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EMBEDDING SEMIGROUPS IN NILPOTENT-GENERATED SEMIGROUPS

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A semigroup with zero is called *nilpotent-generated* if it has a generating set A with the property that for all a in A there exists $n \geqslant 1$ such that $a^n = 0$. Recent work [2, 7, 11] has drawn attention to such semigroups. Also, since it is known [4, 9] that every semigroup is embeddable in an idempotent-generated semigroup it is natural to ask whether nilpotent-generated semigroups have the same universal property. In Section 1 it is shown that this is indeed the case: every semigroup S can be embedded in a nilpotent-generated semigroup T. One can moreover be a good deal more precise about the nature of T (Theorem 1.1) and can arrange for T to inherit from S various special properties (Theorem 1.5). For example, if S is regular, then so is T.

Certain arithmetical aspects of the embedding are explored in Section 2. If n is a positive integer and $\boldsymbol{C}$ is a class of semigroups, then by analogy with the definition in [6] one defines k to be a $\boldsymbol{C}$*NG-cover* of n if every semigroup of order n in the class $\boldsymbol{C}$ is embeddable in a nilpotent-generated semigroup of order at most k. Let $\nu_C(n)$ be the least $\boldsymbol{C}$*NG*-cover of n. It is shown in Theorem 2.4 that if $\boldsymbol{S}$ is the class of all semigroups, then

$$4n + 1 \leqslant \nu_S(n) \leqslant 4n + 2,$$

and in Theorem 2.5 that if $\boldsymbol{Z}$ is the class of semigroups with zero, then

$$4n - 3 \leqslant \nu_Z(n) \leqslant 4n - 2.$$

These inequalities are tantalisingly close to an exact specification for the functions ν_S and ν_Z, but at the moment I am unable to be any more exact. It is of course perfectly possible *a priori* that the upper bound is attained for some values of n and the lower bound for others.

For certain classes $\boldsymbol{C}$ we can specify ν_C precisely. For example, if $\boldsymbol{G}$ is the class of groups and $\boldsymbol{Q}$ is the class of semigroups S such that $S^2 = S$, then $\nu_C(n) = 4n + 1$ for every class $\boldsymbol{C}$ such that $\boldsymbol{G} \subseteq \boldsymbol{C} \subseteq \boldsymbol{Q}$.

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1. The embedding method

For unexplained terms in semigroup theory see [5].

The set of all nilpotent elements of a semigroup S will be denoted by $N(S)$, or just by N if the context allows. If $a \in N(S)$ is such that $a^n = 0$, $a^{n-1} \neq 0$, we call n the *index* of a and write $i(a) = n$. If $S = \langle N(S) \rangle$ is nilpotent-generated, define $i(S) = \max\{i(a): a \in N(S)\}$ if this is finite; otherwise define $i(S) = \infty$. Note that if S has finite index $i(S)$, then $a^{i(S)} = 0$ for all a in $N(S)$, but this does *not* imply that $(N(S))^{i(S)} = 0$.

Let $S = \langle N \rangle$ be nilpotent-generated; then either the ascent

$$N \subset N \cup N^2 \subset N \cup N^2 \cup N^3 \subset \ldots$$

is infinite or there is a unique k for which

$$S = N \cup N^2 \cup \ldots \cup N^k \neq N \cup N^2 \cup \ldots \cup N^{k-1}.$$

In the first case we say that S has *infinite depth* and write $d(S) = \infty$; in the second case we say that S has *depth k* and write $d(S) = k$.

Define the *nilpotent rank* $\mathrm{nr}(S)$ by

$$\mathrm{nr}(S) = \min\{|A|: A \subseteq N \text{ and } \langle A \rangle = S\}.$$

This may well be greater than the *rank* $r(S) = \min\{|A|: \langle A \rangle = S\}$. (See the example in [3, Section 2].)

Theorem 1.1. *Let S be a semigroup. Then S can be embedded in a nilpotent-generated semigroup T. Moreover, T can be chosen so that* $\mathrm{nr}(T) \leqslant 3$ *and* $i(T) = d(T) = 2$.

