Juraj Bosák
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DIRECTED GRAPHS AND MATRIX EQUATIONS

JURAJ BOSÁK

1. Introduction

Throughout the paper the symbols a, b, c, d denote non-negative integers such that $a \leq b$, and i, j, p denote positive integers. All considered matrices are square and all graphs are finite; loops and multiple edges are allowed.

A directed graph G is said to be a *W_a^b-graph* (cf. [2]) if for any two vertices u and v of G there is in G exactly one (directed) walk [3] from u to v whose length c fulfils the inequalities $a \leq c \leq b$.

A directed graph G is said to be *regular of degree d* (or, briefly, a *graph of degree d*) if for every vertex v of G there exists in G just d edges directed from v and just d edges directed to v.

In this paper we prove that any W_a^b-graph is regular. Moreover, we prove that a W_a^b-graph of degree d has $d^a + d^{a+1} + \dots + d^b$ vertices (we put $0^0 = 1$) and we deduce a necessary and sufficient condition for the existence of a W_a^b-graph of degree d. Thus, some results of [7] and those announced in [2] are generalized. We use standard matrix methods (see, e.g., [11]).

By the *adjacency matrix* of a directed graph G with vertices $v_1, v_2, \dots, v_p$ we mean the $p \times p$ matrix $A = (a_{ij})$, where a_{ij} is the number of edges of G directed from v_i to v_j. It is well-known that the (i, j) entry of A^c is the number of walks of length c from v_i to v_j ([3], Theorem 16.8; [1], Chapter 14; [11]). Consequently, we have:

Lemma 1. *A directed graph G is a W_a^b-graph if and only if the adjacency matrix A of G satisfies the equation*

$$(1) \qquad A^a + A^{a+1} + \dots + A^b = J,$$

where J is the matrix each entry of which is 1.

(For every matrix A we put $A^0 = I$, the identity matrix, and we suppose the matrices I and J to be of the same order as A is.)

Lemma 1 enables us to express some considerations concerning W_a^b-graphs in matrix terms, and conversely. We start with some simple auxiliary results concerning matrices.

2. Results concerning matrices

Lemma 2. *Let A be a matrix with non-negative integer entries such that* (1) *holds. Then we have:*

I. *If $b > a$, then all diagonal entries of the matrix $A^c (1 \leq c \leq b-a)$ are equal to zero.*

II. *Every matrix $A^c (0 \leq c \leq b)$ is a 0–1 matrix.*

Proof. I. The equation (1) can be written in the form

$$(I + A + A^2 + \ldots + A^{b-a})A^a = J. \tag{2}$$

Suppose that there is a non-zero diagonal entry in some A^c $(1 \leq c \leq b-a)$. Then the corresponding entry of $I + A + A^2 + \ldots + A^{b-a}$ is ≥ 2. From (2) it follows that in J there exists also an entry ≥ 2: in the case $a = 0$ this evident; in the case $a \geq 1$ this follows from the fact that in every row of A (and, consequently, of A^a as well) there is an entry ≥ 1 (otherwise (1) cannot be true).

II. Suppose that some A^c, $0 \leq c \leq b$, has an entry ≥ 2. Then evidently $0 < c < a$, so that $a - c \geq 1$. Obviously, A (and, consequently, A^{a-c} as well) has in every row a non-zero entry. Therefore $A^a = A^c A^{a-c}$ has an entry ≥ 2, which is impossible. Q.E.D.

Lemma 3. *Let f be a polynomial and let A be a $p \times p$ complex matrix such that $f(A) = J$. Then all row and column sums of A are equal to a constant δ and $p = f(\delta)$.*

Proof. Denote the row sums of A by $\alpha_1, \alpha_2, \ldots, \alpha_p$ and the column sums of A by $\beta_1, \beta_2, \ldots, \beta_p$. We shall prove that $\alpha_i = \beta_j$ for every $i, j \in \{1, 2, \ldots, p\}$. Obviously,

$$AJ = Af(A) = f(A)A = JA.$$

Denote by a_{ij} the (i, j) entry of A, and by b_{ij} the (i, j) entry of $AJ = JA$. Then

$$\alpha_i = \sum_{n=1}^{p} (a_{in} \, . \, 1) = b_{ij} = \sum_{n=1}^{p} (1 \, . \, a_{nj}) = \beta_j .$$

Thus all row and column sums of A are equal to the same number, say δ. Hence the row and column sums of A^c are δ^c. It follows that all row and column sums of $f(A) = J$ are $f(\delta) = p$. Q.E.D.

