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ON THE ORDER AND THE NUMBER OF CLIQUES IN A RANDOM GRAPH

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ABSTRACT. A clique of a graph G is a complete subgraph of G maximal under inclusion. We study the numbers of cliques of various orders in a random graph $G_{n,p}$ and prove that almost all cliques of $G_{n,p}$ have the same order, which is approximately $\log n - \log\log\log n$, where log means the logarithm to the base $1/p$, and estimate the total number of cliques in $G_{n,p}$.

1. Introduction

The clique is a basic structure of a graph and has been investigated very intensively. We study cliques in a random graph in this paper and answer the following two questions:

(1) how many cliques are there in a random graph, and
(2) how large are the cliques which appear most frequently in a random graph?

The first general results on the number of cliques $Y(G)$ in a graph G were proved by Moon and Moser [11]. They proved that $Y(G_n) \le 3^{n/3}$, where G_n is a graph of order n. Since their result is the best possible, further research concentrated on classes of graphs with upper bound on $Y(G_n)$ lower than $3^{n/3}$ ([4], [5]). The latest result is due to Farber, Hujter, Tuza, who described a large class of graphs with the parameter $Y(G_n)$ polynomially bounded ([3]).

We have proved that the random graph $G_{n,p}$ is (with respect to the total number of cliques) closer to Tuza's graphs than to Moon's graphs since a random graph of order n contains

$$Y(n) = n^{\frac{1}{2}\log_b n - \log_b \log_b n + \frac{1}{2} + \log_b e} \times (\log_b n)^{\log_b \log_b \log_b n + O(1)}$$

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cliques, where $b \approx 1/p$.

The investigation of cliques in random graphs concentrated mainly on the behaviour of the clique number $\kappa(n)$ (the order of the largest clique). Matula [10] proved that the clique number of a random graph is highly concentrated (it attains one of at most two values) and established an upper bound on $\kappa(n)$. Kalbfleisch [6] found a lower bound on the order of cliques in a random graph and Bollobás and Erdős [2] determined the orders of cliques which appear in a random graph.

We prove that not only is the clique number of a random graph highly concentrated, but that something very similar holds for the order of cliques in a random graph: almost all the cliques of a random graph $G_{n,p}$ have the same order $\lfloor r_3 \rfloor$ or $\lceil r_3 \rceil$, where

$$r_3 = \log_b n - \log_b \log_b \log_b n + \log_b \log_b \mathrm{e} + \log_b (1-p) + 1 .$$

2. Preliminaries

The standard terminology and notation of random graph theory will be used in this paper. In order to make this paper self contained, we now introduce the most important notions. The notions not mentioned here can be found in [1] or [12].

Throughout this paper, the graphs of order n are considered. Let G be a graph. The vertex set of G will be denoted by $V(G)$, and the edge set by $H(G)$. The set of all graphs of order n will be denoted by $\mathfrak{G}^n$. We shall consider the *model* (probability space) $\mathfrak{G}(n, P(\text{edge}) = p)$. In this model, we have $0 < p < 1$, p is a constant, and the model consists of all graphs with vertex set $V = \{1, \dots, n\}$ in which edges are chosen independently and with probability p. In other words, if G_0 is a graph of order n and *size* (the number of edges) m, then

$$P\{G_0\} = P(G = G_0) = p^m (1-p)^{N-m} , \tag{2.1}$$

where $N = \binom{n}{2}$. The parameter p can be considered as a function of n; $p = p(n)$. The cases $p(n) \to 0$ and $p(n) \to 1$ are of special interest and will be studied in another place. Another interesting case is $\mathfrak{G}\big(n, P\big(\text{edge} = \frac{1}{2}\big)\big)$. Note that $\mathfrak{G}\big(n, \frac{1}{2}\big) = \mathfrak{G}^n$ with any two graphs equiprobable.

