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THE BROOKS-JEWETT THEOREM FOR k-TRIANGULAR FUNCTIONS ON DIFFERENCE POSETS AND ORTHOALGEBRAS

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ABSTRACT. We introduce k-triangular functions on difference posets and we prove a Brooks-Jewett-type theorem for such functions that are defined on a difference poset (or an effect algebra) satisfying the weak subsequential interpolation property. This result enables us to obtain the previously known Brooks-Jewett theorems for orthoalgebras and orthomodular lattices.

1. Introduction

The events of a quantum-mechanical system $\mathcal{S}$ can be represented by (self-adjoint) projections on a separable complex Hilbert space $\mathcal{H}$ ([8]). The set $L(\mathcal{H})$ of all such projections forms a (complete) lattice which is the prototypical example of orthomodular lattices and is used as a mathematical model in the quantum logic approach to the mathematical foundations of quantum mechanics ([1], [12]).

On the other hand, the effects of the quantum-mechanical system $\mathcal{S}$ can be represented by self-adjoint operators A on $\mathcal{H}$ such that $O \leq A \leq I$, where O, I are respectively the zero and identity operators on $\mathcal{H}$ ([5]). The set $\mathcal{E}(H)$ of all such operators A forms a weaker algebraic structure which is the prototypical example of the effect algebras and difference posets discussed in this paper and originally introduced in [5], [14], [13], [3], and it provides a mathematical model for the study of unsharp quantum logics ([5]).

In this paper, we introduce a weak notion of σ-orthocompleteness for difference posets (or effect algebras), namely, the Weak Subsequential Interpolation

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Property (see Definition 3.2), and we prove a Brooks-Jewett theorem (see Theorem 4.4) for k-triangular and s-bounded (i.e., exhaustive) functions defined on such difference posets with values in a triangular semigroup (see Definition 4.1). This result generalizes Guariglia's result [7; (3.2)]. Furthermore, we obtain, as a consequence of this result, a Brooks-Jewett theorem for semigroup-valued additive and s-bounded measures defined on difference posets having the Weak Subsequential Interpolation Property (see Theorem 4.7 and the remarks following it) which yields Theorem 4.1 of [9], the result (5.1) of [2], and the result (4.2) of [7] as special cases.

Throughout this paper, the symbols $\mathcal{P}(X)$, $\mathcal{F}(X)$, and $\mathcal{I}(X)$ denote, respectively, the set of all subsets, all finite subsets, and all infinite subsets of a set X. The symbols $\mathbb{R}$, $\mathbb{Z}$ and ω denote, respectively, the set of all real numbers, all integers, and all nonnegative integers. The notation $:=$ means "equals by definition".

2. Effect algebras and difference posets

Foulis and Bennett [5] have introduced the following definition.

2.1. Definition. An *effect algebra* is a system $(L, \oplus, 0, 1)$ consisting of a set L containing two special elements 0, 1 and equipped with a partially defined binary operation $\oplus$ satisfying the following conditions $\forall a, b, c \in L$:

(EA1) (*Commutative Law*) If $a \oplus b$ is defined, then $b \oplus a$ is defined and $a \oplus b = b \oplus a$.

(EA2) (*Associative Law*) If $b \oplus c$ is defined and $a \oplus (b \oplus c)$ is defined, then $a \oplus b$ is defined, $(a \oplus b) \oplus c$ is defined, and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

(EA3) (*Orthocomplementation Law*) For every $a \in L$ there exists a unique $b \in L$ such that $a \oplus b$ is defined and $a \oplus b = 1$.

(EA4) (*Zero-One Law*) If $1 \oplus a$ is defined, then $a = 0$.

