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Mathematica Slovaca, Vol. 36 (1986), No. 2, 105--108

Persistent URL: http://dml.cz/dmlcz/129017

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LINEAR ARBORICITY OF GRAPHS

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In the note presented we mean by a graph an undirected, loopless, finite graph without multiple edges. Our terminology is based on Harary [5].

The concept of the linear arboricity of a graph G, denoted $\Xi(G)$, was introduced by Harary [6] as the minimum number of linear forests, i.e. unions of vertex-disjoint paths into which a graph G can be decomposed. Obviously, if the maximum degree of G is r then $\lceil r/2\rceil \leqslant \Xi(G)$ and if G is r-regular, then $\lceil (r+1)/2\rceil \leqslant \Xi(G)$. It was conjectured by Akiyama et al. [1] (and independently by Peroche [9] and by Hilton [7]) that for an r-regular graph G the equality $\Xi(G) = \lceil (r+1)/2\rceil$ holds, and it was proved for $r = 2,3$ and 4 (see [4], [2]) and for $r = 5,6$ and 8 (see [4] and [10]) In the sequel we shall refer to this conjecture as Linear Arboricity Conjecture (LAC in short).

The linear arboricity of G is closely related to the older concept of arboricity of G, denoted $\gamma(G)$, defined to be the minimum number of forests into which a graph G can be decomposed. Clearly $\gamma(G) \leqslant \Xi(G)$ for every graph G. Using the well--known result of Nash—Williams [8] one can easily prove (see[3]) that for an r-regular graph G, $\gamma(G) = \lceil (r+1)/2\rceil$. Hence, if true, the assertion of the LAC would be somewhat surprising since it would mean that the linear arboricity and the arboricity of a regular graph are equal.

In the note we consider the linear arboricity of the cartesian product and we prove that if the LAC holds for regular graphs G and H, then it holds for their cartesian product $G \times H$, as well.

Let us recall that the cartesian product $G \times H$ of graphs G and H is defined as the graph with the vertex set $V(G) \times V(H)$, in which two vertices (x, y) and (v, w) are joined with an edge if and only if either $xv \in E(G)$ and $y = w$, or $x = v$ and $yw \in E(H)$.

Lemma 1. *If H is a $2k$-regular graph with $\Xi(H) = k + 1$, and F is a linear forest, then $\Xi(F \times H) = k + 1$.*

Proof. To prove the lemma we shall construct a colouring of the edges of $F \times H$ with $k + 1$ colours such that each monochromatic set spans a linear forest. Clearly we can restrict ourselves to the case when F is a path $x_1x_2 \dots x_p$. Suppose the edges of H are coloured with $k + 1$ colours $c_1, \dots, c_{k+1}$ so that each colour spans a linear forest in H. Since H is 2k-regular, for every vertex a of H either each colour appears among the colours of the edges incident with a and some two of them appear exactly once, let us denote the set of all such vertices by X, or there is a colour which is missing at a, let us denote the set of such vertices by Y. To construct a suitable colouring of $F \times H$ we shall need a certain labelling of the vertices of H. Elements of X will be labelled with ordered pairs of colours. To introduce this labelling let us consider a multigraph M with the vertex set X in which two vertices a and b are joined with λ multiple edges if and only if there are λ maximal monochromatic paths starting in a and ending in b. Clearly M is 2-regular, since for every $a \in X$ there are exactly two colours which appear once among the colours of the edges incident with a and, consequently, exactly two maximal monochromatic paths start in a. Let $a_0 a_1 \dots a_{s-1} a_s$, where $a_0 = a_s$, be a cycle of M and let c_i, $i = 0, 1, \dots, s-1$, be the colour of the maximal monochromatic path which starts in a_i and ends in a_{i+1} (if $s \geqslant 3$, there is exactly one such a path in M, if $s = 2$, there are two such paths in M between a_0 and a_1 and we take for c_0 the colour of an arbitrary one of them and for c_1 the colour of the other). Now we label each vertex a_{i+1}, $i = 0, 1, 2, \dots, s-2$, with the ordered pair (c_i, c_{i+1}) and $a_0 = a_s$ with (c_{s-1}, c_0). In this way we label all vertices of X. Finally, we label each vertex y of Y with the colour which is missing at y, let us denote it by c_y.

We are ready now to define a suitable colouring of the edges of $F \times H$.

1. All edges of $F \times H$ parallel to an edge $e = ab$ of H, i.e. the edges $(x_i, a)(x_i, b)$, $i = 1, \dots, p$, are coloured with the same colour with which e is coloured in H.

