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ON SINGULAR SOLUTIONS OF THIRD ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. The structure of singular solutions, defined on $\mathbb{R}_+$, of the nonlinear differential equation $y''' + py'' + qy' + rf(y, y', y'') = 0$ is studied. Sufficient conditions are given under which a set of all zeros of a singular solution is a neighbourhood of ∞.

1. Introduction

Consider the third-order differential equation

$$y''' + py'' + qy' + rf(y, y', y'') = 0, \tag{1}$$

where $p \in C^1(\mathbb{R}_+)$, $q \in C^0(\mathbb{R}_+)$, $r \in L_{\mathrm{loc}}(\mathbb{R}_+)$, $f \in C^0(\mathbb{R}^3)$, $r \geq 0$ on $\mathbb{R}_+$ and

$$f(x_1, x_2, x_3)x_1 \geq 0 \qquad \text{on} \quad \mathbb{R}^3. \tag{2}$$

DEFINITION 1. Let $I \subset \mathbb{R}_+$, $y \in C^2(I)$ and y'' be absolutely continuous on I. Then y is called a *solution of* (1) if equation (1) is valid for almost all t on I.

A solution y is called *proper* if $I = \mathbb{R}_+$ and $\sup\limits_{\tau \leq t < \infty} |y(t)| > 0$ for all $\tau \in \mathbb{R}_+$.

A proper solution is called *oscillatory* if it has infinitely many zeros tending to ∞.

A solution y defined on $\mathbb{R}_+$ is called *singular* if there exists $T_y > 0$, such that $y \equiv 0$ on $[T_y, \infty)$ and y is not trivial in any left neighbourhood of T_y.

Many authors studied the following problem, see e.g. [3] – [7], [10] – [14]:

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To give sufficient conditions on p, q, r and f under which a solution y of (1), defined on $\mathbb{R}_+$, with a zero is oscillatory.

Note, that they defined oscillatory solution y as a solution defined on $\mathbb{R}_+$, that has a sequence of zeros tending to ∞. Thus oscillatory solution in this sense is, according to Definition 1, either proper oscillatory or singular.

One of significant problems is to study the existence of (proper) oscillatory solutions. To use the results of the above mentioned authors, it is useful to seek conditions under which a solution of (1) with a zero

— is defined on $\mathbb{R}_+$;
— is not singular.

In this paper we give sufficient conditions under which for a singular solution y of (1)

$$y(t) \neq 0\,, \qquad t \in [0, T_y)\,, \tag{3}$$

holds, where T_y is given by Definition 1.

The following example shows that the singular solution may exist, for which (3) does not hold.

EXAMPLE. Let

$$y(t) = \begin{cases} t\{-15(t-1)^3 + 11(t-1)^2 - 5(t-1) + 1\} & \text{for } t \in [0,1]\,, \\ (2-t)^4 & \text{for } t \in (1,2]\,, \\ 0 & \text{for } t > 2\,, \end{cases}$$

$$r(t) \equiv 24\,,$$

$$q(t) \equiv \tfrac{9}{2}\Big[3 + \big(\tfrac{15}{4}\big)^{\frac{1}{4}}\Big] \qquad \text{on} \quad \mathbb{R}_+\,,$$

$$p(t) = \begin{cases} \big(-y''' - qy' - ry^{\frac{1}{4}}\big)(y'')^{-1} & \text{for } t \in [0,1]\,,\ t \neq \frac{2}{3}\,, \\ \frac{15}{4} + \big(\frac{4}{15}\big)^{\frac{1}{4}} & \text{for } t = \frac{2}{3}\,, \\ \frac{q}{3}(2-t) & \text{for } t \in [1,2]\,, \\ 0 & \text{for } t > 2\,. \end{cases}$$

Then $p \in C^0(\mathbb{R}_+)$, $y(0) = 0$, $y > 0$ on $(0,2)$ and y is the singular solution of $y''' + py'' + qy' + r|y|^\lambda \operatorname{sgn} y = 0$ with $\lambda = \frac{1}{4}$. Thus (3) is not valid for $t \in [0,2)$.

2. Structure of singular solutions

The following theorem is often used and gives the sufficient condition under which singular solutions do not exist at all.