We shall write L for the effect algebra $(L, \oplus, 0, 1)$ if there is no danger of misunderstanding. Let L be an effect algebra and $a, b \in L$. Following [5], we say that a is *orthogonal* to b in L and write $a \perp b$ if and only if $a \oplus b$ is defined in L. We define $a \leq b$ to mean that there exists $c \in L$ such that $a \perp c$ and $b = a \oplus c$. The unique element $b \in L$ corresponding to a in Condition (EA3) above is called the *orthocomplement* of a and is written as $a' := b$. For any effect algebra L, it can be easily proved (see [5]) that $0 \leq a \leq 1$ holds for all $a \in L$, that $a \perp b$ if and only if $a \leq b'$, that, with $\leq$ as defined above, $(L, \leq, 0, 1)$ is a partially ordered set (poset), and that L satisfies the so-called *orthomodular* *identity* (OMI):

$$\forall\, a,b \in L\,, \quad a \le b \implies b = a \oplus (a \oplus b')'\,.$$

For $a,b \in L$, a is called a *subelement* of b if and only if $a \le b$. If a is a subelement of b, then, by the OMI, $b = a \oplus (a \oplus b')'$. In this case, we define the *difference* $b \ominus a$ by

$$b \ominus a := (a \oplus b')'\,. \tag{1}$$

2.2. EXAMPLE. Consider the set $\mathcal{E}(\mathcal{H})$ of all self-adjoint operators A on a Hilbert space $\mathcal{H}$ with $O \le A \le I$, where O and I are the zero and identity operators, respectively, on $\mathcal{H}$. For $A, B \in \mathcal{E}(\mathcal{H})$, define

$$A \oplus B := A + B \iff A + B \le I\,.$$

It is not difficult to show that, under this $\oplus$, the system $\big(\mathcal{E}(H), \oplus, O, I\big)$ forms an effect algebra [5].

More generally, if V is an ordered real vector space ordered by the usual positive cone $V^+ = \{x \in V:\ x \ge 0\}$, then

$$V^+[0,y] := \{x \in V^+:\ 0 \le x \le y\}$$

forms an effect algebra under the obvious $\oplus$ operation. In particular, the interval $\mathbb{R}^+[0,1] = \{r \in \mathbb{R}:\ 0 \le r \le 1\}$ forms an effect algebra.

According to [5], the algebra $\mathcal{E}(H)$ serves as the archetypical effect algebra, which motivates the study of effect algebras and unsharp quantum logics. Navara and Pták [14], Dvurečenskij and Riečan [3], and Kôpka and Chovanec [13] have introduced the following definition, which is also motivated by the structure of $\mathcal{E}(H)$.

2.3. DEFINITION. Let $(P, \le, 0, 1)$ be a poset with 0, 1 and define

$$D(\ominus) := \big\{(a,b):\ a,b \in P \text{ with } a \le b\big\}\,.$$

The poset $(P, \le, 0, 1)$ is called a *difference poset* (DP) if $\ominus\colon D(\ominus) \to P$ satisfies

(DP1) $a \ominus 0 = a \quad \forall\, a \in P$,

(DP2) if $a \le b \le c$, then $c \ominus b \le c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.

2.4. PROPOSITION. *Let P be a DP and let $a,b \in P$ with $a \le b$. Then*

(i) $b \ominus a \le b$ *and* $b \ominus (b \ominus a) = a$;

(ii) $b \ominus b = 0$;

(iii) $1 \ominus b \le 1 \ominus a$;

(iv) $1 \ominus (1 \ominus b) = b$.

Proof.

(i) By (DP2), $0 \le a \le b$ implies $b \ominus a \le b \ominus 0$ and $(b \ominus 0) \ominus (b \ominus a) = a \ominus 0$, so (DP1) implies $b \ominus a \le b$ and $b \ominus (b \ominus a) = a$.

Statements (ii) – (iv) have been proved in [13]. □

Let P be a DP. Define a unary operation $'$: $P \to P$ by $a' := 1 \ominus a$. By Proposition 2.4, we have $a'' = a \quad \forall a \in P$ and $b' \le a'$ whenever $a \le b$ in P. Two elements $a, b \in P$ are said to be *orthogonal* and we write $a \perp b$ if and only if $a \le b'$ (if and only if $b \le a'$). Define

$$D(\oplus) := \{(a,b) : \ a, b \in P \text{ with } a \perp b\},$$

and define $\oplus$: $D(\oplus) \to P$ by

$$a \oplus b := (b' \ominus a)' . \tag{2}$$

The following result has been proven in [14], [5].

