_Y$. A d-algebra $(X; *, 0_X)$ is said to be *d-transitive* ([NK]) if $x * z = 0$ and $z * y = 0$ imply $x * y = 0$. Every BCK-algebra is a d-transitive d-algebra, but the converse does not hold in general. See Example 3.2.

Let I be a d^*-ideal of a d-algebra $(X; *, 0_X)$. For any x, y in X, we define $x \sim y$ if and only if $x * y \in I$ and $y * x \in I$. We claim that $\sim$ is an equivalence relation on X. Since $0 \in I$, we have $x * x = 0 \in I$, i.e., $x \sim x$, for any $x \in X$. If $x \sim y$ and $y \sim z$, then $x * y$, $y * x \in I$ and $y * z$, $z * y \in I$. By (D_3) $x * z$, $z * x \in I$ and hence $x \sim z$. This proves that $\sim$ is transitive. The symmetry of $\sim$ is trivial. By (D_4) we can easily see that $\sim$ is a congruence relation on X. Using the notion of d-transitivity we obtain:

PROPOSITION 4.1. *Let $f\colon X \to Y$ be a d-morphism from a d-algebra X into a d-transitive d-algebra Y. Then* $\operatorname{Ker} f$ *is a d^*-ideal of X.*

Proof. The properties (D_1) and (D_2) are simple. If $x*y, y*z \in \operatorname{Ker} f$, then $f(x)*f(y) = 0_Y = f(y)*f(z)$. Since Y is d-transitive, we obtain $f(x)*f(z) = 0$ and hence $x*z \in \operatorname{Ker} f$, which proves (D_3). Let $x*y$, $y*x \in \operatorname{Ker} f$. Then $f(x)*f(y) = 0_Y = f(y)*f(x)$. By (III) we obtain $f(x) = f(y)$. It follows that $f\big((x*z)*(y*z)\big) = f(x*z)*f(y*z) = \big(f(x)*f(z)\big)*\big(f(y)*f(z)\big) = 0_Y$ and hence $(x*z)*(y*z) \in \operatorname{Ker} f$. Similarly, $(z*x)*(z*y) \in \operatorname{Ker} f$, which proves (D_4). □

EXAMPLE 4.2. Let X be a d-algebra as in Example 3.3, and let Y be a d-transitive d-algebra as in Example 3.2. Define a map $f\colon X \to Y$ by $f(0) = f(a) = 0$, $f(b) = f(c) = a$. Then f is a d-morphism. Obviously, $\operatorname{Ker} f = \{0, a\}$ is a d^*-ideal of X.

We denote the congruence class containing x by $[x]_I$, i.e., $[x]_I = \{y \in X \mid x \sim y\}$. We see that $x \sim y$ if and only if $[x]_I = [y]_I$. Denote the set of all equivalence classes of X by X/I, i.e., $X/I = \big\{[x]_I \mid x \in X\big\}$.

LEMMA 4.3. *Let I be a d^*-ideal of a d-algebra $(X;*,0)$. Then $I = [0]_I$.*

Proof. If $x \in I$, then $x*0 \in I*X \subseteq I$ and hence $x \in [0]_I$, i.e., $I \subseteq [0]_I$. Since

$$\begin{aligned}
[0]_I &= \{x \in X \mid x \sim 0\} \\
&= \{x \in X \mid x*0,\ 0*x \in I\} \\
&= \{x \in X \mid x*0 \in I\} && (0 \in I) \\
&\subseteq I, && ((\mathrm{D1}))
\end{aligned}$$

it follows that $I = [0]_I$. □

THEOREM 4.4. *Let $(X;*,0)$ be a d-algebra and I be a d^*-ideal of X. If we define $[x]_I * [y]_I := [x*y]_I$ $(x, y \in X)$, then $(X/I;*,0)$ is a d-algebra, called the quotient d-algebra.*

Proof. Since $\sim$ is a congruence relation on X, $x*y \sim x'*y'$ for any $x \sim x'$, $y \sim y'$. This means that $[x]_I * [y]_I = [x*y]_I$ is well-defined. Let $[x]_I, [y]_I \in X/I$ with $[x]_I * [y]_I = [0]_I = [y]_I * [y]_I$. Then $[x*y]_I = [0]_I = [y*x]_I$ and $x*y$, $y*x \in I$. Thus $x \sim y$ and $[x]_I = [y]_I$. The rest is trivial, and so we omit the proof. □

PROPOSITION 4.5. *Let I be a d^*-ideal of the d-algebra X. Then the mapping $\pi\colon X \to X/I$ defined by $\pi(x) = [x]_I$ is a d-morphism of X onto the quotient d-algebra X/I and the kernel of π is precisely the set I.*

Proof. Since $[x*y]_I = [x]_I * [y]_I$, π is a d-morphism. By Lemma 4.3 we know that

$$\begin{aligned}\operatorname{Ker}\pi &= \{x \in X \mid \pi(x) = [0]_I\}\\ &= \{x \in X \mid [x]_I = [0]_I\}\\ &= \{x \in X \mid x \sim 0\}\\ &= [0]_I\\ &= I.\end{aligned}$$

□

THEOREM 4.6. *If $f\colon X \to Y$ is a d-morphism from a d-algebra X onto a d-transitive d-algebra Y, then $X/\operatorname{Ker} f \cong Y$.*

Proof. Assume $\mu\colon X/\operatorname{Ker} f \to Y$ such that $\mu([x]_{\operatorname{Ker} f}) = f(x)$. If $[x]_{\operatorname{Ker} f} = [y]_{\operatorname{Ker} f}$ then $x*y$, $y*x \in \operatorname{Ker} f$, and so $f(x)*f(y) = 0 = f(y)*f(x)$. By (III) we have $f(x) = f(y)$, i.e., $\mu([x]_{\operatorname{Ker} f}) = \mu([y]_{\operatorname{Ker} f})$. This means that μ is well-defined. For any $y \in Y$, there is an $x \in X$ such that $y = f(x)$ since f is onto. Hence $\mu([x]_{\operatorname{Ker} f}) = f(x) = y$, which means that μ is onto. If $\mu([x]_{\operatorname{Ker} f}) \neq \mu([y]_{\operatorname{Ker} f})$ then either $x*y \notin \operatorname{Ker} f$ or $y*x \notin \operatorname{Ker} f$. Without loss of generality, we may assume $x*y \notin \operatorname{Ker} f$. It follows that $f(x)*f(y) = f(x*y) \neq 0$ and hence $f(x) \neq f(y)$. This means that μ is one-one. Since $\mu([x]_{\operatorname{Ker} f} * [y]_{\operatorname{Ker} f}) = \mu([x*y]_{\operatorname{Ker} f}) = f(x*y) = f(x)*f(y) = \mu([x]_{\operatorname{Ker} f}) * \mu([y]_{\operatorname{Ker} f})$, μ is a d-morphism. Thus we have $X/\operatorname{Ker} f \cong Y$, completing the proof. □

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