Proof. Let T be the Rees matrix semigroup $\mathbf{M}^0[S^1; 2, 2; I]$, where I is the 2×2 identity matrix. That is to say,

$$T = (\{1, 2\} \times S^1 \times \{1, 2\}) \cup \{0\},$$

where

$$(i, a, j)(k, b, l) = \begin{cases} (i, ab, l) & \text{if } j = k \\ 0 & \text{if } j \neq k, \end{cases}$$

and

$$(i, a, j)0 = 0(i, a, j) = 00 = 0.$$

Then

$$N(T) = \{(i, a, j) \in T: i \neq j\} \cup \{0\}.$$

Since

$$(1, a, 1) = (1, 1, 2)(2, a, 1) \tag{1.2}$$

and

$$(2, a, 2) = (2, a, 1)(1, 1, 2) \tag{1.3}$$

for all a in S^1, it follows that T is nilpotent-generated and that $i(T) = d(T) = 2$. It is now easily verified that $s \mapsto (1, s, 1)$ embeds S in T.

In order to obtain a T such that $\operatorname{nr}(T) = 3$ we must first use the result of Evans [1] (see also Neumann [8], Subbiah [10]) to embed S in a semigroup $U = \langle u_1, u_2 \rangle$ of rank 2. If $U^1 = U$, we take $T = \mathbf{M}^0[U; 2, 2; I]$; if $U^1 \supset U$ we take

$$T = \mathbf{M}^0[U^1; 2, 2; I] \setminus \{(1, 1, 1), (2, 1, 2), (2, 1, 1)\}.$$

Now consider the subset

$$A = \{(2, u_1, 1), (2, u_2, 1), (1, 1, 2)\}$$

of $N(T)$. For $i = 1, 2$,

$$(1, u_i, 1) = (1, 1, 2)(2, u_i, 1) \in \langle A \rangle;$$

hence $(1, u, 1) \in \langle A \rangle$ for all u in U. Similarly $(2, u, 2) \in \langle A \rangle$ for all u in U. It now follows that

$$(1, u, 2) = (1, u, 1)(1, 1, 2) \in \langle A \rangle$$

for all u in U. Finally, consider an element of the form $(2, u, 1)$, where $u \in U$. If $u \in \{u_1, u_2\}$, then $(2, u, 1) \in A \subset \langle A \rangle$. Otherwise $u = wu_i$, where $w \in U$ and $i \in \{1, 2\}$, and then

$$(2, u, 1) = (2, w, 2)(2, u_i, 1) \in \langle A \rangle.$$

Thus $\langle A \rangle = T$ and so $\operatorname{nr}(T) \leqslant 3$.

Remark. The values for $i(T)$ and $d(T)$ are clearly as small as possible. Also, it is not possible to have $i(T) = \operatorname{nr}(T) = 2$. To see this, consider a semigroup T generated by two elements a, b such that $a^2 = b^2 = 0$. Then the elements of T are

$$0, a, b, ab, ba, aba, bab, (ab)^2, (ba)^2, \ldots .$$

The elements in $aTa \cup bTb$ are nilpotent, and either ab has infinite order or ab has index m and period r:

$$(ab)^{m+r} = (ab)^m.$$

In this latter case ba also has finite order, since

$$(ba)^{m+r+1} = b(ab)^{m+r}a = b(ab)^m a = (ba)^{m+1}.$$

In fact, the period of ba must be r, and the index must be m or $m - 1$ or $m + 1$. For our purposes the most important conclusion is that whether the order of ab is finite or infinite the number of non-zero idempotents of T is at most 2. It follows that any S with more than 3 idempotents cannot be embedded in T.

For the next theorem it is convenient to make a small alteration in our embedding technique. If S is a semigroup without zero, define

$$\Gamma(S) = \mathbf{M}^0[S^1; 2, 2; I]$$

as before. If S is a semigroup with zero, define

$$\Gamma(S) = \mathbf{M}[S^1; 2, 2; I]/Z,$$

a Rees quotient by the ideal

$$Z = \{(1, 0, 1), (1, 0, 2), (2, 0, 1), (2, 0, 2)\}. \tag{1.4}$$

In effect

$$\Gamma(S) = \{(i, s, j): i, j \in \{1, 2\}, s \in S^1, s \neq 0\} \cup \{0\}$$

in both cases; the difference is that in the second case it can happen that $(i, s, j)(j, t, k) = 0$. Then we have

Theorem 1.5. *Let S be a semigroup. Then S is embedded in the nilpotent-generated semigroup $\Gamma(S)$. Also,*

