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SELF-DUAL NON-HAMILTONIAN POLYHEDRA WITH ONLY TWO TYPES OF FACES

PETER JOHN OWENS

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ABSTRACT. For all $q \geq 8$ it is shown that the family of all self-dual polyhedral graphs whose faces are all triangles or q-gons contains non-hamiltonian graphs and even has shortness exponent less than one.

1. Introduction

Graphs are assumed to be simple (without loops or multiple edges). For concepts and notation not explained here, see Bondy and Murty [1].

For a graph G, $v(G) = |V(G)|$ and $h(G)$ denotes the circumference (length of a maximum cycle). For a family of graphs Γ, the *shortness exponent* $\sigma(\Gamma)$ is defined, as in [2], by

$$\sigma(\Gamma) = \liminf_{G\in\Gamma} \frac{\log h(G)}{\log v(G)}\,.$$

Also, $\sigma(\langle G_n\rangle)$ denotes the shortness exponent of the family of all graphs occurring in the sequence $\langle G_n\rangle$.

For any polyhedral (3-connected planar) graph G, G^* denotes the dual graph, and v^*, e^*, f^* denote the face, edge, vertex of G^* corresponding to the vertex v, edge e, face f of G (respectively).

Let $\Gamma(r;p,q)$ be the family of all r-regular polyhedral graphs whose faces are all p-gons or q-gons, $p<q$. Let $\Gamma(p,q;r)$ be the dual family of all r-gonal polyhedral graphs whose vertices are all p-valent or q-valent, $p<q$. A number of papers have given results of the form $\sigma \leq \sigma_0 < 1$ for such families (or some of their subfamilies) for various values of p, q and r. See [3], [4], [5], [6], [7], [8], [10], [11].

Suppose that G is a self-dual polyhedral graph whose faces are all p-gons or q-gons, $p<q$ and whose vertices are, therefore, all p-valent or q-valent. It is

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easy to show with the aid of Euler's formula that $p = 3$. Let $D(q)$ denote the family of all self-dual polyhedral graphs whose faces are all triangles or q-gons, $q > 3$. It is a subfamily of $\Gamma(3, q; 3, q)$, the family of all polyhedral graphs whose faces are all triangles or q-gons and whose vertices are all 3-valent or q-valent, $q > 3$.

In the main part of the paper, we construct, for all $q \geq 8$, an infinite sequence $\langle M_n \rangle$ of graphs in $\Gamma(3, q; 3, q)$ such that $\sigma(\langle M_n \rangle) < 1$ and also $\sigma(\langle M_n^* \rangle) < 1$. In the terminology of Walther [11], this means that $\Gamma(3, q; 3, q)$ is *minishort*. Thus, we have extended to all $q \geq 8$ a result proved in [11] for $q = 8, 10$.

We then prove a lemma that enables us to obtain a sequence of self-dual non-hamiltonian graphs in $D(q)$. This leads to our main result, namely:

THEOREM. *$\sigma(D(q)) < 1$ for all $q \geq 8$.*

In conclusion, two open problems are presented.

2. Main construction

We begin by obtaining, for all $q \geq 8$, a graph L in $\Gamma(3, q; 3, q)$ and a set $X \subset V(L)$ such that $s > r$, where $r = |X|$ and $s = \omega(L - X)$. The graph L is non-hamiltonian, and, in fact, no cycle in L contains vertices of more than r of the s components of $L - X$.

There are three special cases $q = 8, 9, 10$. In each of these cases, L is actually in $\Gamma(3, q; 3)$, X is defined as the set of all q-valent vertices and all the components of $L - X$ are isolated vertices. For $q = 8$, L is the well-known graph (see, for instance, [7], [11]) obtained from the graph of the octahedron by inserting a new vertex in each face and joining it by new edges to the three vertices of that face. Here $r = 6$ and $s = 8$. For $q = 10$, L is obtained similarly but starting with the icosahedron, so $r = 12$ and $s = 20$. For $q = 9$, L is again obtained from the icosahedron, but vertices are inserted in only 16 faces. The 4 faces to be left empty are chosen so that no two have a vertex in common. Here, $r = 12$ and $s = 16$.

