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A NOTE ON THE INTERSECTION MULTIPLICITY

EDUARD BOĎA

Let V and W be irreducible algebraic projective varieties of the projective space p_k^n over any (algebraically closed) field k with associated prime ideals p_V, $p_W \subset k[X_0, \ldots, X_n]$. Let C be an irreducible component of the intersection $V \cap W$ with the property $\dim(C) = \dim(V) + \dim(W) - n$ (that is to say, V and W are cutting proper in C). Let $i(V, W; C)$ denotes a multiplicity of the component C in $V \cap W$ (see [7]). Without loss of generality we can assume, that W is a complete intersection (see [3], [5]). That means algebraicaly

$$p_W = (F_1, \ldots, F_d) \quad \text{and} \quad \dim(W) = n - d.$$

This implies that

$$R =: (k[X_0, \ldots, X_n]/p_V)_{p_C \cdot (k[X_0, \ldots, X_n]/p_V)}$$

is a Noetherian local ring of dimension d with a maximal ideal $p_C \cdot R$ and $p_C \cdot R$-primary ideal $p_W \cdot R$ generated by a system of parameters $f_1, \ldots, f_d$ ($f_i =: F_i \cdot R$ for any $1 \leqq i \leqq d$). The Samuel's Theorem of reduction ([5], Chap. II, §7, b) says, that

$$i(V, W; C) = e_0(\boldsymbol{n}, R)$$

($\boldsymbol{n} =: (f_1, \ldots, f_d)$ and $e_0(\boldsymbol{n}, R)$ denotes the leading coefficient of the Hilbert—Samuel polynomial $l(R/\boldsymbol{n}^n)$ $n \gg 0$). The last equation shows that the intersection multiplicity can be counted by the multiplicity in local algebra.

There exists a various methods to count this multiplicity (see [1], [3], [6], [8]). In this note we give further method of the calculation of $i(V, W; C)$. I would like to thank Prof. W. Vogel (Halle) for helpful discussions.

Let $(A, \boldsymbol{m})$ be a commutative Noetherian local ring of dimension d. For any ideal $\boldsymbol{a}$ of A, $\dim(\boldsymbol{a})$ means the dimension of the ring $A/\boldsymbol{a}$. Let $\mathrm{Ass}(\boldsymbol{a})$ denotes the set of all the prime ideals which belong to any irredundant primary decomposition of an ideal $\boldsymbol{a}$. $\mathrm{Assi}(\boldsymbol{a})$ indicate the set of all the isolated prime ideals of $\mathrm{Ass}(\boldsymbol{a})$. Let $\mathrm{Assh}(\boldsymbol{a})$ be defined by

$$\mathrm{Assh}(\boldsymbol{a}) =: \{\boldsymbol{p} \in \mathrm{Ass}(\boldsymbol{a}); \dim(\boldsymbol{p}) = \dim(\boldsymbol{a})\}$$

and $U(\boldsymbol{a})$ indicate the intersection of all the primary ideals whose associated prime ideals belong to Assh $(\mathfrak{a})$. At last $Ui\ (\mathfrak{a})$ denotes the intersection of all the primary ideals, whose associated prime ideals belong to Assi $(\mathfrak{a})$.

Let $\mathfrak{q}$ be an ideal generated by a system of parameters $a_1, \ldots, a_d$ in A. Let us construct the following ideals $\mathfrak{q}'_k$ for all $k=0, \ldots, d$ by

$$\begin{gathered} \mathfrak{q}'_0=(0) \\ \mathfrak{q}'_k=(a_k)+U_i(\mathfrak{q}'_{k-1}) \end{gathered} \tag{1}$$

Lemma 1. *For all* $k=0, \ldots, d$ *it holds*

(i) $(a_1, \ldots, a_k)\subseteq \mathfrak{q}'_k$

(ii) $\mathrm{Assi}\ ((a_1, \ldots, a_k))=\mathrm{Assi}\ (\mathfrak{q}'_k)$.

Proof. Part (i) is clear. (ii) will follow by induction on k. If $k=0$ it is obvious. Assume that

$$\mathrm{Assi}\ ((a_1, \ldots, a_k))=\mathrm{Assi}\ (\mathfrak{q}'_k),\ \ 0<k\leqq d-1$$

and $\mathfrak{p}\in \mathrm{Assi}\ ((a_1, \ldots, a_{k+1}))$. Then we have

$$\mathfrak{p}\supseteq\mathfrak{p}'\in \mathrm{Assi}\ ((a_1, \ldots, a_k)),$$

so by the induction hypothesis

$$\mathfrak{p}\supseteq(a_{k+1},\ Ui(\mathfrak{q}'_k))=\mathfrak{q}'_{k+1}.$$

If $\mathfrak{p}\notin \mathrm{Assi}\ (\mathfrak{q}'_{k+1})$, then

$$\mathfrak{p}\supset\mathfrak{p}_0\supseteq\mathfrak{q}'_{k+1},\quad \text{so}$$