(i) *S is regular [orthodox, inverse] if and only if $\Gamma(S)$ is regular [orthodox, inverse];*

(ii) *if $S = S^1$ is without zero, then S is (completely) simple if and only if $\Gamma(S)$ is (completely) 0-simple;*

(iii) *if $S = S^1$ has a zero, then S is (completely) 0-simple if and only if $\Gamma(S)$ is (completely) 0-simple;*

(iv) *if $S = S^1$ is without zero, then S is bisimple if and only if $\Gamma(S)$ is 0-bisimple;*

(v) *if $S = S^1$ has a zero, then S is 0-bisimple if and only if $\Gamma(S)$ is 0-bisimple.*

Proof. This is all fairly routine. If we use the characterization of a completely (0-)simple semigroup as a (0-)simple semigroup containing a primitive idempotent [5, Theorem III.3.1 and Corollary III.3.4], then the key to the proof is the following lemma, whose proof is omitted. We use superscripts S^1 and T to distinguish between Green's relations in S^1 and in $T = \Gamma(S)$.

Lemma 1.6. *Let $i, j, k, l \in \{1, 2\}$, $a, b \in S^1$, $a, b \neq 0$. Then*

(i) *$(i, a, j)\,\mathbf{R}^T(k, b, l)$ if and only if $i = k$ and $a\,\mathbf{R}^{S^1}b$;*

(ii) *$(i, a, j)\,\mathbf{L}^T(k, b, l)$ if and only if $j = l$ and $a\,\mathbf{L}^{S^1}b$;*

(iii) *$(i, a, j)\,\mathbf{D}^T(k, b, l)$ if and only if $a\,\mathbf{D}^{S^1}b$;*

(iv) *$(i, a, j)\,\mathbf{J}^T(k, b, l)$ if and only if $a\,\mathbf{J}^{S^1}b$;*

(v) *(i, a, j) is a non-zero (primitive) idempotent in T if and only if $i = j$ and a is a non-zero (primitive) idempotent in S^1.*

2. Arithmetical aspects

Let us now turn to the definition of $v_C(n)$ in the introduction. For each finite semigroup S we now define a nilpotent-generated semigroup $\psi(S)$ containing S. First, if $S^2 \neq S$ and S has no zero, let

$$\psi(S) = \mathbf{M}^0[S^1; 2, 2; \Pi]\backslash\{(1, 1, 1), (2, 1, 2), (2, 1, 1)\}.$$

If $S^2 \neq S$ and S has a zero, let

$$\psi(S) = (\mathbf{M}[S^1; 2, 2; \Pi]/Z)\backslash\{(1, 1, 1), (2, 1, 2), (2, 1, 1)\},$$

where Z is as defined in (1.4). If $S^2 = S$ and S has no zero, let

$$\psi(S) = \mathbf{M}^0[S; 2, 2; \Pi].$$

If $S^2 = S$ and S has a zero, let

$$\psi(S) = \mathbf{M}[S; 2, 2; \Pi]/Z.$$

The important point to note here is that if $S^2 = S$, then the adjunction of an identity to S is unnecessary. Since every a in S has a factorization $a = bc$ with b, c in S, the crucial equations (1.2) and (1.3) can be replaced by

$$(1, a, 1) = (1, b, 2)(2, c, 1), \quad (2, a, 2) = (2, b, 1)(1, c, 2).$$

Notice now that if $|S| = n$, then $|\psi(S)| \leqslant 4n + 2$. If S has a zero, then $|\psi(S)| \leqslant 4n - 2$. If $S^2 = S$, then $|\psi(S)| \leqslant 4n + 1$. If $S^2 = S$ and S has a zero, then $|\psi(S)| = 4n - 3$. Thus we have

Theorem 2.1. *Let $\mathbf{S}$, $\mathbf{Z}$, $\mathbf{Q}$ denote respectively the class of all semigroups, the class of semigroups with zero, and the class of all semigroups S such that $S^2 = S$. Then, with the definitions as in the introduction,*

$$v_S(n) \leqslant 4n + 2, \quad v_Z(n) \leqslant 4n - 2, \quad v_Q(n) \leqslant 4n + 1,$$
$$v_{Z \cap Q}(n) \leqslant 4n - 3.$$