We call a subset Q of $\mathfrak{G}^n$ a *property* of graphs of order n. We shall say that *almost every* (a.e.) graph in $\mathfrak{G}(n,p)$ has a certain property Q (or, $G \in \mathfrak{G}(n,p)$ has the property Q *almost surely*) if $P(Q) \to 1$ as $n \to \infty$. A real-valued *random variable* (r.v.) X is a measurable real-valued function on a probability space; $X \colon \mathfrak{G}(n,p) \to R$. Since most of the r.v.'s we encounter are non-negative integer valued, unless otherwise indicated, we assume that the random variables take only non-negative integer values. Let X be a r.v., the expectation, and the variance of the r.v. X will be denoted by $E(X)$ and $\mathrm{Var}(X)$ respectively. The variance of a r.v. X can be expressed as follows:

$$\mathrm{Var}(X) = E(X^2) - E(X)^2 . \tag{2.2}$$

Let X be a nonnegative r.v. with expectation $E(X)$ and let $t > 0$. Then we have (*Markov's inequality*)

$$P\big(X \geqslant t \cdot E(X)\big) \leqslant \frac{1}{t} . \tag{2.3}$$

Now, let X be a real-valued r.v. If $d > 0$, we have (*Chebyshev's inequality*)

$$P\big(|X - E(X)| \geqslant d\big) \leqslant \frac{\mathrm{Var}(X)}{d^2} . \tag{2.4}$$

The big O, Ω, Θ notation will be used in standard way. The symbols $\ln x$, $\lg x$ denote the natural and the binary logarithm of x respectively. The symbol $\log x$ will be used in big O terms and will denote the logarithm to an arbitrary base (greater than 1). The bases of other logarithms will be written explicitly. We shall often use the logarithm to the base $\frac{1}{p}$. To simplify the notations, we shall put $b = \frac{1}{p}$ and write $\log_b x$ instead of $\log_{\frac{1}{p}} x$. Let x be a real number. The symbols $\lfloor x \rfloor$, $\lceil x \rceil$, $\{x\}$ denote the floor, the ceiling and the fractional part of x respectively.

Finally, we shall need the following simple but useful asymptotic estimations:

Proposition 2.1. *Let* $k = o(\sqrt{n})$, *then*

$$n^{\underline{k}} = n(n-1)\dots(n-k+1) = n^k\left(1 - \binom{k}{2}\frac{1}{n} + O\left(\frac{k^4}{n^2}\right)\right) . \tag{2.5}$$

P r o o f. The kth falling factorial power of n can be expressed in the following way

$$\begin{aligned}
\prod_{i=0}^{k-1}(n-i) &= n^k \prod_{i=1}^{k-1}\left(1 - \frac{i}{n}\right) \\
&= n^k\Bigg(1 - \frac{1}{n}\sum_{i=1}^{k-1} i + \frac{1}{n^2}\sum_{1\leqslant i_1<i_2<k} i_1 i_2 - \frac{1}{n^3}\times \\
&\quad\times \sum_{1\leqslant i_1<i_2<i_3<k} i_1 i_2 i_3 + \dots + \frac{(-1)^{k-1}}{n^{k-1}}\sum_{1\leqslant i_1<i_2<\dots<i_{k-1}<k} i_1 i_2 \dots i_{k-1}\Bigg) \\
&= n^k \sum_{j=0}^{k-1}\frac{(-1)^j}{n^j}\sum_{1\leqslant i_1<i_2\dots<i_j<k}\ \prod_{l=1}^{j} i_l .
\end{aligned} \tag{2.6}$$

Since $k = o(\sqrt{n})$, we can cut off the tail of (2.6) and obtain even a more precise estimate than (2.5). □

PROPOSITION 2.2. *Let $0 < p < 1$ and $k = \frac{1}{2}\log_{1/p} n + \omega_n$, where $\omega_n \to \infty$ as $n \to \infty$. Then*

$$(1-p^k)^n = \exp(-np^k)\big(1 + O(np^{2k})\big). \tag{2.7}$$

Proof.

$$\begin{aligned}(1-p^k)^n &= \exp\{n \cdot \ln(1-p^k)\} = \exp\left\{n \cdot \left(-p^k - \frac{p^{2k}}{2} - \frac{p^{3k}}{3} - \dots\right)\right\} \\ &= \exp(-np^k) \cdot \exp\big(O(np^{2k})\big) = \exp(-np^k) \cdot \big(1 + O(np^{2k})\big).\end{aligned}$$

□

3. The number of cliques in a random graph

We start our study of cliques in a random graph by investigating the number of cliques of a given order in a random graph. Let us introduce the principal notions first.

