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B-GROUPS OF ORDER A PRODUCT OF TWO DISTINCT PRIMES

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ABSTRACT. An abstract group H is said to be a B-group if every primitive permutation group containing H as a regular subgroup is 2-transitive. Non-abelian B-groups of order pq, where p, q are two distinct primes, are characterized.

1. Introduction

An abstract group H is said to be a *B-group* (*Burnside group*) if every primitive permutation group containing H as a regular subgroup is 2-transitive. Primitive permutation groups which are not 2-transitive are called *uniprimitive groups*. The first examples of B-groups were given by Burnside in 1911. He proved that every cyclic group of order p^m (p prime, $m > 1$) is a B-group ([2]). Later on Schur showed that every cyclic group of composite order is a B-group ([11]). These results have been generalized by Wielandt and Bercov who proved that every abelian group which has a Sylow subgroup isomorphic to $\mathbb{Z}_{p^a} \times \mathbb{Z}_{p^b}$ (p odd prime, $a > b$) is a B-group ([1]).

Some partial results about nonabelian groups are also known. Dihedral groups and generalized dicyclic groups $\langle x, y \mid x^{2n} = 1,\ y^2 = x^n,\ y^{-1}xy = x \rangle$, were shown to be B-groups, respectively, by Wielandt [16] in 1949 and by Scott [12] in 1957. Furthermore, groups of order $3p$ (p a prime) have also been dealt with. Combining works of several authors (see [7], [13], [8]) we can deduce that a nonabelian group of order $3p$ is a B-group provided one of the following holds:

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i) $p = 2 \cdot 3^a + 1$, $a > 2$,

or

ii) $4p = 3c^2 + 1$ and $p > 37$,

or

iii) $p = 6q + 1$, $q > 7$ a prime.

Most of these results were proved using character theory or the Schur method ([17]). With the classification of finite simple groups a different approach is possible. Based on the work of Marušič and Scapellato [4], [5], [6] and Praeger, Wang and Xu [9], [10], [14] on vertex-transitive pq-graphs, a classification of B-groups of order pq, where $p > q$ are two primes, can be obtained.

THEOREM 1.1. *A group of order pq, where $p > q$ are primes, is not a B-group if and only if it is not abelian and one of the following holds:*

(i) $q = \frac{p-1}{2}$ *and* $p > 5$,

or

(ii) $pq = 31 \cdot 5$,

or

(iii) $pq = 29 \cdot 7$.

2. Proof of Theorem 1.1

Since every abelian group of order pq is cyclic and therefore a B-group, we can restrict ourselves to nonabelian groups. Since a nonabelian group of order pq exists if and only if $q \mid p-1$ (and is unique), we will assume that $q \mid p-1$.

A group H of order n is not a B-group if and only if there exists a uniprimitive group G of degree n containing H as a regular subgroup. Therefore, to find all non-abelian groups of degree pq which are not B-groups it is enough to find all regular subgroups of uniprimitive groups of degree pq with $q \mid p-1$. Moreover, the socle of a uniprimitive group G of order pq cannot be abelian. Assume on the contrary that $H = \operatorname{soc} G$ is abelian. Since H is a non-trivial normal subgroup of a primitive group, it is transitive, and since it is abelian, it must be regular and therefore a cyclic group of composite order. But such groups are B-groups and cannot occur as regular subgroups of uniprimitive groups. In [6] Marušič and Scapellato gave the list of non-abelian socles of uniprimitive groups of degree pq. Excluding the cases where $q \nmid p-1$, Table 1 is obtained.

TABLE 1. Socles of uniprimitive groups of degree pq, $q \mid p-1$.

row	soc G	(p, q)	action	comment
1	A_p	$(p, \frac{p-1}{2})$	pairs	$p \geq 5$
2	$PSL(2,p)$	$(p, \frac{p-1}{2})$	cosets of D_{p+1}	$p \equiv 3\ (4)$
3	$PSL(2,59)$	$(59,29)$	cosets of A_5	
4	$PSL(2,23)$	$(23,11)$	cosets of S_4	
5	$PSL(2,11)$	$(11,5)$	cosets of A_4	
6	M_{23}	$(23,11)$		
7	M_{11}	$(11,5)$		
8	$PSL(5,2)$	$(31,5)$	2-spaces	
9	$PSL(2,29)$	$(29,7)$	cosets of A_5	
10	$PSL(2,19)$	$(19,3)$	cosets of A_5	

