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ON MINIMAL GRAPHS OF DIAMETER 2 WITH EVERY EDGE IN A 3-CYCLE

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1. Introduction

Given a graph G (in the sense of [1] or [8]), $V(G)$ and $E(G)$ denote its vertex-set and edge-set, respectively. The distance of two vertices u and v is denoted by $d(u, v)$ and the diameter of G by $\operatorname{diam}(G)$. A graph G with $\operatorname{diam}(G)=k$ is called a *minimal graph of diameter k* if $\operatorname{diam}(G-e)>k$ for every edge $e \in E(G)$. These graphs (often called diameter-critical graphs) have been studied by several authors. See, for example, [4], [6], [7], [9], and [10] and certain parts of the surveys [2] and [3]. The characterization of these graphs seems to be a difficult problem. Nevertheless, there are some partial results. For example, those minimal graphs of diameter 2 which are planar and contain no 3-cycle are completely described in [10]. Also (analogously defined) minimal tournaments are fully characterized [11]. In several papers there are considered graphs of diameter k without cycles of length 3, 4, ..., $k+1$. Clearly, such graphs are minimal. In particular, for $k=2$ we have minimal graphs of diameter 2 without 3-cycles. On the other hand, one can require minimal graphs of diameter 2 with every edge in a 3-cycle. A few years ago I conjectured that such graphs do not exist. Note that the validity of this conjecture would imply a simple proof of a result from [5]: Every bridgeless graph G of diameter 2 admits an orientation of diameter at most 6. Actually, there are two possibilities. (1) The radius of G is 1; then a desired orientation can be found very easily (even by Th. 2 of [5] G admits an orientation of radius 2 and thus of diameter at most 4). (2) The radius of G is 2; then any minimal spanning subgraph G' of G with $\operatorname{diam}(G')=2$ is bridgeless and hence the simple proof of Th. 5 from [5] applies whenever G' has at least one edge not contained in a 3-cycle. If every edge of G' lies in a 3-cycle, then the authors of [5] use a more complicated proof. Unfortunately, as we will see, our conjecture is not valid.

For brevity, a minimal graph of diameter k is called a k-MT graph if every its edge lies in a 3-cycle. It is the main purpose of this paper to present an infinite class of 2-MT graphs. Some remarks and open questions involve also the planarity and outerplanarity, and k-MT graphs with $k \geq 3$.

2. Diameter two

Here we give two classes of examples of 2-MT graphs. The first consists of graphs $A(r)$ $(r = 1, 2, \ldots)$ in Fig. 1. One can easily verify that the diameter of $A(r)$ is 2, the deletion of any edge increases the diameter, and any edge lies in a 3-cycle. Thus $A(r)$ is a 2-MT graph.

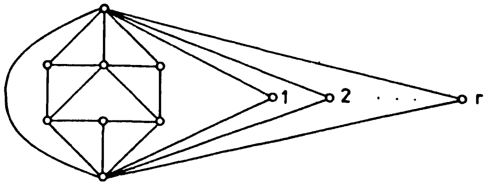

Fig. 1. The graph $A(r)$

The second class is more complicated and therefore we give its members, denoted by $B(s, t)$, in detail. The graph $B(2,3)$ is in Fig. 2 and generally graphs $B(s, t)$ with $s \geqslant 2$ and $t \geqslant 2$ can be described as follows.

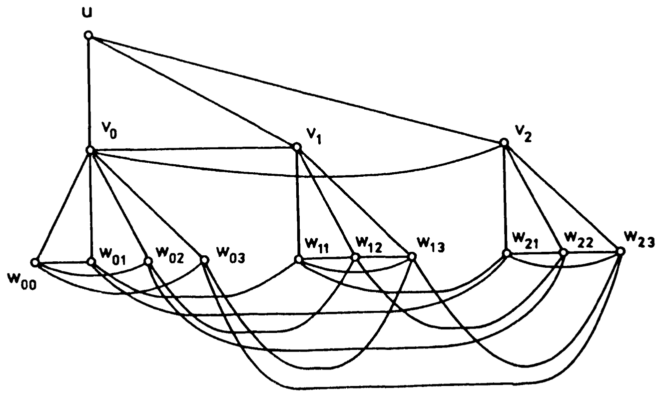

Fig. 2. The graph $B(2,3)$

$$V(B(s, t)) = \{u, v_0, v_1, \ldots, v_s, v_{00}, w_{01}, \ldots, w_{0t}, w_{11}, \ldots, w_{1t}, \ldots, w_{s1}, \ldots, w_{st}\},$$

