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ON WEAKLY RIGID MONOUNARY ALGEBRAS

DANICA JAKUBÍKOVÁ-STUDENOVSKÁ

An algebra $\mathcal{A}=(A, F)$ is said to be rigid if it has no endomorphisms except the identity mapping. The notion of rigidity has been applied for several types of algebraic structures (cf., e.g., [3] for the case of Boolean algebras and [2] for the case of order types).

A monounary algebra $\mathcal{A}=(A, f)$ is rigid if and only if A is a one-element set. The algebra $\mathcal{A}=(A, f)$ will be said to be weakly rigid if there does not exist any isomorphism of $\mathcal{A}$ into $\mathcal{A}$ except the identity mapping.

Let α be an infinite cardinal. Consider the following condition for the cardinal α:

(c(α)) *There exists a system* $\mathcal{S}=\{(A_i, f)\colon i\in I\}$ *of connected monounary algebras such that*

(i) card $I=2^\alpha$ *and* card $A_i=\alpha$ *for each* $i\in I$;

(ii) *if* $i\in I$, *then there does not exist any isomorphism of* (A_i, f) *onto* (A_i, f) *except the identity mapping;*

(iii) *if* i, j *are distinct elements of* I, *then there does not exist any isomorphism of* (A_i, f) *onto* (A_j, f).

S. D. Comer and J. J. LeTourneau [1] proved that the condition (c(α)) holds for each infinite cardinal α. In this paper it will be shown that for a rather large class of cardinals α a stronger result than (c(α)) is valid.

Let us denote by (d(α)) the condition that we obtain from (c(α)) if we replace in (ii) and (iii) the word „*onto*" by „*into*" (thus, instead of (ii) we use the assumption that each (A_i, f) is weakly rigid). Let M be the class of all infinite cardinals having the property of (d(α)) being valid.

The question whether (d(α)) holds for each infinite cardinal remains open. In this paper there are investigated conditions for a cardinal α under which (d(α)) is valid; the results are summarized in Thm. 2.6.

Let us recall some notions concerning unary algebras. By a monounary algebra we understand a pair (A, f), where A is a nonempty set and f is a unary operation defined on A (i.e., f is a mapping of A into A). A monounary algebra is said to be connected if for each $x, y\in A$ there are positive integers m, n with $f^n(x)=f^m(y)$. By a root monounary algebra (or a root) we mean a connected monounary algebra (A, f) such that A contains an element x with $f(x)=x$.

Let N be the set of all positive integers. We put $N_0 = N \cup \{0\}$.

Let β be a cardinal. We denote $\beta(0) = \beta$ and, for each $n \in N$, we set $\beta(n) = 2^{\beta(n-1)}$. Further we put $\beta(\aleph_0) = \sup \{\beta(n) \colon n \in N_0\}$.

§ 1.

In this paragraph it will be shown that $\aleph_0 \in M$, and that $2^\alpha \in M$ whenever $\alpha \in M$.

Let F be the system of all mappings of N into $\{0, 1\}$.

1.1. Construction. *Let $\bar{f} \in F$. Put $J = \{i \in N \colon \bar{f}(i) = 1\}$. We denote by $D(\bar{f}) = (B, f)$ a root monounary algebra such that*
(1) $B = \{x_i \colon i \in N_0\} \cup \{y_i \colon i \in N_0\} \cup \{z_i \colon i \in N\} \cup \{a, b, c, d\}$;
(2) $f(x_i) = x_{i-1}$, $f(y_i) = y_{i-1}$ *for each* $i \in N$, $f(x_0) = b$, $f(y_0) = d$;
(3) $f(z_i) = x_i$ *for each* $i \in J$, $f(z_i) = y_i$ *for each* $i \in N - J$;
(4) $f(a) = b$, $f(b) = f(d) = f(c) = c$.

(Here and below distinct symbols denote distinct elements. Cf. Fig. 1.) We assume that for $\bar{f}, \bar{g} \in F, \bar{f} \neq \bar{g}$, the universes of the algebras $D(\bar{f})$ and $D(\bar{g})$ are disjoint.

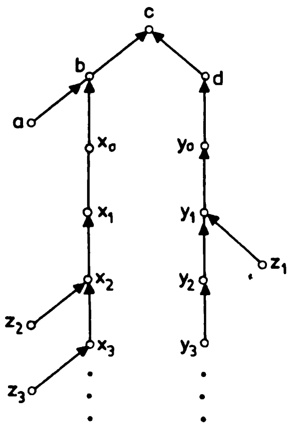

Fig. 1. e.g., $\bar{f} = (0, 1, 1, \ldots)$

1.2. Lemma. *Let $\bar{f}, \bar{g} \in F$. Then we have:*
(a) *If φ is an isomorphism of $D(\bar{f})$ into $D(\bar{g})$, then $\bar{f} = \bar{g}$.*
(b) *If φ is an isomorphism of $D(\bar{f})$ into $D(\bar{g})$, then φ is the identity mapping.*