Theorem 1. *Let A be a $p \times p$ matrix with non-negative integer entries such that* (1) *holds. Then the row and column sums of A are equal to a non-negative integer constant d and*

(3) $$p = d^a + d^{a+1} + \dots + d^b.$$

Moreover, if $p \neq 1$, then A is a 0–1 matrix.

Proof. The first part follows from Lemma 3 for

(4) $$f(x) = x^a + x^{a+1} + \dots + x^b.$$

(Evidently, now $\delta = d$ is a non-negative integer.)

Let $p \neq 1$. Then $b \neq 0$ ($b = 0$ implies $J = A^0 = I$ so that $p = 1$). According to Lemma 2, part II, A is a 0–1 matrix. Q.E.D.

3. Results concerning graphs

By a *pair of oppositely directed edges* in a graph we mean a set consisting of two edges joining two different vertices u and v such that one edge is directed from u to v and the other one from v to u.

If we express Theorem 1 in terms of graphs, we get:

Theorem 2. *Let G be a W_a^b-graph with p vertices. Then G is regular and* (3) *holds, where d is the degree of G. Moreover, if $p \neq 1$, then G has no multiple edges except, possibly, for pairs of oppositely directed edges.*

Theorem 2 allows us to consider, when studying W_a^b-graphs, regular graphs only.

Given integers d and a such that $d \geq 1$ and $a \geq 0$, we shall define two directed graphs $A(d, a)$ and $B(d, a)$ and study some basic properties of them.

The graph $A(d, a)$ is defined as follows. If $a = 0$, then $A(d, a)$ is the one-vertex graph with d loops. If $a \geq 1$, then the vertex set of $A(d, a)$ is $\{1, 2, 3, \dots, d^a\}$. From a vertex y a directed edge goes to all vertices z such that

(A_1) $$y = sd + g,$$

(A_2) $$z = (h-1)d^{a-1} + s + 1,$$

where s, g, h are integers satisfying the inequalities

(A_3) $$0 \leq s \leq d^{a-1} - 1$$

(A_4) $$1 \leq g \leq d,$$

(A_5) $$1 \leq h \leq d.$$

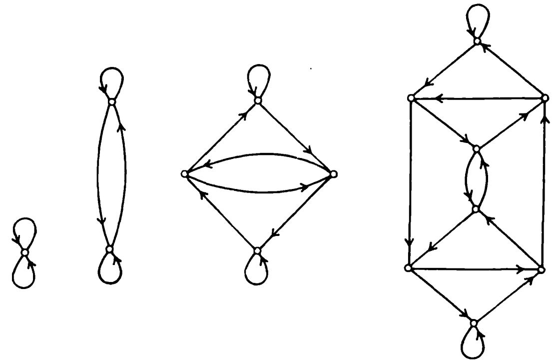
Graphs $A(2, a)$ for $a = 0, 1, 2, 3$ are drawn in Fig. 1.

Theorem 3. *Let $d \geq 1$ and $a \geq 0$. Then $A(d, a)$ is a W_a^a-graph of degree d.*

Proof. For $a = 0$ the assertion is trivial. Therefore we suppose $a \geq 1$. The proof will be divided into five parts.

I. Let u be a vertex of $A(d, a)$. Denote by $V_c(u)$ the set of vertices v of $A(d, a)$ such that in $A(d, a)$ there is a walk of length c from u to v. We prove by induction on c that the following implication holds for $c=0, 1, 2, \ldots, a$:

$$\text{(Y)} \qquad y, y' \in V_c(u) \Rightarrow y \equiv y' \pmod{d^{a-c}}.$$



A(2,0) A(2,1) A(2,2) A(2,3)

Fig. 1. W_a^a-graphs of degree two ($0 \leq a \leq 3$).