$$\mathfrak{p}\supset\mathfrak{p}_0\supseteq(a_1, \ldots, a_{k+1}).$$

This is a contradiction, since $\mathfrak{p}\in \mathrm{Assi}\ ((a_1, \ldots, a_{k+1}))$, so it holds $\mathfrak{p}\in \mathrm{Assi}\ (\mathfrak{q}'_{k+1})$. Conversely let $\mathfrak{p}\in \mathrm{Assi}\ (\mathfrak{q}'_{k+1})$. Then $\mathfrak{p}\supseteq(a_1, \ldots, a_{k+1})$. If $\mathfrak{p}\notin \mathrm{Assi}\ ((a_1, \ldots, a_{k+1}))$, then

$$\mathfrak{p}\supset\mathfrak{p}'\supseteq U_i(\mathfrak{q}'_k)$$

(by the induction hypothesis). This implies

$$\mathfrak{p}\supset\mathfrak{p}'\supseteq(a_{k+1},\ Ui(\mathfrak{q}'_k))=\mathfrak{q}'_{k+1},$$

which is a contradiction. So we have $\mathfrak{p}\in \mathrm{Assi}\ ((a_1, \ldots, a_{k+1}))$. This completes the proof.

Corollary 1. Assh $((a_1, \ldots, a_k))$ = Assh $(\mathfrak{q}'_k)$ *for all* $k=0, \ldots, d$.

Let us return to a ring R and an ideal $\mathfrak{n}=(f_1, \ldots, f_d)$ of R. Construct ideals $\mathfrak{n}'_k$ for all $k=0, \ldots, d$ by process (1). We are going to show, that

$$i(V,\ W;\ C)=l(R/\mathfrak{n}'_d).$$

The following Proposition is implied by [1] and [2].

Proposition 1. *Let q_k be ideals which are constructed from any ideal q generated by a system of parameters $a_1, \ldots, a_d$ (of any local Noetherian ring A) by*

$$q_0 = (0)$$
$$q_k = (a_k) + U(q_{k-1}) \tag{2}$$

for all $k = 0, \ldots, d$. Then

(i) $(a_1, \ldots, a_k) \subseteq q_k$

(ii) $\mathrm{Assh}\,(q_k) = \{p \in \mathrm{Assh}\,((a_1, \ldots, a_k));\ h(p) = k\}$

(iii) $e_0(q, A) = l(A/q_d)$

for all $k = 0, \ldots, d$.

Remark 1. Part (iii) shows one of the methods to count multiplicity.

Proposition 2. *For all $k = 0, \ldots, d$ there holds*

$$q'_k \subseteq q_k .$$

Proof. We use the induction on k. The case $k = 0$ is easy. Let now $q'_k \subseteq q_k$, $0 < k \leqq d - 1$. Since $\mathrm{Assh}\,(q_k) \subseteq \mathrm{Assh}\,(q'_k)$, the induction hypothesis implies

$$Ui(q'_k) \subseteq U(q'_k) \subseteq U(q_k).$$

Then we have $(a_{k+1}) + Ui(q'_k) \subseteq (a_{k+1}) + U(q_k)$, so

$$q'_{k+1} \subseteq q_{k+1} .$$

Corollary 2. $q \subseteq q'_d \subseteq q_d$.

Corollary 3. $e_0(q, A) = l(A/q'_d)$ if and only if $q_d = q'_d$.

Proposition 3. *There is a local Noetherian ring (A, m) and an ideal q generated by a system of parameters of A, for which*

$$q_d \neq q'_d .$$

Proof. Let k be any (algebraically closed) field. Let us observe the ideal $a = (X_2, X_1X_3, X_1X_4)$ of the polynomial ring $Q =: k[X_1, X_2, X_3, X_4]$. Let

$$A =: Q_{(X_1, X_2, X_3, X_4)}/a \cdot Q_{(X_1, X_2, X_3, X_4)} .$$

Since the ideal $(X_3, X_1 + X_4, a) = (X_2, X_3, X_1X_4, X_1 + X_4)$ is (X_1, X_2, X_3, X_4)-primary and $\dim (A) = 2$, the ideal $(X_3, X_1 + X_4) \cdot A$ is generated by a system of parameters. We count immediately

$$(0) = ((X_1, X_2) \cap (X_2, X_3, X_4)) \cdot A$$

$q_1 = (X_1, X_2, X_3) \cdot A$ $\qquad$ $q'_1 = ((X_1, X_2, X_3) \cap (X_2, X_3, X_4)) \cdot A$

$q_2 = (X_1, X_2, X_3, X_4) \cdot A$ $\qquad$ $q'_2 = (X_2, X_3, X_1X_4, X_1 + X_4) \cdot A$

We are going to give a sufficient condition for equality of ideals q_d and q'_d now.