THEOREM 1. ([9]) *Let $\varepsilon > 0$ exist such that*

$$|f(x_1, x_2, x_3)| \le \sum_{i=1}^{3} |x_i| \qquad on \quad [-\varepsilon, \varepsilon]^3 . \tag{4}$$

Then there exists no singular solution of (1).

It is evident that the assumption of Theorem 1 is not valid for Emden-Fowler equation:

$$y''' + py'' + qy' + r|y|^\lambda \operatorname{sgn} y = 0 , \qquad \lambda \in (0,1) . \tag{5}$$

Moreover, singular solutions may exist, see [9] for $p \equiv q \equiv 0$.

In [1] the structure of a singular solution y is studied for two-terms equation (1), i.e. for $p \equiv 0$ and $q \equiv 0$. It was proved that

$$y(t)y'(t) < 0 , \quad y(t)y''(t) > 0 \qquad \text{on} \quad [0, T_y) .$$

In the following theorem the same result is proved for singular solutions of (1) in a neighbourhood of T_y.

First, sum up the needed results in the following lemma:

LEMMA 1. *Let $I \subset \mathbb{R}_+$ and let there exist positive solution h of*

$$h'' + ph' + qh = 0 \tag{6}$$

on I. Then (1) *can be expresed on I in an equivalent form*

$$y^{[3]} + f_1\big(t, y^{[0]}, y^{[1]}, y^{[2]}\big) = 0 , \tag{7}$$

where $R(t) = \mathrm{e}^{\int_0^t p(s)\, \mathrm{d}s}$,

$$\begin{aligned} y^{[0]} &= y , \qquad & y^{[2]} &= Rh^2\big(y^{[1]}\big)' = R(y''h - y'h') , \\ y^{[1]} &= \frac{y'}{h} , \qquad & y^{[3]} &= \big(y^{[2]}\big)' , \end{aligned} \tag{8}$$

$$f_1(t, x_1, x_2, x_3) = r(t)h(t)R(t) f\Big(x_1, h(t)x_2, \frac{x_3}{R(t)h(t)} + h'(t)x_2\Big) .$$

Moreover, if $t_0 \in I$ and y is a solution of equation (7) (*and thus of equation* (1), *too*) *such that*

$$y(t_0)y^{[1]}(t_0) < 0 , \qquad y(t_0)y^{[2]}(t_0) > 0 , \tag{9}$$

then

$$y(t)y^{[1]}(t) < 0 , \quad y(t)y^{[2]}(t) > 0 \qquad for \quad t \in I , \quad t \le t_0 . \tag{10}$$

P r o o f . The transformation of (1) into (7) can be obtained by the direct computation, see [2]. Further (2), (7) and (8) yield

$$y^{[3]}(t)y(t) \le 0 \qquad \text{a.e. on} \quad I \tag{11}$$

and

$$y^{[i]}(t) \geq 0 \; (\leq 0) \quad \text{on } I_1 \subset I, \quad i \in \{1,2,3\},$$

implies (12)

$$y^{[i-1]} \text{ is nondecreasing (nonincreasing) on } I_1 .$$

Let (9) be valid with $y(t_0) > 0$ for simplicity. Put $I_2 = \{t : \; t \in I, \; t < t_0\}$. We prove that

$$\prod_{i=0}^{2} |y^{[i]}(t)| > 0 \qquad \text{on} \quad I_2 . \tag{13}$$

Thus, suppose indirectly, that $\tau \in I_2$ is the maximal number, such that $\prod\limits_{i=0}^{2} |y^{[i]}(\tau)| = 0$. From this, from (9), (11) and (12)

$$y(t) > 0, \quad y^{[1]}(t) < 0, \quad y^{[2]}(t) > 0, \quad y^{[3]}(t) \leq 0 \qquad \text{on} \quad (\tau, t_0]$$

and, evidently, $y^{[2]}(\tau) = 0$. Thus

$$y^{[2]}(t_0) = y^{[2]}(t_0) - y^{[2]}(\tau) = \int_{\tau}^{t_0} y^{[3]}(s) \, \mathrm{d}s \leq 0 .$$

The contradiction with (9) and $y(t_0) > 0$ proves that (13) holds. The conclusion (10) follows from (9), (11), (12) and (13). □