Proof. Suppose that $D(\bar{f}) = (B, f)$, $D(\bar{g}) = (B', g)$, where

$$B = \{x_i \colon i \in N_0\} \cup \{y_i \colon i \in N_0\} \cup \{z_i \colon i \in N\} \cup \{a, b, c, d\},$$
$$B' = \{x'_i \colon i \in N_0\} \cup \{y'_i \colon i \in N_0\} \cup \{z'_i \colon i \in N\} \cup \{a', b', c', d'\}$$

and the operations f, g are defined according to 1.1. Since c (resp. c') is the only element in $D(\bar{f})$ resp. $D(\bar{g})$) with $f(c)=c$ (resp. $f(c')=c'$) and φ is an isomorphism of $D(\bar{f})$ into $D(\bar{g})$, we obtain $\varphi(c)=c'$. Further, $f^{-1}(c)=\{b, d\}$, $g^{-1}(c')=\{b', d'\}$, $\operatorname{card} f^{-1}(b)=2$, $\operatorname{card} f^{-1}(d)=1$, $\operatorname{card} g^{-1}(b')=2$, $\operatorname{card} g^{-1}(d')=1$, and this implies $\varphi(b)=b'$, $\varphi(d)=d'$. Similarly we can prove that $\varphi(a)=a'$, $\varphi(x_i)=x'_i$, $\varphi(y_i)=y'_i$ for each $i\in N_0$ hold. Further we obtain that $\bar{f}(i)\leqq\bar{g}(i)$ and $(1-\bar{f}(i))\leqq(1-\bar{g}(i))$ for each $i\in N$, since $\varphi(z_i)=z'_i$ for each $i\in N$. Thus we get $\bar{f}=\bar{g}$. Then it is obvious that φ is the identity mapping.

1.3. Lemma. $\aleph_0\in M$.

Proof. Put $\mathcal{S}=\{D(\bar{f})\colon \bar{f}\in F\}$. Then $\operatorname{card}\mathcal{S}=\operatorname{card} F=2^{\aleph_0}$. According to the construction 1.1 we have $\operatorname{card} D(\bar{f})=\aleph_0$ for each $\bar{f}\in F$. By Lemma 1.2 there does not exist any isomorphism of $D(\bar{f})$ into $D(\bar{g})$ whenever $\bar{f}$, $\bar{g}$ are distinct elements of F and Lemma 1.2 implies also that each algebra $D(\bar{f})\in\mathcal{S}$ is weakly rigid. Thus $\aleph_0\in M$.

1.4. Construction. *Suppose that $\mathcal{V}$ is a system of roots with mutually disjoint universes, $\mathcal{V}=\{(B_i, g_i)\colon i\in I\}$. If $\emptyset\neq J\subseteq I$, $\mathcal{U}=\{(B_i, g_i)\colon i\in J\}$, then we denote by $D(\mathcal{U}, \mathcal{V})=(B, g)$ a root monounary algebra such that*

(1) $B=\bigcup_{i\in I}B_i\cup\{a, b, c, d, e, f\}$;

(2) $g(x)=g_i(x)$ *for each* $x\in B_i$, $i\in I$, *if* $g_i(x)\neq x$;

(3) $g(x)=b$ *for each* $x\in B_i$, *whenever* $i\in J$ *and* $g_i(x)=x$;

(4) $g(x)=f$ *for each* $x\in B_i$, *whenever* $i\in I-J$ *and* $g_i(x)=x$;

(5) $g(a)=g(b)=c$, $g(e)=g(c)=g(d)=d$, $g(f)=e$.

(Cf. Fig. 2.) We assume that for each nonempty $\mathcal{U}$, $\mathcal{W}\subseteq\mathcal{V}$, $\mathcal{U}\neq\mathcal{W}$ the algebras $D(\mathcal{U}, \mathcal{V})$ and $D(\mathcal{W}, \mathcal{V})$ have disjoint universes.

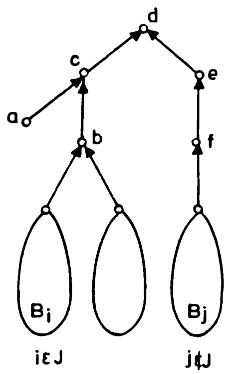

Fig. 2

1.5. Lemma. *Let $\mathcal{V}=\{(B_i, g_i)\colon i\in I\}$ be a system of weakly rigid roots with disjoint universes such that if i, j are distinct elements of I, then there does not exist* *any isomorphism of* (B_i, g_i) *into* (B_j, g_j). *Suppose that* $\mathscr{U}$ *and* $\mathscr{W}$ *are nonempty subsystems of* $\mathscr{V}$. *If* φ *is an isomorphism of* $D(\mathscr{U}, \mathscr{V})$ *into* $D(\mathscr{W}, \mathscr{V})$, *then* $\mathscr{U} = \mathscr{W}$ *and* φ *is the identity mapping.*