For $c=0$ (Y) evidently holds. Let (Y) hold for $c=n$, where n is an integer such that $0 \leq n \leq a-1$. We want to prove that (Y) holds for $c=n+1$, i.e.

$$\text{(Z)} \qquad z, z' \in V_{n+1}(u) \Rightarrow z \equiv z' \pmod{d^{a-n-1}}.$$

Let $z, z' \in V_{n+1}(u)$. Then in $A(d, a)$ there are directed edges (y, z) and (y', z') such that (A_1)—(A_5) and the following relations (A_1')—(A_5') hold (where s, g, h, s', g', h' are integers):

$$(A_1') \qquad y' = s'd + g',$$

$$(A_2') \qquad z' = (h'-1)d^{a-1} + s' + 1,$$

$$(A_3') \qquad 0 \leq s' \leq d^{a-1} - 1,$$

$$(A_4') \qquad 1 \leq g' \leq d,$$

$$(A_5') \qquad 1 \leq h' \leq d.$$

As $y, y' \in V_n(u)$, the induction hypothesis implies that there is an integer l such that

$$(A_6) \qquad y - y' = ld^{a-n}.$$

Then

$$g - g' = (y - sd) - (y' - s'd) = ld^{a-n} - d(s - s'),$$

so that

$$g \equiv g' \pmod{d}.$$

However, (A_4) and (A_4') imply $g = g'$. Therefore

$$(A_7) \qquad y - y' = (s - s')d.$$

If we compare (A_6) and (A_7), we get

$$s - s' = ld^{a-n-1}.$$

But then

$$z - z' = (h - h')d^{a-1} + (s - s') = d^{a-n-1}(l + hd^n - h'd^n),$$

thus (Z) holds. Hence (Y) has been proved.

II. We prove by induction that

$$|V_c(u)| = d^c$$

for $c = 0, 1, 2, \ldots, a$.

For $c = 0$ the assertion is true. Let it hold for $c = n$, i.e. $|V_n(u)| = d^n$, where $0 \leq n \leq a - 1$. We show the assertion to be true for $c = n + 1$. Evidently, $z \in V_{n+1}(u)$ if and only if there exists an edge (y, z) such that $y \in V_n(u)$. The induction hypothesis implies that the vertex y can be chosen in d^n ways. According to (A_2) and (A_5) from every vertex y of $A(d, a)$ there go exactly d edges ending in d mutually different vertices of $A(d, a)$. Therefore it is sufficient to prove that if $A(d, a)$ has edges (y, z), (y', z'), where $y, y' \in V_n(u)$, $z, z' \in V_{n+1}(u)$ and $y \neq y'$, then $z \neq z'$. Suppose again that (A_1)—(A_5) and (A_1')—(A_5') hold. Admit that $z = z'$, i.e.

$$s - s' = (h' - h)d^{a-1}.$$

(A_3) and (A_3') imply $s - s' = 0$ so that

$$y - y' = (s - s')d + g - g' = g - g'.$$

Putting $c = n$ in (Y) we get (A_6) so that

$$g - g' = y - y' = ld^{a-n}.$$

As $a - n \geq 1$, it follows that $g \equiv g' \pmod{d}$ and according to (A_4) and (A_4') we have $g = g'$. Hence $y - y' = (s - s')d + (g - g') = 0$, i.e. $y = y'$, a contradiction.

III. We prove that for $c = 0, 1, 2, \ldots, a$ there are exactly d^c walks of length c from u to a vertex of $V_c(u)$ and these d^c walks end in mutually different vertices of $A(d, a)$. For $c = 0$ the assertion holds. Suppose it to be true for $c = n$, where $n \leq a - 1$. According to (A_2) and (A_5) each of the walks of length n from u to some of d^n vertices of $V_n(u)$ can be prolonged in d ways. We get d^{n+1} walks. They are mutually different and they end in different vertices of $V_{n+1}(u)$, as $V_{n+1}(u)$ has according to II d^{n+1} vertices.

IV. We prove that $A(d, a)$ is a W_a^a-graph. Without loss of generality it is sufficient to prove that from u there exists to every vertex of $A(d, a)$ exactly one walk of length a. According to II we have $|V_a(u)| = d^a$. Thus $V_a(u)$ contains all vertices of $A(d, a)$. By III there exist from u to the vertices of $V_a(u)$ just d^a walks of length a, thus to every vertex of $A(d, a)$ exactly one walk of length a.

V. Now Theorem 2 implies that $A(d, a)$ is regular of degree d. Q.E.D.

The graph $B(d, a)$ is defined as follows. If $a=0$, then $B(d, a)$ is the complete digraph (without loops) with $d+1$ vertices ($d \geqslant 1$). If $a \geqslant 1$, $d \geqslant 1$, then the vertex set of $B(d, a)$ is $\{1, 2, ..., d^a + d^{a+1}\}$. From a vertex y a directed edge goes to all vertices z such that

(B₁) $$y = sd + g,$$

(B₂) $$z = h(d^{a-1} + d^a) - s,$$

where s, g, h are integers satisfying the inequalities

(B₃) $$0 \leqslant s \leqslant d^{a-1} + d^a - 1,$$

(B₄) $$1 \leqslant g \leqslant d,$$

(B₅) $$1 \leqslant h \leqslant d.$$

Graphs $B(2, a)$, $a = 0, 1, 2$ are drawn in Fig. 2.