For $q \geq 11$ there must be some q-gons in L because then $\Gamma(3, q; 3)$ is null. Our construction of L depends on the value of q modulo 6. In each of the six cases a fragment J is defined and then k copies of J, where k is 3 or 4, are joined together in a standard way to form L. A special notation is used in the diagrams that specify J (see Figures 3.1 to 3.6). Numbers inside a 4-gon and near to its corners indicate the presence of vertices and edges, not shown explicitly, that contribute these numbers to the valencies of the four vertices. Figure 1 shows what is actually inside the 4-gon when the numbers are $2, 4, 2, 5$ as shown. The construction is similar whenever the numbers take the form $2, a, 2, b$. In order that the new non-triangular face inside the 4-gon shall be a q-gon, the condition $a + b + 2 = q$ must be satisfied. The significance of numbers $1, 3, 1, 3$ inside a 4-gon is shown in Figure 2. All the vertices inside 4-gons are 3-valent and all the vertices of the 4-gons themselves become q-valent in L.

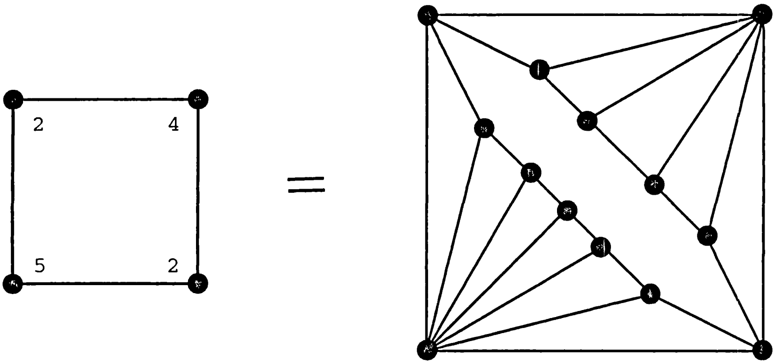

FIGURE 1. Part of J when $q = 11$. Numbers $2, 4, 2, 5$ in a 4-gon.

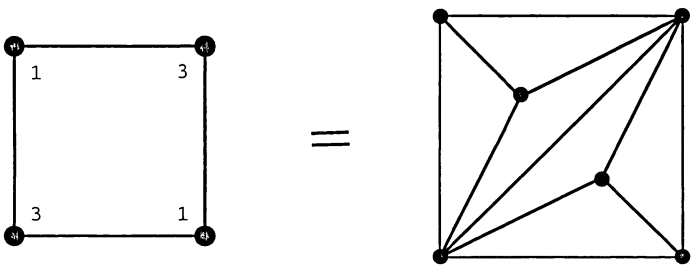

FIGURE 2. Numbers $1, 3, 1, 3$ in a 4-gon.

In all six cases, we now construct L, specify X and give the values of r and s. Vertices of J that become vertices of the set X in L are denoted in the diagrams by large dots. Any component of $L - X$ that is neither an isolated vertex nor a (non-trivial) path will be called *large*.

Case 1: $q = 6t + 5$, $t \geq 1$.
In Figure 3.1, the part of J lying to the right of the line $C_{t+1}D_{t+1}$ is the mirror image of the part to the left of C_tD_t, except that it does not contain any vertices of X. The numbers in the ith 4-gon from the left are 2, $6i - 2$, 2, $6t - 6i + 5$, reading clockwise from the top left, $1 \leq i \leq t$. To obtain L, take k copies of J and identify the edges $AD_0, D_0D_1, \ldots, D_{2t}D_{2t+1}, D_{2t+1}B$ along one

"side" of the jth copy of J with the edges $AC_0, C_0C_1, \ldots, C_{2t}C_{2t+1}, C_{2t+1}B$, respectively, along the other side of the $(j+1)$th copy (or the first copy if $j = k$), $1 \leq j \leq k$. It is easy to check that all the vertices of the 4-gons become $(6t+5)$-valent in L and that all non-triangular faces are $(6t+5)$-gons. Take $k = 3$ in order to make the vertices A, B at the "ends" of L 3-valent. Although 4 vertices of J are shown as large dots and there are 3 copies of J, there are only 6 vertices in X because pairs of vertices are identified when copies of J are joined. Hence $r = 6$. One component of $L - X$ is the isolated vertex A, six components are paths inside the three 4-gons $C_0C_1D_1D_0$ and there is one large component that contains all vertices to the right of C_1D_1 in all three copies of J. Hence $s = 8$.