THEOREM 2. *Let y be a singular solution of* (1) *and let T_y be the number from its definition. Then there exists left open neighbourhood I of T_y such that*

$$y(t)y'(t) < 0, \quad y(t)y''(t) > 0 \qquad \text{on} \quad I. \tag{14}$$

Proof. Let h be a solution of equation (6) such that $h(T_y) = 1$, $h'(T_y) = 0$, and let J be an open left neighbourhood of T_y on which $h > 0$ holds. Then according to Lemma 1, (1) can be expresed in the equivalent form (7). Further

$$y^{(i)}(T_y) = 0, \qquad i = 0,1,2, \tag{15}$$

and we prove that

$$y(t)y^{[1]}(t) < 0, \quad y(t)y^{[2]}(t) > 0 \qquad \text{on} \quad J. \tag{16}$$

First suppose that $y \neq 0$, say $y > 0$, in a left neighbourhood J_1 of T_y, $J_1 \subset J$. Then (11) yields $y^{[3]} \leq 0$ a.e. on J_1. Moreover, $y^{[3]} \not\equiv 0$ a.e. in any left neighbourhood J_2 of T_y as, otherwise, (8) and (15) yield $y \equiv 0$ in J_2 that contradicts $y \neq 0$. From this

$$y^{[2]}(t) = -\int_{t}^{T_y} y^{[3]}(s) \, \mathrm{d}s > 0 \qquad \text{for} \quad t \in J_1$$

and, using (8) and (15), $y^{[1]}(t) < 0$, $y(t) > 0$ on J_1. The validity of (16) follows from Lemma 1.

The last possible case consists in the existence of an increasing sequence $\{\tau_k\}_1^\infty$ such that

$$y \not\equiv 0 \qquad \text{in any left neighbourhood of } \ T_y , \tag{17}$$

and $\tau_k \in J$, $\lim_{k\to\infty} \tau_k = T_y$ and $y(\tau_k) = 0$ for $k = 1, 2, \dots$.

Let $J_2 = [\alpha, T_y]$, $\alpha \in J$ be such an interval that

$$W = \frac{3}{2} \min_{s\in\bar J} h(s) - \frac{1}{2} \max_{s\in\bar J} \big| \big(R(s)h^3(s)\big)' \big| A(T_y) > 0 , \tag{18}$$

where $A(t) = \int\limits_\alpha^t \frac{d\tau}{R(\tau)h^2(\tau)}$ and $\bar J$ is the closure of J. If

$$F(t) = -A(t)y^{[2]}(t)y(t) + \frac{1}{2} A(t)R(t)h^3(t)\big(y^{[1]}(t)\big)^2 + y(t)y^{[1]}(t) , \qquad t \in J_2 ,$$

then, using (8), (11) and (18), we have

$$\begin{aligned} F'(t) &= -A(t)y^{[3]}(t)y(t) + \big[\tfrac{3}{2}h(t) + \tfrac{1}{2}A(t)\big(R(t)h^3(t)\big)'\big]\big(y^{[1]}(t)\big)^2 \\ &\geq W\big(y^{[1]}(t)\big)^2 \geq 0 \qquad \text{on} \quad J_2 . \end{aligned} \tag{19}$$

Thus F is nondecreasing and $F(T_y) = 0$ yields $F(t) \leq 0$ for $t \in J_2$. On the other hand, if $\tau_k \in J_2$, then $F(\tau_k) \geq 0$, and thus $F(t) \equiv 0$ on $[\tau_k, T_y]$. From this, and from (19) we have

$$0 = \int\limits_{\tau_k}^{T_y} F'(s)\, ds \geq W \int\limits_{\tau_k}^{T_y} \big(y^{[1]}(s)\big)^2\, ds \geq 0 .$$

Thus, using (18), $y'(t) \equiv 0$ on $[\tau_k, T_y]$, which contradicts (15) and (17). The contradiction proves that (16) is valid and applying (8)

$$y(t)y'(t) < 0 , \qquad t \in J . \tag{20}$$

Further suppose that there exists a sequence $\{\tau_k\}_1^\infty$ such that $\tau_k < T_y$, $k = 1, 2, \dots$, $\lim_{k\to\infty} \tau_k = T_y$, $y''(\tau_k) = 0$.