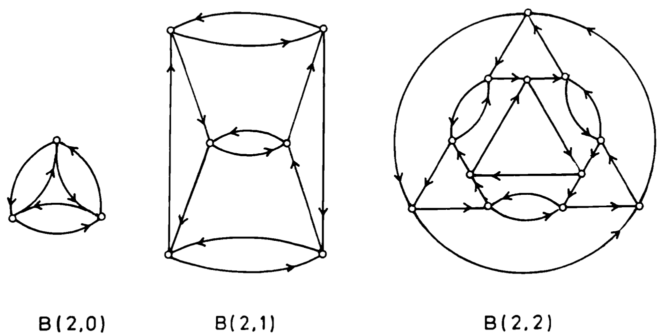


Fig. 2. W_a^{a+1}-graphs of degree two ($0 \leqslant a \leqslant 2$).

Theorem 4. *Let $d \geqslant 1$ and $a \geqslant 0$. Then $B(d, a)$ is a W_a^{a+1}-graph of degree d.*

Proof. The proof is analogous to that of Theorem 3, therefore we indicate only changes to be made and we have left the details to a reader. Relations (A_1)—(A_5) are always replaced by (B_1)—(B_5), and those of (A_1')—(A_5') by

(B$_1'$) $$y'=s'd+g',$$

(B$_2'$) $$z'=h'(d^{a-1}+d^a)-s',$$

(B$_3'$) $$0\leq s'\leq d^{a-1}+d^a-1,$$

(B$_4'$) $$1\leq g'\leq d,$$

(B$_5'$) $$1\leq h'\leq d,$$

respectively.

I. We prove the following implication

(Y*) $$y, y'\in V_c(u)\Rightarrow y\equiv y' \pmod{d^{a-c}+d^{a-c+1}}$$

for $c=0, 1, 2, \ldots, a$.

II. We prove by induction that $|V_c(u)|=d^c$ for $c=0, 1, 2, \ldots, a+1$.

III. We prove that for $c=0, 1, 2, \ldots, a+1$ there are exactly d^c walks of length c from u to a vertex of $V_c(u)$ and these d^c walks end in mutually different vertices of $B(d, a)$.

IV. To prove that $B(d, a)$ is a W_a^{a+1}-graph, we firstly prove that $V_c(u)\cap V_{c+1}(u)=\emptyset$ for $c=0, 1, 2, \ldots, a$. Let us admit the existence of $y\in V_c(u)\cap V_{c+1}(u)$ and suppose (B$_1$), (B$_3$), (B$_4$) to be true. As $y\in V_{c+1}(u)$, in $B(d, a)$ there is an edge $(y'\ y)$ such that $y'\in V_c(u)$ and (B$_1'$), (B$_3'$), (B$_4'$) hold. According to (Y*) we have

$$sd+g\equiv s'd+g' \pmod{d^{a-c}+d^{a-c+1}}$$

so that there is an integer l such that

$$y=sd+g=s'd+g'+l(d^{a-c}+d^{a-c+1}).$$

As (y', y) is an edge of $B(d, a)$, we have

$$y=h'(d^a+d^{a-1})-s'$$

with h' satisfying (B$_5'$). Comparing the last two equalities, we get

$$g'=(d+1)(h'd^{a-1}-s'-ld^{a-c}).$$

Thus g' is a multiple of $d+1$, a contradiction to (B$_4'$). We have proved $V_c(u)\cap V_{c+1}(u)=\emptyset$.

Now II implies that $V_a(u)\cup V_{a+1}(u)$ has d^a+d^{a+1} different vertices, i.e., all the vertices of $B(d, a)$, and the assertion follows from III.

V. Theorem 2 implies that $B(d, a)$ is regular of degree d. Q.E.D.

4. Main results

Theorem 5. *The following three assertions are equivalent:*

I. *There exists a W_a^b-graph of degree d with p vertices.*

II. *There exists a $p \times p$ matrix A with non-negative integer entries such that all row and column sums of A are d and (1) holds.*

III. *One of the following conditions holds:*

(i) $b = a$, $d \geq 1$, $p = d^a$.