Proof. Assume that $\mathscr{U} = \{(B_i, g_i): i \in J\}$, $\mathscr{W} = \{(B_i, g_i): i \in K\}$, $D(\mathscr{U}, \mathscr{V}) = (C, h)$, $D(\mathscr{W}, \mathscr{V}) = (C', h')$, where

$$C = \bigcup_{i \in I} B_i \cup \{a, b, c, d, e, f\},$$
$$C' = \bigcup_{i \in I} B_i \cup \{a', b', c', e', f'\}$$

and the operations h, h' are defined according to the construction 1.4. Let φ be an isomorphism of $D(\mathscr{U}, \mathscr{V})$ into $D(\mathscr{W}, \mathscr{V})$. From 1.4 it follows that $\varphi(d) = d'$, $\varphi(c) = c'$, $\varphi(e) = e'$, $\varphi(b) = b'$, $\varphi(a) = a'$ and $\varphi(f) = f'$. Let $i \in I$ and $x \in B_i$ with $g_i(x) = x$. If $i \in J$, then $h(x) = b$, and since $\varphi(b) = b'$, there exist $k \in K$ and $y \in B_k$ such that $g_k(y) = y$ and $\varphi(x) = y$. For each $v \in B_i$ there is $w \in B_k$ with $\varphi(v) = w$. We put $\varphi'(v) = w$. Then φ' is an isomorphism of (B_i, g_i) into (B_k, g_k). From the fact that there is no isomorphism of one member of $\mathscr{V}$ into another it follows that $i = k$. Thus $J \subseteq K$. Similarly we obtain that $(I - J) \subseteq (I - K)$. Hence $J = K$. Moreover, the mapping φ is the identity, according to the fact that each algebra belonging to $\mathscr{V}$ is weakly rigid.

1.6. Lemma. *If* $\alpha \in M$, *then* $2^\alpha \in M$.

Proof. Let $\alpha \in M$ and let $\mathscr{S}$ be the system of weakly rigid monounary algebras corresponding to α. We denote

$$\mathscr{S}' = \{D(\mathscr{U}, \mathscr{S}) : \emptyset \neq \mathscr{U} \subseteq \mathscr{S}\}.$$

Then we have card $\mathscr{S}' = 2^{\operatorname{card} \mathscr{S}} = 2^{2^\alpha}$. From the construction 1.4 it follows that, for each $D(\mathscr{U}, \mathscr{S}) \in \mathscr{S}'$, card $D(\mathscr{U}, \mathscr{S}) = 2^\alpha$ holds. If $D(\mathscr{U}, \mathscr{S})$, $D(\mathscr{W}, \mathscr{S})$ are distinct algebras belonging to $\mathscr{S}'$, then according to Lemma 1.5 there does not exist any isomorphism of $D(\mathscr{U}, \mathscr{S})$ into $D(\mathscr{W}, \mathscr{S})$. The fact that each algebra $D(\mathscr{U}, \mathscr{S}) \in \mathscr{S}'$ is weakly rigid follows from Lemma 1.5.

§ 2.

In this paragraph we shall use the previous results from § 1 in order to establish two generalizations of Lemma 1.6 (the main results are the assertions (c) and (d) of Thm. 2.6).

2.1. Construction. *We define a fixed monounary algebra* (A, g) *as follows:*

(1) $A = \{a_0, b_0, c_0, d_0, e_0, f_0\} \cup \bigcup_{n \in N} \{a_n, b_n, c_n, d_n, e_n, f_n, a'_n, b'_n, c'_n, d'_n, e'_n, f'_n\}$;

(2) $g(a_i) = g(b_i) = c_i$, $g(a'_i) = g(b'_i) = c'_i$, $g(f_i) = e_i$, $g(f'_i) = e'_i$, $g(c_i) = g(e_i) = d_i$, $g(c'_i) = g(e'_i) = d'_i$ *for each* $i \in N_0$;

(3) $g(d_0) = d_0$, $g(d_1) = b_0$, $g(d'_1) = f_0$, $g(d_i) = b'_{i-1}$, $g(d'_i) = f'_{i-1}$ *for each* $i \in N$, $i > 1$. *(Cf. Fig. 3.)*

2.2. Lemma. *If $\beta \in M$, then $\beta(\aleph_0) \in M$.*

Proof. We suppose that $\beta = \beta(0) \in M$. From Lemma 1.6 it follows (by induction) that $\beta(n) \in M$ for each $n \in N_0$. For $n \in N_0$ let $\mathscr{S}(n)$ be the system of monounary algebras corresponding to $\beta(n)$. We can assume that all algebras of these systems have disjoint universes and we can denote the corresponding unary operation in each of these monounary algebras by the same symbol g. Let Γ by the system of all sequences $\mathscr{T} = \{\mathscr{T}(n)\}_{n \in N_0}$ such that $\emptyset \neq \mathscr{T}(n) \subseteq \mathscr{S}(n)$.

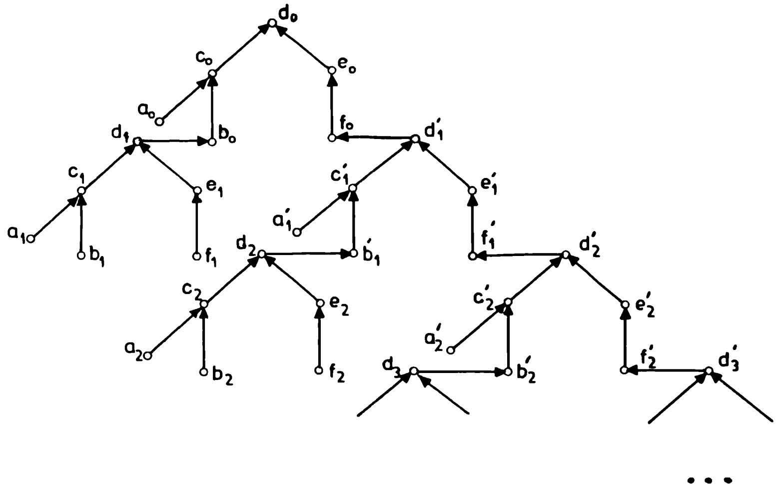

Fig. 3

We need the following construction:

2.2.1. Let $\mathscr{T} \in \Gamma$. Consider the set $A(\mathscr{T})$ of all ordered pairs of the form $(x, \mathscr{T})$, where

$$x \in A \cup \bigcup \{B: (B, g) \in \mathscr{S}(n),\ n \in N_0\}.$$

We introduce a unary operation $g = g(\mathscr{T})$ on the set $A(\mathscr{T})$ as follows:

(a) If $x \in A$, then we put $g((x, \mathscr{T})) = (g(x), \mathscr{T})$.

(b) Let $x \in A(\mathscr{T}) - A$. Then there are $n \in N_0$ and $(B, g) \in \mathscr{S}(n)$ with $x \in B$. We distinguish two cases:

(b 1) If $g(x) \neq x$, then we put $g((x, \mathscr{T})) = (g(x), \mathscr{T})$.

(b 2) Suppose that $g(x) = x$. If $(B, g) \in \mathscr{T}(n)$, then $g((x, \mathscr{T})) = (b_{n+1}, \mathscr{T})$. If $(B, g) \notin \mathscr{T}(n)$, then $g((x, \mathscr{T})) = (f_{n+1}, \mathscr{T})$.

Since for each $n \in N_0$ and each $(B, g) \in \mathscr{S}(n)$ the algebra (B, g) is a root algebra, from the construction of $(A(\mathscr{T}), g)$ it follows that $(A(\mathscr{T}), g)$ is connected and it is a root monounary algebra.

Now we prove the following statement:

2.2.2. *Le* $\mathcal{T}$, $\mathcal{R} \in \Gamma$. *Suppo e that* φ *is an isomorphism of* $(A(\mathcal{T}), g)$ *into* $(A(\mathcal{R}), g)$. *Then* $\mathcal{R} = \mathcal{T}$ *and* φ *is the identity mapping.*

Proof. From the construction 2.2.1 it follows that $\varphi((d_0, \mathcal{T})) = (d_0, \mathcal{R})$, $\varphi((c_0, \mathcal{T})) = (c_0, \mathcal{R})$, $\varphi((e_0, \mathcal{T})) = (e_0, \mathcal{R})$, $\varphi((a_0, \mathcal{T})) = (a_0, \mathcal{R})$, $\varphi((b_0, \mathcal{T})) = (b_0, \mathcal{R})$ and $\varphi((f_0, \mathcal{T})) = (f_0, \mathcal{R})$. Then $\varphi((d_1, \mathcal{T})) = (d_1, \mathcal{R})$ and $\varphi((d'_1, \mathcal{T})) = (d'_1, \mathcal{R})$. By induction we obtain that $\varphi((x, \mathcal{T})) = (x, \mathcal{R})$ for each $x \in A$.

Let $n \in N_0$, $(B, g) \in \mathcal{S}(n)$, $x \in B$ with $g(x) = x$. If $(B, g) \in \mathcal{T}(n)$, then $g((x, \mathcal{T})) = (b_{n+1}, \mathcal{T})$ and since $\varphi((b_{n+1}, \mathcal{T})) = (b_{n+1}, \mathcal{R})$, there exist $(C, g) \in \mathcal{R}(n)$ and $y \in C$ such that $g(y) - y$ and $\varphi((x, \mathcal{T})) = (y, \mathcal{R})$. If $v \in B$, then there is $w \in C$ with $\varphi((v, \mathcal{T})) = (w, \mathcal{R})$. We put $\varphi'(v) = w$. Hence φ' is an isomorphism of (B, g) into (C, g). From the fact that there is no isomorphism of one member of $\mathcal{S}(n)$ into another and that each algebra belonging to $\mathcal{S}(n)$ is weakly rigid it follows that $(B, g) = (C, g)$ and that φ' is the identity mapping. Thus $\varphi((v, \mathcal{T})) = (v, \mathcal{R})$ for each $v \in B$ and we obtain $\mathcal{T}(n) \subseteq \mathcal{R}(n)$. In the case when $(B, g) \in \mathcal{S}(n) - \mathcal{T}(n)$, the relat on $\varphi((v, \mathcal{T})) = (v, \mathcal{R})$ for each $v \in B$ can be obtained in the same manner (with the distinction that e use the relations $g((x, \mathcal{T})) = (f_{n+1}, \mathcal{T})$ and $\varphi((f_{n+1}, \mathcal{T})) = (f_{n+1}, \mathcal{R})$); then we have $(\mathcal{S}(n) - \mathcal{T}(n)) \subseteq (\mathcal{S}(n) - \mathcal{R}(n))$. Hence $\mathcal{T} = \mathcal{R}$ and φ is the identity mapping.