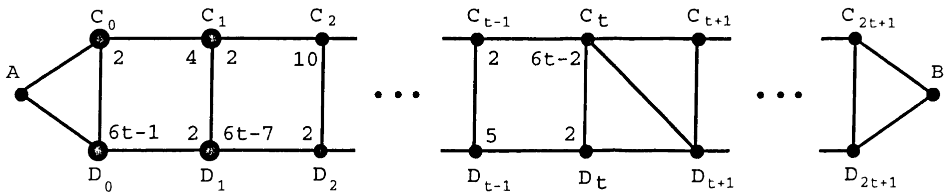

FIGURE 3.1. J when $q = 6t + 5$.

Case 2: $q = 6t + 6$, $t \geq 1$.

Exceptionally, $k = 4$ in this case and the construction of L involves two different forms of J, used alternately. In Figure 3.2, two adjacent copies of J are shown.

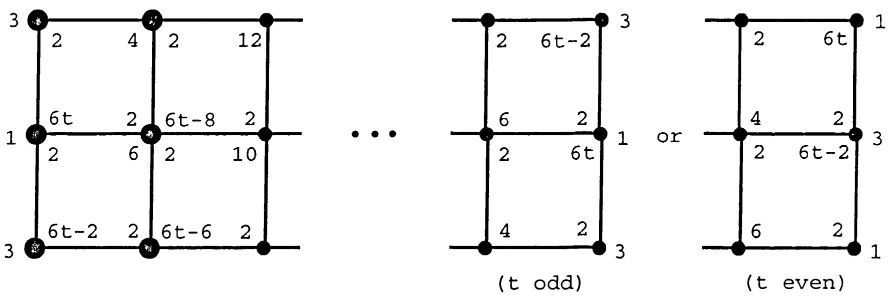

FIGURE 3.2. J when $q = 6t + 6$.

Note the dependence on whether t is odd or even. When the other two copies of J are joined to those shown (as before, by identifying sides), a graph with a 4-gon at each end is produced. The two 4-gons are then filled as in Figure 2, using the orientation indicated by the numbers 1, 3. In this case, $r = 8$. When

$t > 1$, two components of $L - X$ are isolated vertices, eight components are paths and there is one large component. Hence $s = 11$. In the special case $t = 1$, two extra isolated vertex components replace the large component, so $s = 12$.

Case 3: $q = 6t + 7$, $t \geq 1$.
In Figure 3.3, the part of J to the right of the 4-gon containing $1, 3, 1, 3$ is the mirror image of the part to the left. Take $k = 3$ so that L will have triangular end faces. Here, $r = 6$ and $s = 8$. Six components of $L - X$ are isolated vertices and the other two components are large.

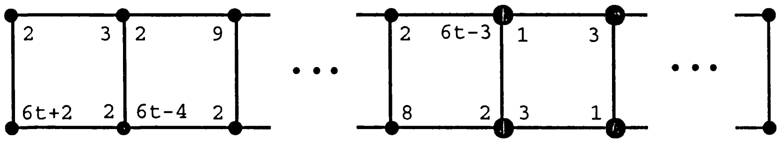

FIGURE 3.3. J when $q = 6t + 7$.

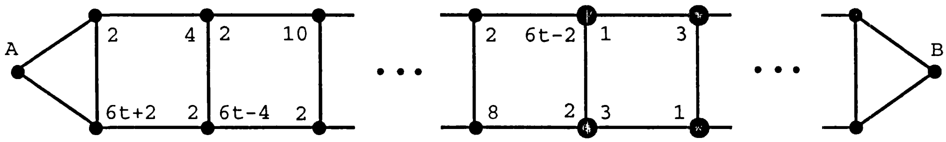

FIGURE 3.4. J when $q = 6t + 8$.

Case 4: $q = 6t + 8$, $t \geq 1$.
This case closely resembles the previous one (compare Figures 3.3, 3.4). Take $k = 3$ so that the end vertices of L will be 3-valent. Here again $r = 6$ and $s = 8$. The components of $L - X$ are of the same types as in Case 3.

Case 5: $q = 6t + 9$, $t \geq 1$.
Take $k = 3$ so that L will have a 3-valent vertex at one end and a triangular face at the other end (see Figure 3.5). Once again $r = 6$ and $s = 8$. One component of $L - X$ is an isolated vertex, six components are paths and there is one large component.