Let z be the solution of

$$z'' - (zp)' + zq = 0 , \qquad z(T_y) = 1 , \quad z'(T_y) < p(T_y) ,$$

and v be the solution of

$$v'' - (vp)' + qv = 3z' - 2zp , \qquad v(T_y) = 1 , \quad v'(T_y) < 1 + p(T_y) .$$

Let $J_3 \subset J$ be an open left neighbourhood of T_y such that

$$\begin{array}{ll} z(t) > 0\,, & z'(t) - z(t)p(t) < 0\,, \\ v(t) > 0\,, & z(t) - v'(t) + v(t)p(t) > 0\,, \end{array} \qquad t \in J_3\,. \tag{21}$$

Define F on $J_3 \cup \{T_y\}$ by

$$F = -zy''y + (z' - zp)y'y + \frac{1}{2}(z - v' + vp)y'^2 + vy''y'\,.$$

Then, using (20) and (2)

$$F' = -vry'f(y, y', y'') + zryf(y, y', y'') + vy''^2 \geq 0 \qquad \text{on} \quad J_3\,.$$

From this, F is nondecreasing and using $F(T_y) = 0$ we have $F(t) \leq 0$, $t \in J_3$. Let $\tau \in J_3$ be a zero of y''. Then

$$F(\tau) = (z' - zp)y'y + \frac{1}{2}(z - v' + vp)y'^2 \big|_{t=\tau} \leq 0$$

that contradicts (20) and (21). Thus $y'' \neq 0$ on J_3 and it follows from (21) and $y^{(i)}(T_y) = 0$, $i = 0, 1, 2$, that $y(t)y''(t) > 0$, $t \in J_3$, must be valid. Put $I = J_3$. □

Remark 1. Note, that for a singular solution y of (1), (16) is valid in a left neighbourhood of T_y.

Remark 2. Singular solution y of (1), fulfilling (14) in an open left neighbourhood of T_y, is called *Kneser singular solution.* Thus every singular solution is Kneser singular solution.

In the light of Theorem 2 our problem can be formulated in the following way:

PROBLEM 1. To give sufficient conditions under which for Kneser solution y the inequality (3) is valid (if (4) does not hold).

3. Problem 1

In the two following theorems, further assumptions are posed only on p and q.

THEOREM 3. *Let $q \leq 0$ on $\mathbb{R}_+$. Then for a singular solution y of* (1), (3) *is valid.*

Proof. Let y be a singular solution of (1). Then, according to [8], there exists a positive solution $h > 0$ of equation (6) on $\mathbb{R}_+$ and the assupmtions of Lemma 1 are fulfilled with $I = \mathbb{R}_+$. As, according to Remark 1, the inequalities (9) are valid for t_0 lying in a left neighbourhood of T_y, (10) yields (3). □

THEOREM 4. *Let $C \in \mathbb{R}$, $p \in C^2(\mathbb{R}_+)$, $q \in C^1(\mathbb{R}_+)$,*

$$(1-C)\{p'' + 3Cpp' + C^2p^3\} - q' - Cpq > 0 \qquad \text{on} \quad \mathbb{R}_+$$

$$\text{and either} \begin{cases} C > \frac{2}{3} \text{ and } p \ge 0 \text{ on } \mathbb{R}_+ \\ \text{or} \\ C < \frac{2}{3} \text{ and } p \le 0 \text{ on } \mathbb{R}_+ \\ \text{or} \\ C = \frac{2}{3}\,. \end{cases} \tag{22}$$

Then for a singular solution y of (1) the inequality (3) holds.

P r o o f . Let y be singular solution of (1) and let T_y be a number from its definition. Contrarily, suppose that there exists $\tau \in [0, T_y)$ such that $y(\tau) = 0$. Put

$$F(t) = \mathrm{e}^{C\int_0^t p(s)\,\mathrm{d}s}\Big\{-2yy'' + y'^2 + \big[-q + (1-C)p' + (C-C^2)p^2\big]y^2 + 2p(C-1)yy'\Big\}\,, \qquad t \ge \tau\,. \tag{23}$$

Then

$$F'(t) = \mathrm{e}^{C\int_0^t p(s)\,\mathrm{d}s}\Big\{y^2\big[-q' - Cpq + 3pp'(C-C^2) + p^3(C^2-C^3) + p''(1-C)\big] + y'^2[3C-2]p + 2ryf(y,y',y'')\Big\}\,. \tag{24}$$