A *clique* of a graph G is the maximal complete subgraph of G. The clique of order r will be denoted by K_r, and the number of K_r's in a graph G will be denoted by $k_r(G)$. In this section, we shall study the random variables

$$Y_r = Y(n,r) = k_r(G_{n,p}),$$

the number of K_r's in $G_{n,p}$. We shall compute the expectation and estimate the variance of Y_r first.

LEMMA 3.1. *We have*

$$E_r = E(Y_r) = E(n,r) = \binom{n}{r} \cdot p^{\binom{r}{2}}(1-p^r)^{n-r}. \tag{3.1}$$

Proof. The vertices of a clique K_r can be chosen in $\binom{n}{r}$ ways. Since K_r is a complete subgraph, each of its r vertices is joined with the remaining $(r-1)$ vertices of K_r. And finally, since K_r is a clique, it cannot be contained in a complete subgraph of some greater order; i.e., we have to exclude the case when one of the $(n-r)$ vertices of the vertex set $V(G) - V(K_r)$ is joined by edges with all the r vertices of K_r. □

COROLLARY. *Let p be fixed, $0 < p < 1$, and let*

$$r_1 = 2\log_b n - 2\log_b \log_b n + 2\log_b e + 1 - 2\log_b 2\,, \tag{3.2}$$

$$r_0 = \log_b n - 2\log_b \log_b n + \log_b 2 + \log_b \log_b e\,. \tag{3.3}$$

A random graph of order n almost surely contains no cliques or order greater than $\lceil r_1 \rceil$ and less or equal than $\lfloor r_0 \rfloor$.

Proof. The upper bound (3.2) was proved by Matula [10] and can be found in [12]. The lower bound (3.3) was proved by Kalbfleisch [6]. □

Remark. If r is close to r_0, the term $(1-p^r)^{n-r}$ influences the value of $E(n,r)$ substantially, but $(1-p^r)^{n-r} = \Theta(1)$ if $r > \log_b n$.

Now we shall estimate the variance of Y_r.

LEMMA 3.2. *Let p be fixed, $0 < p < 1$, $\lfloor r_0 \rfloor \leqslant r \leqslant \lceil r_1 \rceil$. Then*

$$\mathrm{Var}(Y_r) = E(n,r)^2 O\left(\frac{\log^3 n}{n}\right). \tag{3.4}$$

Proof. We estimate $E(Y_r^2)$, the expectation of the number of the ordered pairs of K_r's in a random graph, and then we use (2.2) and (3.1) to obtain the estimate of the variance. We distinguish two cases with respect to the value of the parameter r:

1. If r is close to r_1, then most of complete subgraphs of order r are maximal, and we shall study the random variable X_r denoting the number of complete subgraphs of order r in $G_{n,p}$. Since $E(Y_r) \sim E(X_r)$, it is sufficient to find a good upper bound on $E(X_r^2)$.
2. Otherwise, we need to construct an upper bound on $E(Y_r^2)$.