Now we denote $\mathcal{S} = \{(A(\mathcal{T}), g) : \emptyset \neq \mathcal{T}(n) \subseteq \mathcal{S}(n)$ for each $n \in N_0\}$, $\alpha = \beta(\aleph_0)$. From the construction 2.2.1 it follows that card $A(\mathcal{T}) = \alpha$ and also that card $\mathcal{S} = 2^{\alpha}$. With respect to 2.2.2 we obtain $\alpha \in M$.

Consider the following condition:

2.3. Condition. *For each strictly increasing sequence of cardinals* $\{\alpha_n\}_{n \in N}$ *and for each* $i \in N$ *there is* $j \in N$ *such that* $2^{\alpha_i} \leqq \alpha_j$.

The condition 2.3 is fulfilled if the continuum hypothesis is valid, but it is not equivalent with the continuum hypothesis.

2.4. Remark. *If we suppose that the condition 2.3 is satisfied, the following result can be proved by the same method as in 2.2. Let* $\{\alpha_i\}_{i \in N_0}$ *be a sequence of cardinals belonging to M. Then* $\beta = \sup \{\alpha_i : i \in N_0\}$ *also belongs to M.*

Proof. If $n \in N_0$, we denote by $\mathcal{S}(n)$ the system corresponding to α_n. Construct, like in the proof of Lemma 2.2, the system $\mathcal{S} = \{(A(\mathcal{T}), g) : \emptyset \neq \mathcal{T}(n) \subseteq \mathcal{S}(n)$ for each $n \in N_0\}$. Each algebra $(A(\mathcal{T}), g) \in \mathcal{S}$ is then weakly rigid. If $\mathcal{A}, \mathcal{B} \in \mathcal{S}$, $\mathcal{A} \neq \mathcal{B}$, then there does not exist any isomorphism of $\mathcal{A}$ into $\mathcal{B}$. Since the condition 2.3 is satisfied, we have

$$\begin{aligned} \text{card } A(\mathcal{T}) &= \text{card } A + \sup \{\alpha_n \cdot \text{card } \mathcal{S}(n) : n \in N_0\} = \\ &= \sup \{2^{\alpha_n} : n \in N_0\} = \sup \{\alpha_n : n \in N\} = \beta, \quad \text{card } \mathcal{S} = 2^{\beta}. \end{aligned}$$

Hence $\beta \in M$.

2.5. Lemma. *Suppose that the condition 2.3 is satisfied. Let* γ *be an ordinal such that* card $\gamma \in \{\aleph_0(n) : n \in N_0\}$ *and let* $\alpha_i \in M$ *for each* $i \in \gamma$. *Then* $\sup \{\alpha_i : i \in \gamma\} \in M$.

Proof. Let card $\gamma=\aleph_0(n)$, $n\in N$. (For the case $n=0$ cf. the remark 2.4.) If $i\in\gamma$, let $\mathscr{S}(i)$ be the system of monounary algebras corresponding to α_i. We can assume that all algebras of these systems have disjoint universes and so we can denote the unary operations in them by the same symbol g. Denote $\beta=\sup\{\alpha_i: i\in\gamma\}$.

Let K_1 be the set of all mappings of N into $\{0, 1\}$ and, for each $i\in N$, $i>1$, let K_i be the set of all mappings of the set $K_{i-1}\times K_{i-2}\times\ldots\times K_1\times N$ into $\{0, 1\}$. Obviously we have card $K_i=\aleph_0(i)$ for each $i\in N$. Hence there exists a one-to-one mapping η of γ onto the set $K_n\times\ldots\times K_1\times N$.

2.5.1. Construction. Let o, p, r, s, t, u, w, v be distinct elements. Denote

$$W=\{o, p, r, s, t, u, w\},$$
$$B=\{v\}\cup K_n\cup K_n\times K_{n-1}\cup\ldots\cup K_n\times\ldots\times K_1\cup$$
$$\cup A\times K_n\times\ldots\times K_1\cup W\times K_n\times\ldots\times K_1\times N\times\{1, 2, \ldots, n\}.$$

(We may assume that the summands in the expression defining B are mutually disjoint.) Further let A have the same meaning as in the construction 2.1. We define a unary operation g on the set B as follows:

(a) We put $g(v)=v$, $g(k_n)=v$, $g((k_n, k_{n-1}))=k_n$, $\ldots$ $g((k_n, \ldots, k_2, k_1))=(k_n, \ldots, k_2)$ for each $(k_n, \ldots, k_1)\in K_n\times\ldots\times K_1$.

(b) Let $(k_n, \ldots, k_1, i, j)\in K_n\times\ldots\times K_1\times N\times\{1, \ldots, n\}$. We set $g((d_0, k_n, \ldots, k_1))=(k_n, \ldots, k_1)$. If $x\in A$, $x\neq d_0$, then we put $g((x, k_n, \ldots, k_1))=(g(x), k_n, \ldots, k_1)$. Further we set $g((o, k_n, \ldots, k_1, i, j))=g((p, k_n, \ldots, k_1, i, j))=(r, k_n, \ldots, k_1, i, j)$, $g((u, k_n, \ldots, k_1, i, j))=(t, k_n, \ldots, k_1, i, j)$, $g((r, k_n, \ldots, k_1, i, j))=g((t, k_n, \ldots, k_1, i, j))=(s, k_n, \ldots, k_1, i, j)$.