(ii) $b = a + 1$, $d \geq 1$, $p = d^a + d^b$.

(iii) $b \geq a + 2$, $d = 1$, $p = b - a + 1$.

(iv) $b \geq a = 0$, $d = 0$, $p = 1$.

Proof. III⇒I. In each of cases (i)—(iv) we give an example of a W_a^b-graph of degree d with p vertices:

(i) $A(d, a)$ (see Theorem 3).

(ii) $B(d, a)$ (see Theorem 4).

(iii) Z_{b-a+1} (the graph induced by the edges of a directed cycle on $b - a + 1$ vertices).

(iv) K_1 (the graph with one vertex and no edges).

I⇒II. This implication follows from Lemma 1.

II⇒III. According to Theorem 1 we have (3). If $d = 0$, then $a = 0$ (otherwise $p = 0$, a contradiction) and $p = 0^0 = 1$ so that (iv) holds. Therefore we can suppose $d \geq 1$. If $b = a$ or $b = a + 1$, we have (i) or (ii), respectively.

It remains to deal with the case $b \geq a + 2$ and $d \geq 1$ so that $p \geq 3$. We use the method from the proof of Theorem 3 of [7].

The eigenvalues of J are $\lambda_1 = \lambda_2 = \ldots = \lambda_{p-1} = 0$, $\lambda_p = p$. Then for the eigenvalues $\mu_1, \mu_2, \ldots, \mu_{p-1}, \mu_p$ of A we have

$$\begin{aligned} f(\mu_1) &= \lambda_1 = 0, \\ f(\mu_2) &= \lambda_2 = 0, \\ &\cdots\cdots\cdots\cdots \\ f(\mu_{p-1}) &= \lambda_{p-1} = 0, \\ f(\mu_p) &= \lambda_p = p, \end{aligned} \tag{5}$$

where f is defined by (4). Evidently, d is an eigenvalue of A. According to Theorem 1, $f(d) = p$, therefore $\mu_p = d$.

From (4) and (5) it follows that each of the eigenvalues $\mu_1, \mu_2, \ldots, \mu_{p-1}$ is either zero or a root of the binomial equation $x^{b-a+1} = 1$ different from one. Therefore for every $j \in \{1, 2, \ldots, p-1\}$ either $\mu_j = 0$ or there exists $n \in \{1, 2, \ldots, b-a\}$ such that

$$\mu_j = \omega^n,$$

where

$$\omega = e^{2ri},$$

$$r = \frac{\pi}{b-a+1}.$$

Denote the multiplicity of the eigenvalue ω^n in A by m_n. (The eigenvalue $\mu_p = d$ has multiplicity 1; the eigenvalue 0 has multiplicity $p - m_1 - m_2 - \ldots - m_{b-a} - 1$.)

From Lemma 2 (part I) it follows that for $c = 1, 2, \ldots, b-a$ the trace of a^c is zero so that

$$\mu_1^c + \mu_2^c + \ldots + \mu_p^c = 0.$$

This equality can be written in the form

$$m_1\omega^c + m_2(\omega^2)^c + \ldots + m_{b-a}(\omega^{b-a})^c + d^c = 0, \tag{6}$$

$$(c = 1, 2, \ldots, b-a).$$

(6) can be considered as a system of $b-a$ linear equations for the unknows m_1, m_2, ..., m_{b-a}. The (Vandermonde) determinant of (6) is

$$\left(\left(\prod_{n=1}^{b-a} \omega^n\right)\right) \prod_{1 \le m < n \le b-a} (\omega^n - \omega^m)\Big) \neq 0.$$

However, for our purposes we need to determine only the first unknown m_1:

$$m_1 = -\frac{d(d-\omega^2)(d-\omega^3)\ldots(d-\omega^{b-a})}{\omega^{b-a}(1-\omega)(1-\omega^2)\ldots(1-\omega^{b-a-1})}. \tag{7}$$

As all roots of the equation

$$x^{b-a} + x^{b-a-1} + \ldots + x^2 + x + 1 = 0$$

are ω, ω^2, ..., ω^{b-a}, we have the identity

$$x^{b-a} + x^{b-a-1} + \ldots + x + 1 = (x-\omega)(x-\omega^2)\ldots(x-\omega^{b-a})$$

so that

$$d^{b-a} + d^{b-a-1} + \ldots + d + 1 = (d-\omega)(d-\omega^2)\ldots(d-\omega^{b-a}).$$