The expectation can be expressed in the following way:

$$E(Y_r^2) = \\ = \sum_{j=0}^{r} \binom{n}{r}\binom{r}{j}\binom{n-r}{r-j} \cdot p^{2\binom{r}{2}-\binom{j}{2}} \left[(1-p^j) + p^j(1-p^{r-j})^2\right]^{n-2r+j} \cdot P(K_r^1, K_r^2)\,. \tag{3.5}$$

The vertices of the first clique K_r^1 can be chosen in $\binom{n}{r}$ ways. The cliques K_r^1, K_r^2 can (but need not) have j common vertices. These vertices can be chosen in $\binom{r}{j}$ ways. The remaining $(r-j)$ vertices of the second clique K_r^2 have to be chosen from $(n-r)$ vertices of $V(G) - V(K_r^1)$. Now we shall choose the edges: both cliques are complete graphs of order r, and therefore they contain $2\binom{r}{2}$ edges. But K_r^1, K_r^2 can have a nonempty intersection, a complete subgraph of order j. Therefore $\binom{j}{2}$ edges were counted twice. Both subgraphs K_r^1, K_r^2 are cliques, and so none of the remaining $(n-2r+j)$ vertices of the set $V(G) - \big(V(K_r^1) \cup V(K_r^2)\big)$ may be joined with all the r vertices of the cliques K_r^1, K_r^2. Consequently, for an arbitrary vertex $v \notin \big(V(K_r^1) \cup V(K_r^2)\big)$ either one of the common vertices of $\big(V(K_r^1) \cap V(K_r^2)\big)$ is not joined with v, or all common vertices are joined with v, but there are at least two vertices u, w; $u \in \big(V(K_r^1) - V(K_r^2)\big)$ and $w \in \big(V(K_r^2) - V(K_r^1)\big)$, such that neither u nor w is joined with v. The term $P(K_r^1, K_r^2)$ in (3.5) denotes the probability that none of the vertices of $\big(V(K_r^1) - V(K_r^2)\big) \cup \big(V(K_r^2) - V(K_r^1)\big)$ is joined with all $(r-j)$ vertices of the other clique and can be estimated in the following way

$$P(K_r^1, K_r^2) = P(K_r^1) \cdot P(K_r^1 \mid K_r^2) < (1 - p^{r-j})^{r-j}.$$

We now estimate the expectation $E(Y_r^2)$. Let $\lfloor r_0 \rfloor \leqslant r \leqslant r_2 = 2\log_b n - 3\log_b \log_b n$ (the second case). Since we need to prove that $\operatorname{Var}(Y_r)$ is negligible with respect to $E(n,r)^2$, we extract the expression $E(n,r)^2$ from the sum on the right side of the equation (3.5):

$$E(Y_r^2) \leqslant E(n,r)^2 \sum_j \binom{r}{j}\binom{n-r}{r-j}\binom{n}{r}^{-1} \cdot p^{-\binom{j}{2}} \times \\ \times [1 - 2p^r + p^{2r-j}]^{n-2r+j} [1 - p^{r-j}]^{r-j} [1 - p^r]^{2r-2n}. \tag{3.6}$$

To simplify the notation, let $F(r,j)$ denote the jth term of (3.6), namely

$$F(r,j) = E(n,r)^2 \binom{r}{j}\binom{n-r}{r-j}\binom{n}{r}^{-1} \cdot p^{-\binom{j}{2}} \times \\ \times [1 - 2p^r + p^{2r-j}]^{n-2r+j} [1 - p^{r-j}]^{r-j} [1 - p^r]^{2r-2n}.$$

The first term ($F(r,0)$) brings the largest contribution to the sum in (3.6). To make estimating the tail of the sum (3.6) easier, we estimate separately the contribution of the first two terms ($F(r,0)$, $F(r,1)$) to the expectation $E(Y_r^2)$. Using (2.5) and (2.7) and the fact $r = \Theta(\log n)$, we obtain

$$F(r,0) + F(r,1) = E(n,r)^2 \left(1 + O\left(\frac{\log^3 n}{n}\right)\right). \tag{3.7}$$

Now we prove that the tail of the sum (3.6) is negligible with respect to the value (3.7). At first we simplify our sum by estimating the last three terms of its seven-termed summand, namely the product

$$[1 - 2p^r + p^{2r-j}]^{n-2r+j} \cdot [1 - p^{r-j}]^{r-j} \cdot [1 - p^r]^{2r-2n}, \tag{3.8}$$

where $2 \leqslant j \leqslant r$, $\lfloor r_0 \rfloor \leqslant r \leqslant \lceil r_1 \rceil$.