Observe that in the rows 1 to 7, $q = \frac{p-1}{2}$ holds. So let us first deal with groups of order pq, where $q = \frac{p-1}{2}$. If $p = 5$, then $q = 2$ and the only non-abelian group of order 10 is dihedral group, which is a B-group. For $p > 5$ consider the uniprimitive action of A_p on the set of unordered pairs $X = \{\{x,y\} : 0 \leq x < y \leq p-1\}$ ($|X| = pq$). Let α denote the cyclic permutation $\alpha = (0,1,\ldots,p-1)$. Since the multiplicative group $\mathbb{Z}_p^*$ is cyclic, it contains an element r of order $\frac{p-1}{2}$. Let β denote the permutation which sends $i \in \mathbb{Z}_p$ to $ri \in \mathbb{Z}_p$. The decomposition of β into disjoint cycles equals $(0)(a, ra, \ldots, r^{\frac{p-3}{2}}a)(b, rb, \ldots, r^{\frac{p-3}{2}}b)$, which implies $\beta \in A_p$. Clearly, $\beta^{-1}\alpha\beta = \alpha^r$, showing that $\langle\alpha,\beta\rangle$ is the non-abelian group of order pq. Using the fact that $p \equiv 3\ (4)$ it is easy to see that the group $\langle\alpha,\beta\rangle$ acts transitively and therefore regularly on X, and thus showing that the non-abelian group of order pq, where $q = \frac{p-1}{2}$, is not a B-group.

We are now going to show that the non-abelian groups of order $31 \cdot 5$ and $29 \cdot 7$ are not B-groups. We will need the following lemma from [6].

LEMMA 2.1. ([6; Lemma 2.2]) *Let G be a transitive group on a set X of degree n and order ns. Let H be a subgroup of G such that n divides $|H|$ and $([G:H],|H|)=1$. Then H is transitive.*

Let $G = PSL(5,2)$ acting on 2-spaces. We know $|G| = 2^{10}\cdot 3^2\cdot 5\cdot 7\cdot 31$. Let S denote a Sylow subgroup of order 31. Its normalizer $N = N_G(S)$ is a non-abelian group of order $31\cdot 5$ ([3; 7.3]). The group N is transitive by Lemma 2.1 and therefore regular. Since the action of G is uniprimitive, the nonabelian group of order $31\cdot 5$ is not a B-group.

Let now $G = PSL(2,29)$ acting on the cosets of its subgroup A_5. Its order is $29\cdot 2^2\cdot 3\cdot 5\cdot 7$ and its degree $29\cdot 7$. It contains a Sylow subgroup $S = \left\{\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix} : x\in\mathbb{Z}_{29}\right\}$ of order 29. Its normalizer $N = \left\{\begin{pmatrix} a & x \\ 0 & a^{-1}\end{pmatrix} : a\in\mathbb{Z}_{29}^*, x\in\mathbb{Z}_{29}\right\}$ contains a non-abelian subgroup of order $7\cdot 29$ which is transitive and therefore regular by Lemma 2.1. Since the action of G is uniprimitive, this shows that the non-abelian group of order $7\cdot 29$ is not a B-group.

In order to complete the proof we have to show that the nonabelian group of order $19\cdot 3$ is a B-group. Let G be a uniprimitive group of degree $19\cdot 3$. By [14, Lemma 2.1], it follows that $\operatorname{soc} G = PSL(2,19)$ acting on the cosets of a subgroup $A\cong A_5$, and G is either $PSL(2,19)$ or $\operatorname{Aut} PSL(2,19) = PGL(2,19)$. Suppose that G contains a regular subgroup R. Since the index of $PSL(2,19)$ in G is either 1 or 2 and R is a group of odd order, it is contained in $PSL(2,19)$. Thus without loss of generality we can assume $G = PSL(2,19)$. It is known that a Sylow subgroup of order 9 in $PSL(2,19)$ is cyclic, implying that all subgroups of order 3 in $PSL(2,19)$ are conjugate. It follows that the group R has a non-trivial intersection with some conjugate of A. But conjugates of A are point stabilizers of the action of G, contradicting the fact that R is regular. We have thus proved that the non-abelian group of order $57 = 3\cdot 19$ is a B-group, completing the proof of Theorem 1.1.

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