Therefore (7) can be written thus:

$$m_1 = -\frac{d(d^{b-a} + d^{b-a-1} + \ldots + d + 1)}{(d-\omega)\omega^{b-a}(1-\omega)(1-\omega^2)\ldots(1-\omega^{b-a-1})}.$$

Since m_1 and d are non-zero and real, there is real also the denominator

$$t = (d-\omega)\omega^{b-a}(1-\omega)(1-\omega^2)\ldots(1-\omega^{b-a-1}). \tag{8}$$

We observe that for every integer n we have

$$1 - \omega^n = iq_n e^{rni}, \tag{9}$$

where q_n is real. In fact,

$$1-\omega^n = 1-\cos 2rn - \mathrm{i}\sin 2rn =$$
$$= 2\sin^2 rn - 2\mathrm{i}\sin rn\cos rn =$$
$$= -2\mathrm{i}\sin rn\,(\cos rn + \mathrm{i}\sin rn) =$$
$$= \mathrm{i}\, q_n\, \mathrm{r}^{rn\mathrm{i}},$$

where $q_n = -2\sin rn$. Substituting $n = 1, 2, \ldots, b-a-1$ in (9), we get from (8)

$$t = (d-\omega)\, \mathrm{e}^{2ri(b-a)}\, \mathrm{i}q_1\, \mathrm{e}^{ri}\, \mathrm{i}q_2\, \mathrm{e}^{2ri} \ldots \mathrm{i}q_{b-a-1}\, \mathrm{e}^{(b-a-1)ri} =$$
$$= (d-\omega)\, \mathrm{e}^{ri(b-a)}\, \mathrm{i}^{b-a-1} q_1 q_2 \ldots q_{b-a-1}\, \mathrm{e}^{ri(1+2+\ldots+(b-a))}.$$

However,

$$\mathrm{e}^{ri(b-a)} = \mathrm{e}^{ri(b-a+1)}\, \mathrm{e}^{-ri} = \mathrm{e}^{\pi\mathrm{i}}\mathrm{e}^{-ri} = -\mathrm{e}^{-ri},$$
$$\mathrm{e}^{ri(1+2+\ldots+(b-a))} = (\mathrm{e}^{\pi\mathrm{i}/2})^{b-a} = \mathrm{i}^{b-a},$$

therefore

$$t = -(d-\omega)\, \mathrm{e}^{-ri}\, \mathrm{i}^{2(b-a-1)}\, \mathrm{i}q_1 q_2 \ldots q_{b-a-1} = q(d-\mathrm{e}^{2ri})\, \mathrm{e}^{-ri}\, \mathrm{i},$$

where

$$q = (-1)^{b-a} q_1 q_2 \ldots q_{b-a-1}$$

is non-zero and real. Hence

$$(d-\mathrm{e}^{2ri})\, \mathrm{e}^{-ri}\mathrm{i} = \mathrm{i}(d\mathrm{e}^{-ri} - \mathrm{e}^{ri}) = \mathrm{i}(d\cos r - \mathrm{i}d\sin r -$$
$$-\cos r - \mathrm{i}\sin r) = (d+1)\sin r + \mathrm{i}(d-1)\cos r$$

is a real number so that

$$(d-1)\cos r = 0.$$

However, as $b-a \geqslant 2$, we have $0 < r < \pi/2$, hence $\cos r \neq 0$ and $d = 1$. Substituting this result into (3), we get $p = b-a+1$ and (iii) holds. Q.E.D.

Remark. Evidently, the only W_a^b-graph satisfying (iii) or (iv), is Z_{b-a+1} or K_1, respectively. Thus we have:

Corollary 1. *Every W_a^b-graph with $b \geqslant a+2$ is either Z_{b-a+1} or K_1 (this case can occur only for $a = 0$).*

To find all W_a^b-graphs satisfying (i) or (ii) seems to be a difficult problem. A very special case $a = b = 2$ (corresponding to the matrix equation $A^2 = J$) has been studied by several authors (see, e.g. [5], [8]) but it is still not completely settled. We are able to describe only some general properties of W_a^b-graphs.