(c) Let $(k_n, \ldots, k_1, i)\in K_n\times\ldots\times K_1\times N$. Then we distinguish the following cases:

(c 1) Let $m\in\{2, \ldots, n\}$. If $k_m((k_{m-1}, \ldots, k_1, i))=0$, then $g((s, k_n, \ldots, k_1, i, m))=(o, k_n, \ldots, k_1, i, m-1)$ and $g((w, k_n, \ldots, k_1, i, m))=(u, k_n, \ldots, k_1, i, m-1)$. If $k_m((k_{m-1}, \ldots, k_1, i))=1$, then $g((s, k_n, \ldots, k_1, i, m))=(u, k_n, \ldots, k_1, i, m-1)$ and $g((w, k_n, \ldots, k_1, i, m))=(o, k_n, \ldots, k_1, i, m-1)$.

(c 2) If $k_1(i)=0$, then $g((s, k_n, \ldots, k_1, i, 1))=(b_i, k_n, \ldots, k_1)$ and $g((w, k_n, \ldots, k_1, i, 1))=(f_i, k_n, \ldots, k_1)$. If $k_1(i)=1$, then $g((s, k_n, \ldots, k_1, i, 1))=(f_i, k_n, \ldots, k_1)$ and $g((w, k_n, \ldots, k_1, i, 1))=(b_i, k_n, \ldots, k_1)$.

2.5.2. Construction. Let Γ be the set of all γ-sequences $\mathscr{T}=\{\mathscr{T}(i)\}_{i\in\gamma}$ such that $\emptyset\neq\mathscr{T}(i)\subseteq\mathscr{S}(i)$ for each $i\in\gamma$. Let $\mathscr{T}\in\Gamma$. Consider the set $B(\mathscr{T})$ of all ordered pairs of the form $(y, \mathscr{T})$, where

$$y\in B\cup\bigcup\{C: (C, g)\in\mathscr{S}(i), i\in\gamma\}.$$

(We shall write $(x, k_n, \ldots, k_1, \mathscr{T})$ instead of $((x, k_n, \ldots, k_1), \mathscr{T})$, and similarly for all elements of B.) We introduce a unary operation g on the set $B(\mathscr{T})$ as follows:

(a) If $y \in B$, we put $g((y, \mathcal{T})) = (g(y), \mathcal{T})$.

(b) Let $y \in B(\mathcal{T}) - B$. Then there are $i \in \gamma$ and $(C, g) \in \mathcal{S}(i)'$ with $y \in C$. We distinguish two cases:

(b 1) If $g(y) \neq y$, then we set $g((y, \mathcal{T})) = (g(y), \mathcal{T})$.

(b 2) Suppose that $g(y) = y$. If $(C, g) \in \mathcal{T}(i)$, then $g((y, \mathcal{T})) = (o, \eta(i), n, \mathcal{T})$. If $(C, g) \notin \mathcal{T}(i)$, then $g((y, \mathcal{T})) = (u, \eta(i), n, \mathcal{T})$.

We proceed by proving the following assertion:

2.5.3. *Let $\mathcal{T}$, $\mathcal{R} \in \Gamma$ and suppose that φ is an isomorphism of $(B(\mathcal{T}), g)$ into $(B(\mathcal{R}), g)$. Then $\mathcal{R} = \mathcal{T}$ and φ is the identity mapping.*

Proof. From the constructions 2.5.1 and 2.5.2 it follows that $\varphi((v, \mathcal{T})) = (v, \mathcal{R})$. Let $(k_n, \ldots, k_1) \in K_n \times \ldots \times K_1$. Since $g^{-1}((v, \mathcal{T})) = (K_n, \mathcal{T})$, $g^{-1}((v, \mathcal{R})) = (K_n, \mathcal{R})$, there is $l_n \in K_n$ with $\varphi((k_n, \mathcal{T})) = (l_n, \mathcal{R})$. Further we get that there is $l_{n-1} \in K_{n-1}$ with $\varphi((k_n, k_{n-1}, \mathcal{T})) = (l_n, l_{n-1}, \mathcal{R})$. By induction there is $(l_n, \ldots, l_1) \in K_n \times \ldots \times K_1$ such that

$$\varphi((k_n, \ldots, k_m, \mathcal{T})) = (l_n, \ldots, l_m, \mathcal{R})$$

for each $m \in N$, $1 \leqq m \leqq n$.

By a reasoning analogous to that in the proof of 2.2.2 we obtain $\varphi((x, k_n, \ldots, k_1, \mathcal{T})) = (x, l_n, \ldots, l_1, \mathcal{R})$ for each $x \in A$.