If $r \geqslant \log_b n$, then (3.8) can be bounded by a suitable constant. To simplify the notation, we shall denote by $f(r,j)$ the product

$$f(r,j) = \binom{n-r}{r-j}\binom{r}{j}\binom{n}{r}^{-1} \cdot p^{-\binom{j}{2}}, \qquad 2 \leqslant j \leqslant r, \tag{3.9}$$

and by the symbol $S(n,m,r)$ the sum

$$S(n,m,r) = \sum_{j=m}^{r} f(r,j). \tag{3.10}$$

As can be easily seen, for $\lfloor r_0 \rfloor \leqslant r \leqslant \lceil r_1 \rceil$

$$\max_j f(r,j) = O\left(\frac{\log^4 n}{n^2}\right), \tag{3.11}$$

and therefore

$$S(n,2,r) = O\left(\frac{\log^5 n}{n^2}\right). \tag{3.12}$$

Let us return to the case $\lfloor r_0 \rfloor \leqslant r \leqslant \log_b n$. Then the product (3.8) can attain the value of order $O(\log^2 n)$, but the situation is controlled by the term $f(r,2)$. Since

$$f(r,2) = O\left(\frac{\log^4 n}{n^2}\right),$$

in the case $\lfloor r_0 \rfloor \leqslant r < r_2$, we have

$$S(n,2,r) = O\left(\frac{\log^7 n}{n^2}\right). \tag{3.13}$$

Taking into account (3.10), (3.12) and (3.13), we have

$$E(Y_r^2) = E(n,r)^2\left(1 + O\left(\frac{\log^3 n}{n}\right)\right). \tag{3.14}$$

Let $r_2 \leqslant r \leqslant \lceil r_1 \rceil$. We will study the expectation $E(X_r^2)$ now.

$$E(X_r^2) = \sum_{j=0}^{r} \binom{n}{r}\binom{r}{j}\binom{n-r}{r-j} \cdot p^{2\binom{r}{2}-\binom{j}{2}}.$$

Since

$$E(X_r) = \binom{n}{r} \cdot p^{\binom{r}{2}},$$

the expectation $E(X_r^2)$ can be expressed as follows:

$$E(X_r^2) = E(X_r)^2 \cdot \sum_{j=0}^{r} \binom{n}{r}^{-1}\binom{r}{j}\binom{n-r}{r-j} \cdot p^{-\binom{j}{2}}. \tag{3.15}$$

Now we can proceed in the same way as in the previous case: the first two terms (with $j = 0, 1$) of (3.15) yield the value

$$G(r,0) + G(r,1) = E(X_r)^2 \left(1 + O\left(\frac{\log^4 n}{n^2}\right)\right). \tag{3.16}$$

Let the symbols $S(n,m,r)$ and $f(r,j)$ have the same meaning as in the previous case. The tail of the sum in (3.16) can be expressed as follows

$$S(n,2,r) < S(n,2,r_2) + S(n,r_2,r).$$

The first sum, $S(n,2,r_2)$ can be estimated quite easily. Since (3.11) holds,

$$S(n,2,r_2) = O\left(\frac{\log^5 n}{n^2}\right). \tag{3.17}$$

To estimate the second sum $S(n,r_2,r)$, we change the summation range, then we extract the value $E(X_r)^{-1} = \binom{n}{r}^{-1} \cdot p^{-\binom{r}{2}}$ out of the last sum and change the order of summation by putting $k = r - j$:

$$S(n,r_2,r) \leqslant \binom{n}{r}^{-1} \cdot p^{-\binom{r}{2}} \sum_{k=0}^{\lceil r_1 \rceil - r_2} \binom{r}{k}\binom{n-r}{k} \cdot p^{-\binom{k}{2}+rk-k}.$$

