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# A CHARACTERIZATION OF THE DECAY NUMBER OF A CONNECTED GRAPH

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ABSTRACT. We show that the minimum number of components in a cotree of a connected graph $G$ equals the maximum value of the expression $2c(G-A)-1-|A|$, where $A$ is a set of edges of $G$ and $c(G-A)$ denotes the number of components of $G-A$. This invariant was previously studied in [3].

By a graph, we shall mean a multigraph in the sense of [1]. Assume that $G$ is a graph with vertex set $V(G)$ and edge set $E(G)$. Let $W$ be a nonempty subset of $V(G)$; we denote by $\langle W\rangle_G$ the graph $G-\big(V(G)-W\big)$; in other words, $\langle W\rangle_G$ is the subgraph of $G$ induced by $W$. Moreover, we denote by $c(G)$ and $\mathcal{T}(G)$ the number of components of $G$ and the set of all spanning trees of $G$, respectively.

If $G$ is a connected graph, then the *decay number* $\zeta(G)$ of $G$ is defined as follows:

$$\zeta(G) = \min_{T\in\mathcal{T}(G)} c\big(G-E(T)\big)\,.$$

This concept was introduced by Škoviera in [3] and was used for studying the maximum genus of a graph.

The following theorem gives a characterization of the decay number.

**THEOREM.** *Let $G$ be a connected graph. Then*

$$\zeta(G) = \max_{A\subseteq E(G)} \big(2c(G-A)-1-|A|\big)\,.$$

Proof. For every connected graph $H$, we denote

$$z(H) = \max_{A\subseteq E(H)} \big(2c(H-A)-1-|A|\big)\,.$$

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We wish to prove that $\zeta(G) = z(G)$. We proceed by induction on $|V(G)|$. The case when $|V(G)| = 1$ is obvious. Assume that $|V(G)| \geq 2$.

Consider an arbitrary $T \in \mathcal{T}(G)$ and an arbitrary $A \subseteq E(G)$. We denote by $m$ the number of components $F$ of $G - A$ such that $\langle F \rangle_T$ is connected. It is clear that

$$c\big(G - E(T)\big) \geq m - |A - E(T)|\,.$$

Moreover, we see that

$$c(T - A) \geq 2c(G - A) - m\,.$$

Since $T \in \mathcal{T}(G)$, hence

$$c\big(G - E(T)\big) \geq 2c(G - A) - 1 - |A|\,.$$

The above three inequalities imply that $\zeta(G) \geq z(G)$.

It remains to prove that $\zeta(G) \leq z(G)$. We denote by $\mathcal{R}$ the set of all ordered pairs $(T, F)$ such that $T \in \mathcal{T}(G)$, $F$ is a spanning forest of $G - E(T)$ and $c(F) = \zeta(G)$. Clearly, $\mathcal{R} \neq \emptyset$. We distinguish two cases:

*Case* 1. Assume that there exist $(T, F) \in \mathcal{R}$ and $W \subseteq V(G)$ such that both $\langle W \rangle_T$ and $\langle W \rangle_F$ are connected and $|W| \geq 2$. Let $H$ denote the graph obtained from $G - E\big(\langle W \rangle_G\big)$ by identifying the vertices of $W$ into one vertex. According to the induction hypothesis, $\zeta(H) = z(H)$. It is easy to see that $z(H) \leq z(G)$. Since both $\langle W \rangle_T$ and $\langle W \rangle_F$ are connected, we see that for every $T' \in \mathcal{T}(H)$ there exists $T'' \in \mathcal{T}(G)$ such that $E(T') \subseteq E(T'')$ and $c\big(G - E(T'')\big) = c\big(H - E(T')\big)$. We conclude that $\zeta(G) \leq z(G)$.

*Case* 2. Assume that either $\langle W \rangle_T$ or $\langle W \rangle_F$ is disconnected for any $(T, F) \in \mathcal{R}$ and any $W \subseteq V(G)$, $|W| \geq 2$. This means that $\zeta(G) \geq 2$.

Consider an arbitrary $(T, F) \in \mathcal{R}$. Put $F = J_1 = J_3 = J_5 = \dots$ and $T = J_2 = J_4 = J_6 = \dots$. We shall say that a sequence $(G_1, \dots, G_n)$, $n \geq 1$, is a *key* to $(T, F)$ if

(a) $G_1 = G$,
(b) if $n \geq 2$ and $k \in \{2, \dots, n\}$, then there exists a component $L$ of $\langle V(G_{k-1}) \rangle_{J_{k-1}}$ such that $G_k = \langle V(L) \rangle_G$,
(c) there exists $e \in E(G_n) - E(J_{n+1})$ such that $e$ is incident with vertices of distinct components of $\langle V(G_n) \rangle_{J_n}$.

It follows from the definition of $z(G)$ that

$$2|V(G)| - 1 - |E(G)| \leq z(G)\,.$$

Recall that we wish to prove that $\zeta(G) \leq z(G)$. To the contrary, let us assume that $\zeta(G) > z(G)$. Then

$$|E(G)| > \big(|V(G)| - 1\big) + \big(|V(G)| - \zeta(G)\big)\,.$$

Consider an arbitrary $(T_0, F_0) \in \mathcal{R}$. We have

$$|E(G)| > |E(T_0)| + |E(F_0)|.$$

Combining this statement with the assumption of Case 2, we see that there exists a key to $(T_0, F_0)$.

Let $(T, F) \in \mathcal{R}$, and let $(G_1, \dots, G_n)$ be a key to $(T, F)$. Without loss of generality, we assume that, if $n \geq 2$, then $(G_1, \dots, G_{n-1})$ is a key to no $(T^*, F^*) \in \mathcal{R}$. We put $F = J_1 = J_3 = J_5 = \dots$ and $T = J_2 = J_4 = J_6 = \dots$. By the definition of a key, there exists $e \in E(G_n) - E(J_{n+1})$ such that $e$ is incident with vertices of distinct components of $\langle V(G_n) \rangle_{J_n}$. If $n = 1$, then $F + e$ is a spanning forest of $G - E(T)$ and $c(F + e) = \zeta(G) - 1$, which is a contradiction.

Let $n \geq 2$. Then it follows from the definition of a key that $J_n + e$ contains a cycle passing through an edge $e'$ which is incident with vertices of distinct components of $\langle V(G_{n-1}) \rangle_{J_{n-1}}$. Put $J'_n = (J_n - e') + e$. Certainly, $J'_n$ is a spanning forest of $G - E(J_{n-1})$ and $c(J'_n) = c(J_n)$. This implies that either $(J'_n, J_{n-1}) \in \mathcal{R}$, or $(J_{n-1}, J_n) \in \mathcal{R}$. It is clear that $(G_1, \dots, G_{n-1})$ is a key to $(J'_n, J_{n-1})$ or to $(J_{n-1}, J'_n)$, which is a contradiction.

We conclude that $\zeta(G) \leq z(G)$, and this completes the proof of the theorem. □

Remark 1. Škoviera [3] introduced the notion of the decay number for graphs with possible loops, i.e., for pseudographs in the sense of [1]. It is obvious that our theorem can also be extended to pseudographs.

Remark 2. Let $n$ be a positive integer. Tutte [4] and Nash-Williams [2] proved that a graph $G$ has $n$ edge-disjoint spanning trees if and only if $n\big(c(G - A) - 1\big) \leq |A|$, for every $A \subseteq E(G)$. For $n = 2$ this result immediately follows from our theorem.

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