Theorem 1. *Let $(A, \boldsymbol{m})$ be a commutative Noetherian local ring of dimension d and $\boldsymbol{q}$ an ideal of A generated by a system of parameters $a_1, \ldots, a_d$. Denote $\boldsymbol{q}_k(\boldsymbol{q}'_k)$ the ideals, which are constructed from $\boldsymbol{q}$ by processes (2) ((1)). If $h(\boldsymbol{p}) = k$ for all $\boldsymbol{p} \in \operatorname{Assh}((a_1, \ldots, a_k))$ and $k = 0, \ldots, d$, then*

(i) $\operatorname{Assh}(\boldsymbol{q}_k) = \operatorname{Assh}(\boldsymbol{q}'_k) = \operatorname{Assh}((a_1, \ldots, a_k))$
(ii) $U(\boldsymbol{q}_k) = U(\boldsymbol{q}'_k)$

for all $k = 0, \ldots, d$.

Proof. (i) follows from Proposition 1 and Corollary 1. (ii) follows by induction on k. If $k = 0$, it is obvious. Let now $U(\boldsymbol{q}_k) = U(\boldsymbol{q}'_k)$ $0 < k \leqq d - 1$.

Proposition 2 implies $U(\boldsymbol{q}'_{k+1}) \subseteq U(\boldsymbol{q}_{k+1})$. Take an element $x \in U(\boldsymbol{q}_k)$. Then $x \in U(\boldsymbol{q}'_k)$ by the induction hypothesis, so

$$\boldsymbol{q}'_k : x \not\subset \boldsymbol{p} \quad \text{for all} \quad \boldsymbol{p} \in \operatorname{Assh}(\boldsymbol{q}'_k).$$

Hence by the assumption of the theorem we have

$$Ui(\boldsymbol{q}'_k) : x \not\subset \boldsymbol{p} \quad \text{for all} \quad \boldsymbol{p} \in \operatorname{Assh}(\boldsymbol{q}'_{k+1}), \quad \text{so}$$

$$x \in U(\boldsymbol{q}'_{k+1}).$$

We have proved that $U(\boldsymbol{q}_k) \subseteq U(\boldsymbol{q}'_{k+1})$. Hence we get

$$\boldsymbol{q}_{k+1} = (a_{k+1}) + U(\boldsymbol{q}_k) \subseteq U(\boldsymbol{q}'_{k+1})$$

and, using part (i),

$$U(\boldsymbol{q}_{k+1}) \subseteq U(\boldsymbol{q}'_{k+1}).$$

Corollary 4. *On the assumptions in Theorem 1 it holds*

$$\boldsymbol{q}_d = \boldsymbol{q}'_d, \quad \text{so} \quad l(A/\boldsymbol{q}_d) = l(A/\boldsymbol{q}'_d).$$

Let us return to an ideal $\boldsymbol{n} = (f_1, \ldots, f_d)$ of a ring

$$R = (k[X_0, \ldots, X_n]/\boldsymbol{p}_V)_{\boldsymbol{p}_C \cdot (k[X_0, \ldots, X_n]/\boldsymbol{p}_V)}$$

again. Since the ring R is equidimensional ([4], Chap. II, § 3), it obviously satisfies the conditions of Theorem 1, which yields

Theorem 2. $i(V, W; C) = l(R/\boldsymbol{n}'_d)$.

Proposition 1 and Theorem 2 show, that both processes (1) and (2) are applicable for calculating of the intersection multiplicity.

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ЗАМЕТКА О КРАТНОСТИ ПЕРЕСЕЧЕНИЯ

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Резюме

Пусть $i(V, W; C)$ обозначает кратность компоненты C в пересечении неприводимых алгебраических проективных многоразий V и W. В силу теоремы Самюэля об редукции $i(V, W; C) = e_0(\mathfrak{n}, R)$ для определенного параметрического идеала $\mathfrak{n}$ в определенном локальном нетеровом кольце $R(e_0(\mathfrak{n}, R)$ — старший член многочлена Гильберт—Самюэля $l(R/\mathfrak{n}^n)$, т. н. кратность идеала $\mathfrak{n}$ в кольце R). В работе вводится один практический метод для вычисления $e_0(\mathfrak{n}, R)$.