Lemma 4. *The number of closed walks of a length $c \geqslant 1$ in a W_a^b-graph of degree d is*

$$d^c, \quad \textit{if} \quad b=a;$$

$$d^c + d(-1)^c, \quad \textit{if} \quad b=a+1.$$

Proof. Let A be the adjacency matrix of a W_a^b-graph of degree d. If $b=a$, then $A^a = J$ and the eigenvalues of A are $\mu_1=\mu_2=\ldots=\mu_{p-1}=0$, $\mu_p=d$ (cf. (5)). Thus the eigenvalues of A^c are $\mu_1^c=\mu_2^c=\ldots=\mu_{p-1}^c=0$, $\mu_p^c=d^c$. The number of closed walks of length c is equal to the trace of A^c, $\mathrm{tr}A^c=\mu_1^c+\mu_2^c+\ldots+\mu_{p-1}^c+\mu_p^c=d^c$.

If $b=a+1$, then $A^a+A^{a+1}=J$ and then A has one eigenvalue d, d eigenvalues (-1) and the other eigenvalues are equal to zero. The matrix A^c has one eigenvalue d^c, d eigenvalues $(-1)^c$ and the others are zero. Thus the number of closed walks of length c is $\mathrm{tr}A^c=d^c+d(-1)^c$. Q.E.D.

Theorem 6. *Let G be a W_a^b-graph of degree d. Then we have:*

I. *G has exactly d loops if $a=b$, and no loops if $a<b$.*

II. *The number of pairs of oppositely directed edges of G is*

$$\binom{d}{2} \quad \textit{if} \quad b=a\geqslant 1,$$

$$\binom{d+1}{2} \quad \textit{if} \quad b=a+1,$$

$$0, \quad \textit{otherwise.}$$

III. *G has diameter*

$$b \quad \textit{if} \quad d\geqslant 2,$$

$$b-a \quad \textit{if} \quad d=1,$$

$$0 \quad \textit{if} \quad d=0.$$

Proof. I. If $a\leqslant b\leqslant a+1$, it is sufficient to put $c=1$ in Lemma 4. If $b\geqslant a+2$, the result follows from Corollary 1.

II. If $b=a\geqslant 1$, according to Lemma 4 the number of closed walks of length two in G is d^2. However d of these walks are formed by loops and each pair of oppositely directed edges corresponds to two closed walks. Thus we obtain the number

$$(d^2-d)/2=\binom{d}{2}.$$

For $b=a+1$ the proof is analogous. The rest of the proof follows from Theorem 5 and Corollary 1.

III. For $d=0$ the assertion is evident. If $d=1$, then G is Z_{b-a+1} and has the diameter $b-a$.

Let G be a W_a^b-graph of degree $d \geqslant 2$. Obviously, for the diameter k of G we have $k \leqslant b$. If $k < b$, then every vertex of G is reachable from a fixed vertex of G by a walk of length $\leqslant b-1$. But in a regular directed graph of degree d there exist only

$$1+d+d^2+\ldots+d^{b-1}=\frac{d^b-1}{d-1}$$

such walks, so that

$$p \leqslant \frac{d^b-1}{d-1}$$

and, consequently, $d^b \geqslant 1+p(d-1)$. Thus, according to (3) we have

$$p=d^a+d^{a+1}+\ldots+d^b \geqslant d^b \geqslant 1+p(d-1) \geqslant 1+p,$$

a contradiction. Therefore $k=b$. Q.E.D.

5. Related problems and results

In [7] the following class of graphs has been introduced (we use a somewhat adapted terminology):

A digraph G is said to be a *graph* $G_{b,a}$ if the following conditions hold:

1° The diameter of G is b.

2° G is a W_a^b-graph.

3° G has no closed walks of a length c, where $1 \leqslant c \leqslant b-a$.

[By a *digraph* we mean a (finite) directed graph without loops or multiple edges; however, we admit pairs of oppositely directed edges.]

From Lemmas 1 and 2 (Part I) it follows that 3° is superfluous as it is a consequence of 2°.

From Theorems 5 and 6 we have:

Corollary 2 ([7], Theorem 3).

I. *The graphs* $G_{1,0}$ *are just the complete digraphs.*

II. *For* $b \geqslant 2$ *the only graphs* $G_{b,0}$ *are* Z_{b+1}.

III. *The graphs* $G_{b,a}$ *do not exist if* $a>0$ *and* $b \geqslant a+2$.

The authors of [7] left open the question of existence of graphs $G_{b,b-1}$ $(b \geqslant 2)$ and $G_{b,b}$ $(b \geqslant 0)$ with a given number of vertices (there is given one example of $G_{2,1}$ with 6 vertices). However, from Theorems 2, 5 and 6 it easily follows:

Corollary 3.