Now let G be a finite group and suppose that G is embedded in a finite nilpotent-generated semigroup T. Then G is contained within a single $\mathbf{H}$-class of T and hence certainly within a single $\mathbf{J}$-class J of T. We show now that J must contain at least two $\mathbf{L}$-classes. For suppose by way of contradiction that J contains a single $\mathbf{L}$-class. Then the identity e of G is a right identity for J [5. Proposition II.3.3]. The assumption that T is nilpotent-generated means that $e = a_1 a_2 \ldots a_k$, a product of nilpotents in T. Since

$$R_{ea_1} \leqslant R_e \quad \text{and} \quad R_{ea_1} \geqslant R_{ea_1 \ldots a_k} = R_{e^2} = R_e$$

it follows that $ea_1 \mathbf{R} e$. Hence $ea_1 \in H_e$, since J contains only one $\mathbf{L}$-class. Now

$a_1^m = 0$ for some m. To show that $(ea_1)^m = 0$ assume inductively that $(ea_1)^{m-1} = ea_1^{m-1}$ and then deduce that

$$\begin{aligned}(ea_1)^m &= (ea_1)(ea_1)^{m-1} = (ea_1)e \cdot a_1^{m-1} \\ &= (ea_1)a_1^{m-1} \quad (\text{since } ea_1 \in H_e) \\ &= ea_1^m.\end{aligned}$$

Thus we have $ea_1 \in H_e$ and $(ea_1)^m = 0$, a contradiction.

We deduce that J contains at least two **L**-classes, and a dual argument shows that J contains at least two **R**-classes. Hence J contains at least four **H**-classes, each containing at least n $(= |G|)$ elements. Since T also contains a zero, the order of T must at the very least be $4n + 1$.

Let us say that a class $\boldsymbol{C}$ of semigroups is *group-saturated* if (for every $n \geqslant 2$) $\boldsymbol{C}$ contains at least one group of order n. Then we have

Theorem 2.2. *If $\boldsymbol{C}$ is a group-saturated class of semigroups, then $v_C(n) \geqslant 4n + 1$.*

From Theorems 2.1 and 2.2 we now obtain

Theorem 2.3. *Let $\boldsymbol{C}$ be a group-saturated class of semigroups such that $\boldsymbol{C} \subseteq \boldsymbol{Q}$. Then $v_C(n) = 4n + 1$.*

Among classes $\boldsymbol{C}$ satisfying the conditions for this theorem are the class of all groups, the class of all monoids, and the class of all regular semigroups and the class of all inverse semigroups.

For a group-saturated class $\boldsymbol{C}$ not contained in $\boldsymbol{Q}$ (such as the class $\boldsymbol{S}$ of all semigroups) Theorems 2.1 and 2.2 give a less satisfactory outcome:

Theorem 2.4. *Let $\boldsymbol{C}$ be a group-saturated class of semigroups. Then $4n + 1 \leqslant v_C(n) \leqslant 4n + 2$.*

For semigroups with zero we can obtain closely analogous results. First, let $\boldsymbol{G}^0$ denote class of 0-groups (groups with zero adjoined), and say that a class $\boldsymbol{C}$ of semigroups with zero is *0-group-saturated* if it contains 0-groups of every finite order n. Then a modified version of the proof of Theorem 2.2 leads to the conclusion that

$$v_C(n) \geqslant 4n - 3,$$

for any such class $\boldsymbol{C}$. Hence we obtain

Theorem 2.5. *Let $\boldsymbol{C}$ be a 0-group-saturated class of semigroups with zero. Then*

$$4n - 3 \leqslant v_C(n) \leqslant 4n - 2.$$

If $\boldsymbol{C}$ is also contained in $\boldsymbol{Q}$ then $v_C(n) = 4n - 3$.