Now suppose that $k_1 \neq l_1$, i.e., there is $i \in N$ with $k_1(i) \neq l_1(i)$. Hence one of the following two cases occurs:

(a) $g((s, k_n, \ldots, k_1, i, \mathcal{T})) = (b_i, k_n, \ldots, k_1, \mathcal{T})$, $g((w, k_n, \ldots, k_1, i, 1, \mathcal{T})) = (f_i, k_n, \ldots, k_1, \mathcal{T})$, $g((s, l_n, \ldots, l_1, i, 1, \mathcal{R})) = (f_i, l_n, \ldots, l_1, \mathcal{R})$, $g((w, l_n, \ldots, l_1, i, 1, \mathcal{R})) = (b_i, l_n, \ldots, l_1, \mathcal{R})$;

(b) $g((s, k_n, \ldots, k_1, i, 1, \mathcal{T})) = (f_i, k_n, \ldots, k_1, \mathcal{T})$, $g((w, k_n, \ldots, k_1, i, 1, \mathcal{T})) = (b_i, k_n, \ldots, k_1, \mathcal{T})$, $g((s, l_n, \ldots, l_1, i, 1, \mathcal{R})) = (b_i, l_n, \ldots, l_1, \mathcal{R})$, $g((w, l_n, \ldots, l_1, i, 1, \mathcal{R})) = (f_i, l_n, \ldots, l_1, \mathcal{R})$. We shall consider the case (a) (the case (b) being analogous). In this case we have $g^{-2}((b_i, k_n, \ldots, k_1, \mathcal{T})) = \{(t, k_n, \ldots, k_1, i, 1, \mathcal{T}), (r, k_n, \ldots, k_1, i, 1, \mathcal{T})\}$, $g^{-2}((b_i, l_n, \ldots, l_1, \mathcal{R})) = \emptyset$, which is a contradiction, since we have already proved that $\varphi((b_i, k_n, \ldots, k_1, \mathcal{T})) = (b_i, l_n, \ldots, l_1, \mathcal{R})$. Thus $k_1(i) = l_1(i)$ for each $i \in N$, and according to (c 2) we have also $\varphi((x, k_n, \ldots, k_1, i, 1, \mathcal{T})) = (x, l_n, \ldots, l_1, i, 1, \mathcal{R})$ for each $x \in W$, $i \in N$.

Suppose that $k_2 \neq l_2$, i.e., that there are $i \in N$ and $k_1' \in K_1$ with $k_2((k_1', i)) \neq l_2((k_1', i))$. Consider the element $(k_n, k_{n-1}, \ldots, k_2, k_1')$. Then we have $\varphi((k_n, \mathcal{T})) = (l_n, \mathcal{R})$, $\varphi((k_n, k_{n-1}, \mathcal{T})) = (l_n, l_{n-1}, \mathcal{R})$, $\ldots$, $\varphi((k_n, \ldots, k_2, \mathcal{T})) = (l_n, \ldots, l_2, \mathcal{R})$. Since $(k_n, \ldots, k_2, k_1', \mathcal{T}) \in g^{-1}((k_n, \ldots, k_2, \mathcal{T}))$ and $g^{-1}((l_n, \ldots, l_2, \mathcal{R})) \subseteq (\{(l_n, \ldots, l_2)\} \times K_1, \mathcal{R})$, there exists $l_1' \in K_1$ with $\varphi((k_n, \ldots, k_2, k_1', \mathcal{T})) = (l_n, \ldots, l_2, l_1', \mathcal{R})$. Similarly as above we have $\varphi((x, k_n, \ldots, k_2, k_1', \mathcal{T})) = (x, l_n, \ldots, l_2, l_1', \mathcal{R})$ for each $x \in A$ and we obtain also that $k_1' = l_1'$ and that $\varphi((x, k_n, \ldots, k_2, k_1', i, 1, \mathcal{T})) = (x, l_n, \ldots, l_2, l_1', i, 1,$

$\mathscr{R}$) for each $x \in W$. Since we assume that $k_2((k_1', i)) \neq l_2((l_1', i))$, the following two cases are possible:

(a) $g((s, k_n, \ldots, k_2, k_1', i, 2, \mathscr{T})) = (o, k_n, \ldots, k_2, k_1', i, 1, \mathscr{T})$, $g((w, k_n, \ldots, k_2, k_1', i, 2, \mathscr{T})) = (u, k_n, \ldots, k_2, k_1', i, 1, \mathscr{T})$, $g((s, l_n, \ldots, l_2, l_1', i, 2, \mathscr{R})) = (u, l_n, \ldots, l_2, l_1', i, 1, \mathscr{R})$, $g((w, l_n, \ldots, l_2, l_1', i, 2, \mathscr{R})) = (o, l_n, \ldots, l_2, l_1', i, 1, \mathscr{R})$;

(b) $g((s, k_n, \ldots, k_2, k_1', i, 2, \mathscr{T})) = (u, k_n, \ldots, k_2, k_1', i, 1, \mathscr{T})$, $g((w, k_n, \ldots, k_2, k_1', i, 2, \mathscr{T})) = (o, k_n, \ldots, k_2, k_1', i, 1, \mathscr{T})$, $g((s, l_n, \ldots, l_2, l_1', i, 2, \mathscr{R})) = (o, l_n, \ldots, l_2, l_1' i, 1, \mathscr{R})$, $g((w, l_n, \ldots, l_2, l_1', i, 2, \mathscr{R})) = (u, l_n, \ldots, l_2, l_1', i, 1, \mathscr{R})$. In the case (a) (the case (b) being analogous) we have $g^{-2}((o, k_n, \ldots, k_2, k_1', i, 1, \mathscr{T})) = \{(t, k_n, \ldots, k_2, k_1', i, 2, \mathscr{T}), (r, k_n, \ldots, k_2, k_1', i, 2, \mathscr{T})\}$, $g^{-2}((o, l_n, \ldots, l_2, l_1', i, 1, \mathscr{R})) = \emptyset$, and this is a contradiction with $\varphi((o, k_n, \ldots, k_2, k_1', i, 1, \mathscr{T})) = (o, l_n, \ldots, l_2, l_1', i, 1, \mathscr{R})$.