Assumptions of the theorem yield $F'(t) \ge 0$. As $F(T_y) = 0$ and $F(\tau) = [y'(\tau)]^2 \ge 0$, we can conclude that $F \equiv 0$ on $[\tau, T_y]$ and thus, by integration of (24) on $[\tau, T_y]$ and by (22), $y \equiv 0$ on $[\tau, T_y)$. It is in contradiction with the definition of the singular solution y. □

CONSEQUENCE 1. *Let $q \in C^1(\mathbb{R}_+)$ and let one of the following assumptions hold:*

(i) $p \in C^2(\mathbb{R}_+)$, $p \le 0$, $p'' - q' > 0$ *on* $\mathbb{R}_+$;
(ii) $p \ge 0$, $q' + pq < 0$ *on* $\mathbb{R}_+$;
(iii) $p \in C^2(\mathbb{R}_+)$, $9p'' + 18pp' + 4p^3 - 27q' - 18pq > 0$ *on* $\mathbb{R}_+$.

Then for a singular solution y of (1) the relation (3) holds.

P r o o f . It follows from Theorem 4 for $C = 0, 1, \frac{2}{3}$, respectively. Note, that according to (23) the assumption $p \in C^2(\mathbb{R}_+)$ is not needed in case $C = 1$. □

In the following theorem assumptions are posed also on the nonlinearity f.

THEOREM 5. *Let $\alpha > 0$ exist such that*

$$|f(x_1, x_2, x_3)| \geq \alpha |x_1| \qquad on \quad \mathbb{R}^3 . \tag{25}$$

Further, let one of the following assumptions be valid on $\mathbb{R}_+$:

(a) $p \geq 0\,,\ q < \sqrt{\alpha r p}$;
(b) $p \geq 0\,,\ q \in C^1(\mathbb{R}_+)\,,\ 2\alpha r - q' - qp > 0$;
(c) $p \leq 0\,,\ p \in C^2(\mathbb{R}_+)\,,\ q \in C^1(\mathbb{R}_+)\,,\ \alpha r + p'' - q' > 0$.

Then for a singular solution of (1) *the inequality* (3) *holds.*

Proof. Let y be a singular solution, T_y be defined by Definition 1.

a) Suppose, contrarily, that $\tau \in [0, T_y)$ exists such that

$$y(\tau) = 0 .$$

Theorem 2 and Definition 1 yield, for simplicity,

$$y(t) > 0\,, \qquad y'(t) < 0\,, \qquad y''(t) > 0 \tag{26}$$

in a left neighbourhood of T_y, and thus $\tau_1 \in (\tau, T_y)$ must exist such that

$$y'''(\tau_1) = 0\,, \quad \text{and} \quad (26) \text{ is valid on } [\tau_1, T_y) .$$

In both of the following cases

— y''' has the maximal zero on $[\tau, T_y)$,
— there exists an increasing sequence of zeros of y''' tending to T_y,

it is possible to choose τ_1 such that

$$\max_{\tau_1 \leq t \leq T_y} y''(t) = y''(\tau_1) . \tag{27}$$

From this

$$q|y'| = py'' + rf(y, y', y'') \geq py'' + r\alpha y \,\big|_{t=\tau_1} . \tag{28}$$

On the other hand, using $y(T_y) = y'(T_y) = 0$, (26) and (27), we have

$$y'^2(\tau_1) = 2 \int_{\tau_1}^{T_y} |y'(s)| y''(s) \,\mathrm{d}s \leq 2y''(\tau_1) y(\tau_1) .$$

Thus, together with (28),

$$q(2y''y)^{\frac{1}{2}} \geq py'' + r\alpha y \,\big|_{t=\tau_1} ,$$

$$\left(\sqrt{py''} - \sqrt{\alpha r y}\,\right)^2 + \left(\sqrt{2\alpha p r} - q\right)\sqrt{2yy''} \leq 0 \,\big|_{t=\tau_1}$$

that contradicts the assumption $q < \alpha r p$ on $\mathbb{R}_+$.

b), c) The statement can be proved similarly as that of Theorem 4 for $C = 1$ or $C = 0$, respectively. Instead of (22) a more precise estimation of F' must be used, using (22) and (25):

$$F'(t) = \mathrm{e}^{C\int\limits_0^t p(s)\,\mathrm{d}s} \{y^2[-q' - Cpq + 3pp'(C - C^2) + p^3(C^2 - C^3) + p''(1 - C) + 2\alpha r] + y'^2[3C - 2]p\}.$$

□

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