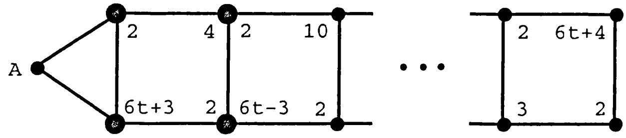

FIGURE 3.5. J when $q = 6t + 9$.

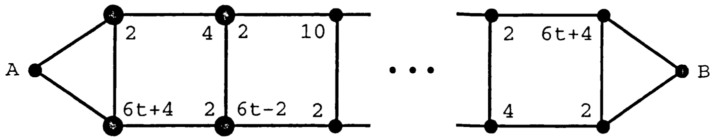

FIGURE 3.6. J when $q = 6t + 10$.

Case 6: $q = 6t + 10$, $t \geq 1$.
This case closely resembles the previous one (compare Figures 3.5, 3.6). Once again $k = 3, r = 6$ and $s = 8$. The components of $L - X$ are of the same types as in Case 5.

This completes the construction of L for all $q \geq 8$. Note that in all cases $L \in \Gamma(3, q; 3, q)$, L is non-hamiltonian because $s > r$ and $L - X$ has at least one isolated vertex component.

For every $q \geq 8$ we now choose s 3-valent vertices of L, one belonging to each component of $L - X$, and call them the *y-vertices* of L. A y-vertex is of *type* 1 if it is incident (in L) with three triangles and adjacent to three q-valent vertices. A y-vertex is of *type* 2 if it is incident with two triangles and a q-gon and adjacent to two 3-valent vertices and a q-valent vertex, where the two triangles are incident with the q-valent vertex. Every isolated vertex component of $L - X$ provides a y-vertex of type 1. By inspection, for all $q \geq 8$ each path component or large component of $L - X$ provides at least one possible y-vertex of type 2 for L. In the case of a path component any vertex other than an end vertex can be the y-vertex.

The triangular face of the dual graph L^* that corresponds to a y-vertex of L is called a *y^*-face*. It is classified as type 1 or type 2 according to the type of the y-vertex. Larger graphs built out of copies of L and L^* contain y-vertices and y^*-faces inherited from them.

Let $H, K \in \Gamma(3, q; 3, q)$, where H has a y-vertex y_1, and K has a y^*-face y_2^*. Form a new planar graph G from H and K by identifying the three vertices of $H - y_1$ that were adjacent (in H) to y_1 with the three vertices of $K - E(y_2^*)$ that were incident (in K) with y_2^* in such a way that no vertex of valency greater than q is produced. In every case, G is not unique. When y_1 and y_2^* are both of type 2, there are two ways to identify the vertices, otherwise there are three ways. We say that G has been formed by *attaching* H to K (or K to H) *using* y_1 and y_2^*. The process is easiest to visualize if H and K are mapped into the plane in such a way that y_1 is incident with the unbounded face of H, and y_2^* is the unbounded face of K.

Lemma 1. *We have:*

(1) $G \in \Gamma(3, q; 3, q)$,
(2) $v(G) = v(H) + v(K) - 4$,
(3) $h(G) \leq h(H) + h(K) - 4$.

Proof.

(1) is immediate.

For (2), note that in forming G, the vertex y_1 has been deleted and three pairs of vertices have been identified.

To prove (3), consider a maximum cycle C in G. If C has non-trivial intersections with both $H - y_1$ and $K - E(y_2^*)$, then these intersections are paths, say P_1 and P_2 respectively. In H, we can form a cycle from P_1, y_1 and two edges incident at y_1, so $|E(P_1)| \leq h(H) - 2$. In K, we can form a cycle from P_2 by adding two edges and one vertex of the face y_2^*, so $|E(P_2)| \leq h(K) - 2$. Hence

$$h(G) = |E(P_1)| + |E(P_2)| \leq h(H) + h(K) - 4.$$

If C lies entirely in H (or in K), then the inequality still holds since $h(K) \geq \delta(K) + 1 = 4$ (or $h(H) \geq 4$, respectively). □

Let y_i, $1 \leq i \leq s$, be the y-vertices of L. For all $q \geq 8$, L has some y-vertices of type 1, so we may assume that y_1 is of type 1. The dual graph L^* has y^*-faces y_i^*, $1 \leq i \leq s$, where y_1^* is of type 1. Obtain M from L by attaching $s-1$ copies of L^* to it, using y_i, $2 \leq i \leq s$, and the face of y_1^* of each copy of L^*. Because the process of attaching graphs to one another is not uniquely defined, neither is M. This does not matter. Whichever of the possible forms of M we use, it has just one y-vertex y_1 and $(s-1)^2$ y^*-faces, $s-1$ in each copy of L^*.