Thus we have proved that $k_2 = l_2$, and by induction it can be shown that $l_m = k_m$ for each $m = 1, 2, \ldots, n$ and that the relation

$$\varphi((x, k_n, \ldots, k_1, i, j, \mathscr{T})) = (x, k_n, \ldots, k_1, i, j, \mathscr{R})$$

for each $x \in W$, $i \in N$, $j = 1, \ldots, n$ holds. Hence we have

$$\varphi((y, \mathscr{T})) = (y, \mathscr{R}) \quad \text{for each} \quad y \in B.$$

Suppose that $\mathscr{T} \neq \mathscr{R}$, i.e., there is $i \in \gamma$ with $\mathscr{T}(i) \neq \mathscr{R}(i)$. Let $(C, g) \in \mathscr{S}(i)$, $y \in C$ such that $g(y) = y$. Assume that $(C, g) \in \mathscr{T}(i)$. Then $g((y, \mathscr{T})) = (o, \eta(i), n, \mathscr{T})$ and since $\varphi((o, \eta(i), n, \mathscr{T})) = (o, \eta(i), n, \mathscr{R})$, there exist $(D, g) \in \mathscr{R}(\imath)$ and $z \in D$ such that $g(z) = z$ and $\varphi((y, \mathscr{T})) = (z, \mathscr{R})$. Let $y' \in C$. Then there is $z' \in D$ with $\varphi((y', \mathscr{T})) = (z', \mathscr{R})$. We put $\varphi'(y') = z'$. Hence φ' is an isomorphism of (C, g) into (D, g). From the fact that there is no isomorphism of one member of $\mathscr{S}(i)$ into another and that each algebra belonging to $\mathscr{S}(i)$ is weakly rigid it follows that $(C, g) = (D, g)$ and that φ' is the identity mapping. Thus $\varphi((y', \mathscr{T})) = (y', \mathscr{R})$ for each $y' \in C$ and we have $\mathscr{T}(i) \subseteq \mathscr{S}(i)$. In the case when $(C, g) \in \mathscr{S}(i) - \mathscr{T}(i)$, the relation $\varphi((y', \mathscr{T})) = (y', \mathscr{R})$ for each $y' \in C$ can be obtained similarly, only we use the fact that $g((y, \mathscr{T})) = (u, \eta(i), n, \mathscr{T})$ and $\varphi((u, \eta(i), n, \mathscr{T})) = (u, \eta(i), n, \mathscr{R}))$; then $(\mathscr{S}(i) - \mathscr{T}(i)) \subseteq (\mathscr{S}(i) - \mathscr{R}(i))$. Hence we have proved that $\mathscr{T} = \mathscr{R}$ and that φ is the identity mapping.

Now we denote $\mathscr{S} = \{(B(\mathscr{T}), g) \colon \mathscr{T} \in \Gamma\}$. Then card $\mathscr{S} =$ card $\Gamma = 2^\beta$ and card $B(\mathscr{T}) = \sup\{\alpha_i \cdot \text{card}\, \mathscr{S}(i) \colon i \in \gamma\} = \sup\{2^{\alpha_i} \colon i \in \gamma\} = \sup\{\alpha_i \colon i \in \gamma\} = \beta$ (we have used the condition 2.3). Further, from 2 5.3 it follows that each algebra belonging to $\mathscr{S}$ is weakly rigid and that there does not exist any isomorphism of one member of $\mathscr{S}$ into another. Therefore $\beta \in M$ and the proof of Lemma 2.5 is complete.

2.6. Theorem. (a) $\aleph_0 \in M$.

(b) *If* $\alpha \in M$, *then* $2^\alpha \in M$.

(c) *If* $\alpha \in M$, *then* $\alpha(\aleph_0) \in M$.

(d) *Let the condition 2.3 be fulfilled and let* γ *be an ordinal such that* card $\gamma \leqq \aleph_0(\aleph_0)$. *If* $\alpha_i \in M$ *for each* $i \in \gamma$, *then* $\sup\{\alpha_i \colon i \in \gamma\} \in M$.

Proof. According to Lemmas 1.3, 1.6 and 2.2 we have (a), (b) and (c). The result (d) follows from Lemma 2.5 and from the fact that

$$\aleph_0(\aleph_0) = \sup \{\aleph_0(k) : k \in N\}.$$

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О СЛАБО ЖЕСТКИХ МОНОУНАРНЫХ АЛГЕБРАХ

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Резюме

Алгебра $\mathcal{A}$ называется слабо жесткой, если не существует изоморфизм $\mathcal{A}$ в $\mathcal{A}$ кроме тождественного изоморфизма. В этой статье исследуются системы $\{\mathcal{A}_i : i \in I\}$ слабо жестких моноунарных алгебр такие, что если $i, j \in I$, $i \neq j$, тогда не существует изоморфизм $\mathcal{A}_i$ в $\mathcal{A}_j$.