I. *There is no graph* $G_{b,b}$ *except for* K_1 *(with* $b=0$*).*

II. *A graph* $G_{b,b-1}$ $(b \geqslant 2)$ *with* p *vertices exists if and only if*

$$p=d^{b-1}(d+1), \tag{10}$$

where d *is an integer,* $d \geqslant 2$ *(and then this digraph is regular of degree* d*).*

(The necessity of (10) has been also mentioned in [7].)
Now we replace equation (1) by a more general equation

$$A^{a_1} + A^{a_2} + \dots + A^{a_n} = \lambda J. \tag{11}$$

It is easy to obtain the following result.

Theorem 7. *Let p, n and λ be positive integers and $a_1, a_2, \dots, a_n$ be non-negative integers with $a_1 < a_2 < \dots < a_n$. Let A be a $p \times p$ matrix with non-negative integer entries satisfying* (11). *Then the row and column sums of A are equal to a non-negative integer d and*

$$p = \frac{1}{\lambda}(d^{a_1} + d^{a_2} + \dots + d^{a_n}).$$

Proof. It is sufficient to use Lemma 3 for

$$f(x) = \frac{1}{\lambda}(x^{a_1} + x^{a_2} + \dots + x^{a_n}).$$

Q.E.D.

Problem. *For what parameters $p, n, d, \lambda, a_1, a_2, \dots, a_n$, satisfying the conditions of Theorem 7, has the equation* (11) *a solution A with non-negative integer entries such that the row and column sums of A are d?*

The problem has also an obvious graph-theoretical interpretation: When does there exist a regular directed graph of degree d with p vertices such that for any two vertices u and v of G there are in G exactly λ walks from u to v whose lengths are in the set $\{a_1, a_2, \dots, a_n\}$?

Theorem 5 answers the question in the special case

$$\lambda = 1, \quad a_2 - a_1 = a_3 - a_2 = \dots = a_n - a_{n-1} = 1.$$

Also other special cases may be of interest.

All graph-theoretical problems studied in this article may be modified in such a way that the conditions concerning the uniqueness (or the number λ) of walks are related only to different vertices u and v of G. This leads to the matrix equation

$$A^{a_1} + A^{a_2} + \dots + A^{a_n} = D + \lambda J$$

with two unknown matrices (having non-negative integer entries) A and D, where D should be diagonal. A special case $n = 1$, $a_1 = 2$ has been studied in [6] and [8]. It is interesting that in this case the assertion concerning the regularity of a graph has some exceptions (see [8]).

Finally, let us mention that (1) can be modified so that it is only demanded that all the entries of $A^a + A^{a+1} + \dots + A^b$ are positive. This leads to the study of irreducible matrices (or relations) and strongly connected directed graphs. These questions have been studied in many papers, see e.g. [4], [9] and [10].

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Matematický ústav SAV
Obrancov mieru 49
886 25 Bratislava

ОРИЕНТИРОВАННЫЕ ГРАФЫ И МАТРИЧНЫЕ УРАВНЕНИЯ

Юрай Босак

Резюме

Пусть a и b – неотрицательные целые числа. Конечный ориентированный граф G называется W_a^b-графом, если для произвольных его вершин u и v существует в G точно один ормаршут из вершины u в вершину v, длина c которого удовлетворяет неравенствам $a \leq c \leq b$.

В работе показано, что W_a^b-граф всегда однородный и следующие условия равносильны:

1. Существует W_a^b-граф степени d с p вершинами.

2. Существует квадратная матрица A порядка p с неотрицательными элементами такая, что сумма всех элементов произвольной строки (произвольного столбца) равна d и $A^a+A^{a+1}+\ldots+A^b=J$, где J – матрица, все элементы которой равны 1.

3. Выполняется одно из условий:

(i) $b=a,\ d\geq 1,\ p=d^a$.

(ii) $b=a+1,\ d\geq 1,\ p=d^a+d^b$.

(iii) $b\geq a+2,\ d=1,\ p=b-a+1$.

(iv) $b\geq a=0,\ d=0,\ p=1$.

Таким образом, обобщены результаты статьи [7]. Кроме того, исследовано несколько смежных вопросов, обобщений и открытых проблем.