$$\begin{aligned} E(B(s, t)) = {} & \{uv_0, uv_1, \ldots, uv_s\} \cup \{v_0v_1, v_0v_2, \ldots, v_0v_t\} \\ & \cup \{v_0 w_{00}\} \cup \bigcup_{i=0}^{s} \bigcup_{j=1}^{t} \{v_i w_{ij}\} \cup \bigcup_{j=1}^{t} \{w_{00} w_{0j}\} \\ & \cup \bigcup_{i=1}^{s} \bigcup_{1 \leqslant j < k \leqslant t} \{w_{ij} w_{ik}\} \\ & \cup \bigcup_{k=1}^{t} \bigcup_{0 \leqslant i < j \leqslant s} \{w_{ik} w_{jk}\}. \end{aligned}$$

And again, it is a routine matter to verify that the diameter of $B(s, t)$ is 2, every edge is contained in a 3-cycle, and no edge can be deleted without increasing the diameter.

We see that the minimum degree of $A(r)$ is 2 and that of $B(s, t)$ is min $\{s+1, t+1\}$. Thus we have established the following assertion.

Theorem 1. *For every integer* $d \geq 2$ *there exist infinitely many* 2-MT *graphs with minimum degree* d.

As every graph $A(r)$ is planar, we have

Theorem 2. *There exist infinitely many planar* 2-MT *graphs with minimum degree* 2.

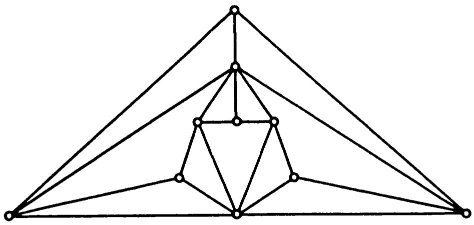

Fig. 3. A planar 2-MT graph with minimum degree 3

Fig. 3 shows a planar 2-MT graph with minimum degree 3 (even it is 3-connected). However, we know no other such graph and therefore we put the following question.

Problem 1. Do there exist infinitely many planar 2-MT graphs with minimum degree at least 3? We conjecture that the answer is negative.

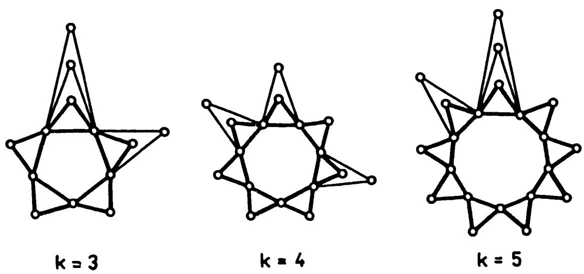

Fig. 4. Examples of k-MT graphs

Theorem 3. *There exists no outerplanar* 2-MT *graph.*

Proof. Suppose that there is an outerplanar 2-MT graph G. Clearly, G is without cutvertices (otherwise G is a star). If G is properly imbedded in the plane, then the boundary of the exterior face is a circuit corresponding to a hamiltonian cycle Z of G. Since G has at least two vertices, say, v_1 and v_2, of degree 2 (see e.g. [8]), it is useful to deal with them. As they are incident only with edges of Z, $d(v_1, v_2) \neq 1$ (otherwise the third vertex of the 3-cycle containing the edge v_1v_2 would be a cutvertex). Thus $d(v_1, v_2) = 2$ in G as well as in Z. A simple case analysis shows that the length of Z cannot be 4, 5, or 6. Therefore let it be at least 7 and let u_1, u_2 and u_0 be such vertices that $u_1v_1u_0v_2u_2$ is a section (a path) of Z. Since G is not a star, there exists a vertex x not adjacent to u_0. Then one sees that at least one of the distances $d(v_1, x)$ and $d(v_2, x)$ exceeds 2. This contradiction completes the proof.