Since

$$\binom{r}{k}\binom{n-r}{k} \cdot p^{-\binom{k}{2}+rk-k} < \left(rnp^{r-(k+1)/2}\right)^k$$

and $r - (k+1)/2 > 2\log_b n - 4\log_b \log_b n + O(1)$, we have

$$rnp^{r-(k+1)/2} = O\left(\frac{\log^5 n}{n}\right).$$

Taking into account these estimates, the last sum does not exceed

$$1 + O\left(\frac{\log^5 n}{n}\right),$$

and therefore

$$S(n,r_2,r) = E(X_r) \cdot \left(1 + O\left(\frac{\log^5 n}{n}\right)\right). \tag{3.18}$$

Combining (3.16), (3.17) and (3.18) yields

$$E(X_r^2) = E(X_r)^2 \cdot \left(1 + O\left(\frac{\log^5 n}{n^2}\right)\right).$$

Since

$$E(X_r)^2 = E(Y_r)^2 \cdot (1 - p^r)^{2r-2n},$$

and if $r \geqslant r_2$, then

$$(1-p^r)^{2r-2n} = \exp(2np^r)\cdot\left(1+O\left(np^{2r}\right)\right) = \left(1+O\left(\frac{\log^3 n}{n}\right)\right),$$

we can write

$$E(X_r^2) = E(Y_r)^2\cdot\left(1+O\left(\frac{\log^3 n}{n}\right)\right). \tag{3.19}$$

Substituting (3.1), (3.14) and (3.19) into (2.2) yields the assertion of our lemma. □

Now we can use the second moment method to estimate the number of cliques of a given order in a random graph.

THEOREM 3.1. *Let $\lfloor r_0\rfloor \leqslant r \leqslant \lceil r_1\rceil$, p be a constant, $0<p<1$, and $G_{n,p}$ be a random graph. Then it holds almost surely*

$$Y_r = \binom{n}{r}\cdot p^{\binom{r}{2}}(1-p^r)^{n-r}\left(1+O\left(\frac{\log^3 n}{\sqrt{n}}\right)\right). \tag{3.20}$$

Proof. The assertion of the theorem follows directly from Chebyshev's inequality, Lemmas 3.1. and 3.2. for $d=\operatorname{Var}(Y_r)\log_b^{3/2} n$. □

The result of the previous theorem enables us to estimate the total number of cliques in a random graph.

COROLLARY. *Let $G_{n,p}$ be a random graph, $Y(n)=\sum_r Y_r$, $\lfloor r_0\rfloor \leqslant r \leqslant \lceil r_1\rceil$, then*

$$Y(n) = n^{\frac{1}{2}\log_b n - \log_b\log_b n + O(1)} \tag{3.21}$$

almost surely.

Proof. Let $G_{n,p}$ be a random graph. According to the Theorem 3.1, the r.v.'s Y_r fulfil the condition (3.20) almost surely (with probability $1-O(1/\log^3 n)$). Therefore the following estimates hold almost surely (with probability $(1-O(1/\log^2 n))$, too:

$$Y(n) = \left(1+O\left(\frac{\log^3 n}{\sqrt{n}}\right)\right)\sum_{\lfloor r_0\rfloor\leqslant r\leqslant\lceil r_1\rceil}\binom{n}{r}\cdot p^{\binom{r}{2}}(1-p^r)^{n-r}.$$