A cycle in a graph *enters* a subgraph if its intersection with the subgraph contains at least one edge. Every cycle in L contains at most r y-vertices because the y-vertices on the cycle are separated by vertices from X. Hence every cycle in M that contains y_1 enters at most $r-1$ of the $s-1$ copies of L^* and therefore enters at most $(r-1)(s-1)$ of the $(s-1)^2$ y^*-faces. Every cycle in M that does not contain y_1 enters at most $r(s-1)$ y^*-faces.

We now used a standard sort of iterative construction. Let $M_1 = M$ and, for $n \geq 1$, let M_{n+1} be a graph obtained from M_n by attaching as many copies of

M as there are y^*-faces in M_n, using these y^*-faces and the vertex y_1 of every copy of M. The following lemma holds:

LEMMA 2.

(1) *M_n has $(s-1)^{2n}$ y-faces and one y-vertex.*
(2) *No cycle in M_n enters more than $r(r-1)^{n-1}(s-1)^n$ y^*-faces.*

This lemma is easy to prove by induction, so the proof is omitted. By the lemma,

$$v(M_{n+1}) - v(M_n) = (s-1)^{2n}\big(v(M)-4\big)\,,$$
$$h(M_{n+1}) - h(M_n) \le r(r-1)^{n-1}(s-1)^n\big(h(M)-4\big)\,.$$

These recurrence relations have solutions of the form

$$v(M_n) = a(s-1)^{2n} + b\,,$$
$$h(M_n) \le c(r-1)^n(s-1)^n + d\,,$$

where a, b, c and d are certain constants such that $a>0$ and $c>0$. Hence

$$\sigma(\langle M_n\rangle) \le \frac{\log((r-1)(s-1))}{\log(s-1)^2} = \frac{1}{2}\left(1+\frac{\log(r-1)}{\log(s-1)}\right) < 1\,.$$

Consider the dual sequence $\langle M_n^*\rangle$. Its first term M^* is obtained from L^* by attaching $s-1$ copies of L to it, using y_i^*, $2\le i\le s$, and the vertex y_1 of each copy of L. Thus M^* has one y^*-face y_1^* and $(s-1)^2$ y-vertices.

Let C be a cycle in M^*. If its intersection with one of the copies of $L-y_1$ is not null, then it is a path P that can be converted into a cycle C' in L by adding the vertex y_1 and two edges incident at y_1. As C' contains at most r of the y-vertices of L, P contains at most $r-1$ of them. Hence C contains at most $(r-1)(s-1)$ y-vertices.

In place of Lemma 2, we have:

LEMMA 2*.

(1) *M_n^* has $(s-1)^{2n}$ y-vertices and one y^*-face.*
(2) *No cycle in M_n^* contains more than $(r-1)^n(s-1)^n$ y-vertices.*

Reasoning similar to that for M_n leads to

$$v(M_n^*) = a^*(s-1)^{2n} + b^*\,,$$
$$h(M_n^*) \le c^*(r-1)^n(s-1)^n + d^*\,,$$

where a^*, b^*, c^* and d^* are constants such that $a^*>0$ and $c^*>0$.

Hence

$$\sigma(\langle M_n^*\rangle) \le \frac{1}{2}\left(1+\frac{\log(r-1)}{\log(s-1)}\right) < 1\,.$$

3. Proof of the theorem

The sequences $\langle M_n \rangle$ and $\langle M_n^* \rangle$ will be combined to give a sequence of self-dual non-hamiltonian graphs by using the following lemma.

LEMMA 3. *Suppose that $H \in \Gamma(3,q;3,q)$ and that H has a y-vertex v of type 1. Let G be obtained by attaching H to H^*, using v and v^*. Then $G \in D(q)$.*

Proof. By Lemma 1, $G \in \Gamma(3,q;3,q)$. To show that G is self-dual, we define an involutory bijection θ from the vertices, edges and faces of G to the faces, edges and vertices of G, respectively, such that θ preserves incidences and adjacencies.