There are two obvious approaches to the problem of resolving the ambiguity exhibited in Theorem 2.4. One might try to find a new embedding method that would give the conclusion $v_S(n) \leqslant 4n + 1$. Or (if $4n + 2$ is in fact the correct answer) one might look for a class $\boldsymbol{C}$ of semigroups (with $\boldsymbol{C} \nsubseteq \boldsymbol{Q}$ obviously) for which $v_C(n) \geqslant 4n + 2$. One obvious such class is the class M of monogenic (one-generator) semigroups. Let $S = \langle a\colon a^{m+r} = a^m \rangle$ be such a semigroup, where a has index m and period r. Then $|S| = m + r - 1 = n$ (say). If S is a cyclic group, then from Theorem 2.4 we know that it can be embedded in a nilpotent-generated semigroup T of order $4n + 1$. So suppose that S is not a group, which happens precisely when $m \geqslant 2$. Let T be the semigroup with zero defined by the presentation

$$T = \langle b, c \mid b^2 = c^2 = 0, (cb)^{m+r-1} = (cb)^{m-1} \rangle.$$

The relation $(cb)^{m+r-1} = (cb)^{m-1}$ implies that

$$(bc)^{m+r} = b(cb)^{m+r-1}c = b(cb)^{m-1}c = (bc)^m;$$

so the elements of T are

$$0, b, c, bc, cb, bcb, cbc, (bc)^2, (cb)^2, \ldots$$
$$\ldots, (bc)^{m+r-2}, (cb)^{m+r-2}, c(bc)^{m+r-2}, b(cb)^{m+r-2}, (bc)^{m+r-1}.$$

Thus

$$|T| = 1 + 2(2m + 2r - 3) + 1 = 4(m + r - 1) = 4n.$$

Also, $a \mapsto bc$ embeds S in T. The conclusion is

Theorem 2.6. *If M is the class of monogenic semigroups, then $v_M(n) = 4n + 1$.*

Though of some interest in its own right, this result is in a sense disappointing, since it contributes nothing to the main question raised by Theorem 2.4. One might of course regard it as 'evidence' in support of a conjecture that $v_S(n) = 4n + 1$, but it is evidence of a very flimsy kind.

REFERENCES

[1] EVANS, T.: Embedding theorems for multiplicative systems and projective geometries. Proc. American Math. Soc. 3, 1952, 614—620.

[2] GOMES, G. M. S.—HOWIE, J. M.: Nilpotents in finite symmetric inverse semigroups. Proc. Edinburgh Math. Soc., (2) 30, 1987, 383—395.

[3] ——————.: On the ranks of certain finite semigroups of transformations Math. Proc. Cambridge Philos. Soc., 101, 1987, 395—403.

[4] HOWIE, J. M.: The subsemigroup generated by the idempotents of a finite full transformation semigroup. J. London Math. Soc. 41, 1966, 707—716.

[5] ——————.: An Introduction to Semigroup Theory. Academic Press, London, 1976.

[6] ——————.: Embedding semigroups in semibands: some arithmetical results. Quart. J. Math. Oxford (2), 32, 1981, 323—337.

[7] HOWIE, J. M.— MARQUES-SMITH, M. P. O.: Inverse semigroups generated by nilpotent transformations. Proc. Royal Soc. Edinburgh A 99, 1984, 152—162.

[8] NEUMANN, B. H.: Embedding theorems for semigroups. J. London Math. Soc. 35, 1960, 183—192.

[9] PASTIJN, F.: Embedding semigroups in semibands. Semigroup Forum 14, 1977, 247—263.
[10] SUBBIAH, S.: Another proof of a theorem of Evans. Semigroup Forum 6, 1973, 93—94.
[11] SULLIVAN, R. P.: Semigroups generated by nilpotent transformations. J. Algebra (to appear).

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ПОГРУЖЕНИЕ ПОЛУГРУПП И НИЛЬПОТЕНТНО ПОРОЖДЕННЫЕ ПОЛУГРУППЫ

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Резюме

Каждую (конечную) полугруппу S можно погрузить в (конечную) нильпотентно порожденную полугруппу T и метод погружения сохраняет некоторые свойства S: например, если S регулярна, то T тоже регулярна.

Если n — положительное целое число и $\boldsymbol{C}$ — класс полугрупп, то определим $\nu_C(n)$ как наименьшее целое число k, для которого верно, что каждая полугруппа порядка n из класса C погружима в нильпотентно порожденную полугруппу не высшего порядка чем k.

Одним из главных результатов является то, что если $\boldsymbol{G}$ — класс всех таких полугрупп S, что $S^2 = S$, то $\nu_C(n) = 4n + 1$ для каждого такого класса $\boldsymbol{C}$, для которого $\boldsymbol{G} \subseteq \boldsymbol{C} \subseteq \boldsymbol{Q}$.