Remark. The reader has certainly observed that the graph $A(r)$ (see Fig. 1) has r vertices with the same neighbourhood. In this way we can sometimes form new minimal graphs from smaller ones, but in general such an operation does not preserve the minimality (cf. [7]). Nevertheless, one can obtain a new 2-MT graph, e.g., from that of Fig. 2 by adding one or more copies of u. Adding a copy of the top vertex in Fig. 3, we also obtain a 2-MT graph, but we lose the planarity.

3. Larger diameters

Now we present classes of k-MT graphs with $k \geqslant 3$. These are illustrated in Fig. 4 for $k = 3$, 4, and 5. Each of these graphs of diameter k consists of the $(2k-1)$-cycle C and one or more internally disjoint paths of length 2 for each edge of C, where the paths join the ends of the edge. If the ends of each edge of C are joined by exactly one path of length 2, then we get outerplanar k-MT graphs. In general we get at least planar k-MT graphs and hence we have

Theorem 4. *For every integer $k \geqslant 3$ there exist an outerplanar and infinitely many planar k-MT graphs with minimum degree 2.*

Problem 2. Describe all outerplanar k-MT graphs for every $k \geqslant 3$.

Problem 3. Do there exist k-MT graphs with $k \geqslant 3$ and minimum degree at least 3?

REFERENCES

[1] BEHZAD, M., CHARTRAND, G., LESNIAK-FOSTER, L.: Graphs and Digraphs. Prindle, Weber and Schmidt, Boston, 1979.

[2] BERMOND, J. C., BOLLOBÁS, B.: The diameter of graphs — a survey. In: Proc. 12th S. E. Conf. Graph Theory and Computing. Congressus Numerantium 32, 1981, 3—27.

[3] BERMOND, J. C., BOND, J., PAOLI, M., PEYRAT, C.: Graphs and interconnection networks: diameter and vulnerability. In: Proc. 9th British Combinatorial Conference. London Math. Society, London, 1983, 1—30.

[4] CACETTA, L., HÄGGKVIST, R.: On diameter critical graphs. Discrete Math. 28, 1979, 223—229.

[5] CHVÁTAL, V., THOMASSEN, C.: Distances in orientations of graphs. J. Combin. Theory B 24, 1978, 61—75.

[6] GLIVIAK, F.: On certain edge-critical graphs of a given diameter. Mat. časopis 25, 1975, 249—263.
[7] GLIVJAK, F., KYŠ, P., PLESNÍK, J.: On the extension of graphs with a given diameter without superfluous edges. Mat. časopis 19, 1969, 92—101.
[8] HARARY, F.: Graph Theory. Addison-Wesley, Reading, MA, 1969.
[9] KRISHNAMOORTHY, V.,NANDAKUMAR, R.: A class of counterexamples to a conjecture on diameter critical graphs. In: Combinatorics and Graph Theory (Proc. Second Symp. Calcutta, 1980). Lecture Notes in Math. 885, Springer, Berlin, 1981, 297—300.
[10] PLESNÍK, J.: Critical graphs of given diameter. Acta Fac. Rerum Natur. Univ. Comen. Math. 30, 1975, 71—93.
[11] PLESNÍK, J.: Diametrically critical tournaments. Časopis pěst. mat. 100, 1975, 361—370.

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О МИНИМАЛЬНЫХ ГРАФАХ ДИАМЕТРА 2 С КАЖДЫМ РЕБРОМ В 3-ЦИКЛЕ

Ján Plesník

Резюме

Показывается, что существует бесконечный класс минимальных графов диаметра 2 с каждым ребром в 3-цикле. Частично исследуется и существование таких графов для больших диаметров.