2. Consider an edge $e = (x_i, a)(x_{i+1}, a)$ of $F \times H$.

(a) If $a \in X$, then it is labelled with an ordered pair, say (c, d). Colour e with c if i is even and with d if i is odd.

(b) If $a \in Y$, then it is labelled with c_a. Colour e with c_a.

Clearly, each subgraph of $F \times H$ spanned by a monochromatic set of edges has its maximum degree less than or equal to 2 and none of them contains cycles (see Figure 1). Hence the obtained colouring gives a decomposition of $F \times H$ into $k+1$ linear forests.

Theorem 2. *Let G and H be k-regular and p-regular graphs, respectively. Suppose $\Xi(G) = \lceil (k+1)/2 \rceil$ and $\Xi(H) = \lceil (p+1)/2 \rceil$. Then $\Xi(G \times H = = \lceil (k+p+1)/2 \rceil$. (In other words, if the LAC holds for G and H, then it holds for $G \times H$, as well.)*

Proof. Let $V(G) = \{x_1, \dots, x_m\}$ and $V(H) = \{y_1, \dots, y_n\}$. Suppose that E is a linear forest of G. Then $E_H = E \times \{y_1\} \cup \dots \cup E \times \{y_n\}$ is a linear forest of $G \times H$. Similarly we can define a linear forest F_G of H. Clearly, E_H and F_G are edge-disjoint for every linear forests E and F of G and H, respectively. Moreover if T_1 and T_2 are two edge-disjoint linear forests of G (resp H) then T_{1H} and T_{2H}

(resp. T_{1G} and T_{2G}) are also edge-disjoint. Denote $[(k+1)/2] = k'$ and $[(p+1)/2] = p'$ and let $E_1, \ldots, E_{k'}$, (resp. $F_1, \ldots, F_{p'}$) be linear forests covering the edges of G (resp. H). If both k and p are odd, then $G \times H$ can be decomposed into $k' + p'$ edge-disjoint linear forests $E_{1H}, \ldots, E_{k'H}, F_{1G}, \ldots, F_{p'G}$. If k or p, say p, is even, then we decompose $G \times H$ into $E_1 \times H$, $E_{2H}, \ldots, E_{k'H}$, and then we decompose $E_1 \times H$ into p' linear forests, which is possible by Lemma 1. This gives a decomposition of $G \times H$ into $k' + p' - 1$ linear forests. In both cases the obtained decomposition consists of $[(k+p+1)/2]$ linear forests, as claimed.

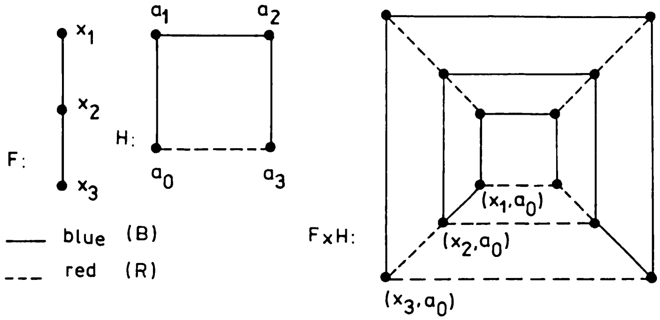

Fig. 1. a_0 is labelled with (R, B), a_3 with (B, R), a_1 and a_2 with $\{R\}$.

This theorem ensures the validity of the LAC for many regular graphs. Below we state just one example.

Corollary 3. *For an n-dimensional cube Q_n we have $\Xi(Q_n) = \lceil (n+1)/2 \rceil$.*

Proof. $Q_n = \underbrace{K_2 \times K_2 \times \ldots \times K_2}_{n \text{ times}}$.

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Received December 9, 1982

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ЛИНЕЙНАЯ ДРЕВЕСНОСТЬ ГРАФОВ

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Резюме

Линейная древесность $\Xi(G)$ графа G это минимальное число линейных лесов, соединение которых равно G. В работах [1], [7] и [9] независимо была высказана гипотеза, что линейная древесность r-регулярного графа G равна $\lceil (r+1)/2 \rceil$. В работах [1], [2], [4], [9], [10] она была доказана для $r = 2, 3, 4, 5, 6, 8$. В настоящей работе исследуется линейная древесность декартова произведения регулярных графов. Показано, что если гипотеза верна для регулярных графов G и H, то она верна для декартова произведения $G \times H$.