Let $g(r)=\binom{n}{r}\cdot p^{\binom{r}{2}}(1-p^r)^{n-r}$. The sequence $g(r)$ is increasing for $\lfloor r_0\rfloor \leqslant r \leqslant r_3$ and decreasing for $r>r_3$. The point r_3 at which $g(r)$ reaches its maximum lies between $\log_b n - \log_b\log_b n$ and $\log_b n$. (The exact value of r_3 will be found later.) Since

$$g(\log_b n + i) = n^{\frac{1}{2}\log_b n - \log_b\log_b n + O(1)} \quad \text{for} \quad i = -\log_b\log_b n,\dots,(1+\varepsilon)\log_b n,$$

and $(r_1 - r_0) = O(\log n) = O(n)$, the estimate (3.21) is proved. □

The previous study of the behaviour of $g(r)$ discovered an interesting fact: $g(r)$ has a sheer peak in r_3, and consequently, although a random graph contains cliques of order r, $\lfloor r_0 \rfloor \leqslant r \leqslant \lceil r_1 \rceil$, most of its cliques have order $\approx r_3$. We shall study this phenomenon in detail now.

THEOREM 3.2. *Let $Y(n)$ be a random variable on $\mathfrak{G}(n,p)$ denoting the number of cliques in a random graph. Let*

$$r_3 = \log_b n - \log_b \log_b \log_b n + \log_b \log_b \mathrm{e} + \log_b (1-p) + 1 .$$

Then a random graph $G_{n,p}$ has the following properties almost surely

(1) *Almost all cliques in $G_{n,p}$ have order $\lfloor r_3 \rfloor$ or $\lceil r_3 \rceil$;*
(2) *there are*

$$Y(n) = n^{\frac{1}{2}\log_b n - \log_b \log_b n + \frac{1}{2} + \log_b \mathrm{e}} \cdot \left(\log_b n\right)^{\log_b \log_b \log_b n + O(1)}$$

cliques in $G_{n,p}$.

Proof. We can concentrate our effort on the interval

$$\langle \log_b n - (1+\varepsilon) \log_b \log_b n,\ \log_b n \rangle \tag{3.22}$$

since

$$\frac{g(r)}{g(\log_b n)} = \begin{cases} n^{-\log_b \mathrm{e} \cdot \log^{\varepsilon} n + O(1)} & \text{if } \lfloor r_0 \rfloor \leqslant r \leqslant \log_b n - (1+\varepsilon)\log_b \log_b n , \\ O(1/\log_b n)^i & \text{if } r = (\log_b n) + i ,\ \ i = 1, \dots, \log_b n , \end{cases} \tag{3.23}$$

and therefore the contribution to $Y(n)$ of cliques to (3.22) of orders not belonging is negligible with respect to the value $g(\log_b n)$. Therefore we can assume that the point r_3, in which $g(r)$ reaches its maximum, can be expressed in the form $\log_b n - j$. For simplicity, let $g(\log_b n - j) = a_j$, $j = 0, \dots, (1+\varepsilon)\log_b \log_b n$. Now we will study the ratio a_{j+1}/a_j:

$$\frac{a_{j+1}}{a_j} = \frac{\log_b n - j}{n - \log_b n - j + 1} \left(\frac{1}{p}\right)^{\log_b n - j - 1} \frac{\left(1 - \frac{(1/p)^{j+1}}{n}\right)^{n - \log_b n + j + 1}}{\left(1 - \frac{(1/p)^{j}}{n}\right)^{n - \log_b n + j}} . \tag{3.24}$$

We need to find good asymptotic estimates of terms of the product (3.24). Since $j = O(\log \log n)$, using (2.7) we have

$$\frac{a_{j+1}}{a_j} = \log_b n \cdot p^{j+1} \exp\left(-(1/p)^j \left(\frac{p}{1-p}\right)\right) \left(1 + O\left(\frac{\log \log n}{\log n}\right)\right) . \tag{3.25}$$

If

$$j = \log_b \log_b \log_b n - \log_b \log_b \mathrm{e} + \log_b \frac{p}{1-p} + \log\left(1 - \frac{\log_b \log_b \log_b n - \log_b \log_b \mathrm{e} - \log_b(1-p)}{\log_b \log_b n}\right), \tag{3.26}$$

the ratio (3.25) equals asymptotically 1. If j is greater (less) than (3.26), the ratio a_{j+1}/a_j is greater (less) than 1 respectively. Therefore $g(r)$ attains its maximum at the point

$$r_3 = \log_b n - \log_b \log_b \log_b n + \log_b \log_b \mathrm{e} - \log_b \frac{p}{1-p} + \log\left(1 - \frac{\log_b \log_b \log_b n - \log_b \log_b \mathrm{e} - \log_b(1-p)}{\log_b \log_b n}\right).$$