Let ϕ be the bijection from the vertices, edges and faces of H to the faces, edges and vertices of H^*, respectively, that is used to define H^*. For every element (vertex, edge or face) e of H that remains unaltered in G, define $e\theta = e\phi$. For every element e of H^* that remains unaltered in G, define $e\theta = e\phi^{-1}$.

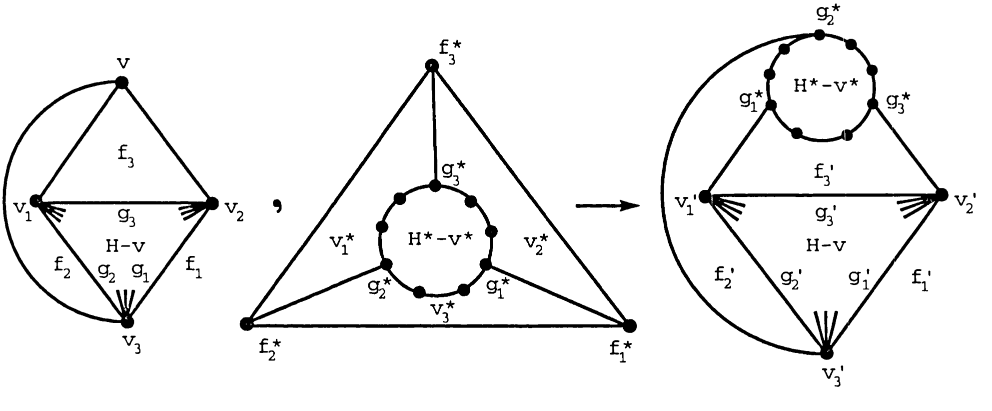

FIGURE 4. Attaching H^* to H.

As in Figure 4, let v_1, v_2, v_3 be the vertices of H adjacent to v and let f_1, f_2, f_3 be the triangular faces, with boundaries vv_2v_3, vv_3v_1, vv_1v_2, respectively, that are incident with v. In H^*, the 3-valent vertices on the boundary of v^* are f_1^*, f_2^*, f_3^* and the faces adjacent to v^* are v_1^*, v_2^*, v_3^*. These three faces are q-gons because v_1, v_2, v_3 are q-valent in H. When H is attached to H^*, the vertex v and the edges vv_1, vv_2, vv_3, $f_2^*f_3^*$, $f_3^*f_1^*$, $f_1^*f_2^*$ disappear. The triples of vertices v_1, v_2, v_3 in $H-v$, and f_1^*, f_2^*, f_3^* in $H^*-E(v^*)$ become identified in G, but in opposite cyclic orders. We may suppose that v_1 is identified with f_1^*, v_2 with f_3^*, v_3 with f_2^* and we denote the resulting q-valent vertices of

G by v_1', v_2', v_3', respectively. The faces f_1, f_2, f_3 of H become merged with the faces v_1^*, v_3^*, v_2^* of H^* to yield q-gons f_1', f_2', f_3' in G, respectively. To complete the definition of θ, let

$$\begin{aligned} v_1'\theta &= f_1', & v_2'\theta &= f_3', & v_3'\theta &= f_2', \\ f_1'\theta &= v_1', & f_3'\theta &= v_2', & f_2'\theta &= v_3'. \end{aligned}$$

Clearly, θ^2 is the identity mapping. We claim that incidences and adjacencies are preserved under θ. Where elements of H only (or H^* only) are involved and they remain unaltered in G, this is clear. In other cases, it is easy to verify and we just give an example. In Figure 4, the faces other than f_1, f_2, f_3 that are incident (in H) with the edges v_2v_3, v_3v_1, v_1v_2 are denoted by g_1, g_2, g_3, respectively. In G, v_2' is incident with f_3', f_1' and adjacent to g_3^*. Hence, f_3' $(= v_2'\theta)$ should be incident with v_2', v_1' and adjacent to g_3. This is indeed the case. □

Choose $q \geq 8$ and any n. Apply Lemma 3, with $H = M_n$, $v = y_1$, and let $G = G_n$. Then $G_n \in D(q)$ and, by Lemma 1,

$$\begin{aligned} v(G_n) &= v(M_n) + v(M_n^*) - 4 = a'(s-1)^{2n} + b', \\ h(G_n) &\leq h(M_n) + h(M_n^*) - 4 \leq c'(r-1)^n(s-1)^n + d', \end{aligned}$$

where $a' = a + a^* > 0$, $b' = b + b^* - 4$, $c' = c + c^* > 0$, and $d' = d + d^* - 4$. Since $\sigma(D(q)) \leq \sigma(\langle G_n \rangle)$, it follows that

$$\sigma(D(q)) \leq \frac{1}{2}\left(1 + \frac{\log(r-1)}{\log(s-1)}\right) < 1$$

for all $q \geq 8$. This completes the proof of the theorem.