Since the value r_3 is not integer and

$$\log\left(1 - \frac{\log_b \log_b \log_b n - \log_b \log_b \mathrm{e} - \log_b(1-p)}{\log_b \log_b n}\right) = o(1),$$

the function $g(r)$ attains its maximum at integer $\lfloor r_3 \rfloor$ or $\lceil r_3 \rceil$.

Now we can estimate the total number of cliques in $G_{n,p}$. For simplicity, let $g(r_3') = \max\{g(\lfloor r_3 \rfloor), g(\lceil r_3 \rceil)\}$. Then

$$Y(n) = g(r_3') \cdot \sum_r \frac{g(r)}{g(r_3')}.$$

Since $g(r)$ is increasing for $r \leqslant \lfloor r_3 \rfloor$ and decreasing for $r \geqslant \lceil r_3 \rceil$, and moreover

$$g(\lfloor r_3 \rfloor - 1) \cdot \log_b \log_b n = o(g(r_3')),$$
$$g(\lceil r_3 \rceil + 1) \cdot \log_b \log_b \log_b n = o(g(r_3')),$$
$$g(\log_b n) \cdot \log_b n = o(g(r_3'));$$
$$Y(n) = (1 + c_r) \cdot g(r_3'),$$

where c_r is a constant, $0 < c_r \leqslant 1$.

To complete the proof of our theorem, it is sufficient to substitute the values $\lfloor r_3 \rfloor$, $\lceil r_3 \rceil$ into (3.1). □

Remark. The asymptotic bound on $Y(n)$ generalizes and improves one of the results published in [7].

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REFERENCES

[1] BOLLOBÁS, B.: *Random Graphs*, Academic Press, New York, 1985.

[2] BOLLOBÁS, B.—ERDÖS, P.: *Cliques in random graphs*, Math. Proc. Cambridge Philos. Soc. **80** (1976), 419–427.

[3] FARBER, M.—HUJTER, M.—TUZA, ZS.: *An upper bound on the number of cliques in a graph*, Networks **23** (1993), 207–210.

[4] FÜREDI, Z.: *The number of maximal independent sets in connected graphs*, J. Graph Theory **11** (1987), 463–470.

[5] HEDMAN, B.: *The maximum number of cliques in dense graphs*, Discrete Math. **54** (1985), 161–166.

[6] KALBFLEISCH, J. G.: *Complete subgraphs of random hypergraphs and bipartite graphs.* In: Proc. of 3rd Southeastern Conference on Combinatorics, Graph Theory and Computing, Florida Atlantic University, 1972, pp. 297–304.

[7] KORSHUNOV, A. D.: *The basic properties of random graphs with large numbers of vertices and edges*, Uspekhi Mat. Nauk **40** (1985), 107–173. (Russian)

[8] MATULA, D. W.: *On the complete subgraphs of a random graph.* In: Proc. 2nd Chapel Hill Conf. Combinatorial Math. and its Applications (R. C. Bose et al., eds.), Univ. North Carolina, Chapel Hill, 1970, pp. 356–369.

[9] MATULA, D. W.: *The employee party problem*, Notices Amer. Math. Soc. **19** (1972), A-382.

[10] MATULA, D. W.: *The largest clique size in a random graph.* Technical report CS 7608, Dept. of Computer Science, Southern Methodist University, Dallas, 1976.

[11] MOON, J. W.—MOSER, L.: *On cliques in graphs*, Israel J. Math. **3** (1965), 22–28.

[12] PALMER, E. M.: *Graphical Evolution: An Introduction to the Theory of Random Graphs*, John Wiley, New York, 1985.

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