The theorem gives an upper bound independent of q for $\sigma(D(q))$. Suppose that we exclude the worst case $q = 9$. Then we have

$$\sigma(D(q)) \leq \frac{1}{2}(1 + \log_{10} 7)$$

for all $q \geq 10$ (and for $q = 8$).

4. Final remarks

Tkáč [9] constructed non-hamiltonian graphs in $\Gamma(3; 3, 7)$ and we can use them to obtain non-hamiltonian graphs in $D(7)$. Take H^* to be the graph that is denoted by G_n in [9]. Then

$$v(H^*) = 12 + 416n, \qquad h(H^*) \leq v(H^*) - n.$$

For the dual graph H, Euler's formula gives $v(H) = \frac{1}{2}v(H^*) + 2$, so

$$v(H) = 8 + 208n\,, \qquad h(H) \le v(H)\,.$$

Graphs in $\Gamma(3;3,7)$ cannot have adjacent triangles because they are 3-connected. Hence, we may choose any triangle in H^* to use as v^* in Lemma 3. Attach H^* to its dual H, using v^* and v, and denote the resulting graph in $D(7)$ by L_n. We have

$$\begin{aligned} v(L_n) &= v(H) + v(H^*) - 4 = 16 + 624n\,, \\ h(L_n) &\le h(H) + h(H^*) - 4 \le 16 + 623n\,, \end{aligned}$$

so L_n is non-hamiltonian. In fact, $h(L_n)/v(L_n) \to 623/624$ as $n \to \infty$, which shows that the shortness coefficient (defined in [2], [9]) is less than one for $D(7)$.

PROBLEM 1. Is $\sigma\big(D(7)\big) < 1$?

An edge of a plane graph is of *type* $(a,b;c,d)$ if it is incident with vertices of valencies a, b ($a \le b$) and with faces having c, d edges ($c \le d$). Our graph G_n has 7 types of edges, for all $q \ge 11$, so we ask:

PROBLEM 2. For $q \ge 11$, is there a non-hamiltonian graph in the family $D(q)$ with fewer than 7 types of edges?

Acknowledgement

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REFERENCES

[1] BONDY, J. A.—MURTY, U. S. R.: *Graph Theory with Applications*, Macmillan, London, 1976.

[2] GRÜNBAUM, B.—WALTHER, H.: *Shortness exponents of families of graphs*, J. Combin. Theory Ser. A **14** (1973), 364–385.

[3] HARANT, J.—WALTHER, H.: *Some new results about the shortness exponent in polyhedral graphs*, Časopis Pěst. Mat. **112** (1987), 114–122.

[4] HARANT, J.—OWENS, P. J.—TKÁČ, M.—WALTHER, H.: *5-regular 3-polytopal graphs with edges of only two types and shortness exponents less than one*, Discrete Math. **150** (1996), 143–153.

[5] JENDROL', S.—OWENS, P. J.: *Pentagonal 3-polytopal graphs with edges of only two types and shortness parameters*, Discrete Math. **137** (1995), 251–263.

[6] OWENS, P. J.: *Regular planar graphs with faces of only two types and shortness parameters*, J. Graph Theory **8** (1984), 253–275.

[7] OWENS, P. J.: *Shortness parameters for planar graphs with faces of only one type*, J. Graph Theory **9** (1985), 381–395.

[8] OWENS, P. J.—WALTHER, H.: *Hamiltonicity in multitriangular graphs*, Discuss. Math. - Graph Theory **15** (1995), 77–88.

[9] TKÁČ, M.: *On shortness coefficients of simple 3-polytopal graphs with only one type of faces besides triangles*, Discrete Math. **128** (1994), 407–413.

[10] WALTHER, H.: *Note on two problems of J. Zaks concerning hamiltonian 3-polytopes*, Discrete Math. **33** (1981), 107–109.

[11] WALTHER, H.: *Longest cycles in polyhedral graphs*, Israel J. Math. **83** (1993), 203 212.

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