2.5. THEOREM. *Let $(P, \le, 0, 1, \ominus)$ be a difference poset. Then $(P, \le, 0, 1, \oplus)$, where $\oplus$ is defined by (2) above, is an effect algebra. Conversely, let $(L, \le, 0, 1, \oplus)$ be an effect algebra. Then $(L, \le, 0, 1, \ominus)$, where $\ominus$ is defined by (1) above, is a difference poset.*

By Theorem 2.5, *difference posets and effect algebras are the same thing.*

2.6. DEFINITION. A subset P_1 of a difference poset P is called a *subdifference poset* (sub-DP) of P if $0, 1 \in P_1$ and whenever $a, b \in P_1$ with $a \le b$, it follows that $b \ominus a \in P_1$.

Clearly, a sub-DP P_1 of a DP P is a DP in its own right. Also, P_1 is closed under the unary operation $a \mapsto a' := 1 \ominus a$. It follows from (2) above that $a \oplus b = (b' \ominus a)' \in P_1$ whenever $a, b \in P_1$ and $a \perp b$. Consequently, every sub-DP of a DP is also closed under the induced operation $\oplus$.

3. Orthoalgebras, orthomodular posets, orthomodular lattices, and Boolean algebras

We note that an *orthoalgebra* ([4], [8]) is an effect algebra L in which the zero-one law (Condition (EA4) of Definition 2.1) is replaced by the stronger condition:

(OA4) (*Consistency Law*) $a \in L$, $a \oplus a$ defined $\Longrightarrow a = 0$.

Consequently, *every orthoalgebra is an effect algebra* (*or a difference poset*). There are many effect algebras (or difference posets) that are not orthoalgebras [14], [5]. The effect algebra $\mathcal{E}(H)$ of Example 2.2 is one such [13], as well as the interval effect algebra $\mathbb{R}^+[0,1]$ (see [5]).

Recall that an *orthomodular poset* (OMP) [8] may be regarded as an orthoalgebra L that satisfies the following additional condition ([4]):

$$a, b \in L\,, \ \ a \perp b \implies a \vee b \text{ exists and } a \vee b = a \oplus b\,.$$

It can be shown (see [4]) that this condition is equivalent to the *coherence law*:

$$a, b, c \in L\,, \ \ a \perp b \perp c \perp a \implies a \oplus b \perp c\,.$$

It can also be shown (see [5]) that an effect algebra L is an OMP if and only if it satisfies the coherence law. An *orthomodular lattice* (OML) may be defined as an OMP which is also a lattice. A *Boolean algebra* may be defined as a *distributive* OML. It has been shown in [5] that every Boolean algebra is an effect algebra L that satisfies the coherence law and the following *law of compatibility*:

For all $a, b \in L$, there exist $a_1, b_1, c \in L$ such that $b_1 \oplus c$ and $a_1 \oplus (b_1 \oplus c)$ are defined,

$$a = a_1 \oplus c \quad \text{ and } \quad b = b_1 \oplus c\,.$$

Let P_1 be a sub-DP of a DP P. For $a, b, c \in P_1$, we write $c = a \vee^{P_1} b$ (resp., $c = a \wedge^{P_1} b$) to indicate that c is the least upper bound (resp., greatest lower bound) of a and b in the poset $(P_1, \leq)$.

For the remainder of this paper, we assume that P is a difference poset (i.e., an effect algebra).

3.1. DEFINITION. Let $P_1 \subseteq P$ be a sub-DP. Then P_1 is called

1. a *sub-OMP* if $a, b \in P_1\,, \ a \perp b \implies a \vee^{P_1} b$ exists;
2. a *sub-OML* if $a, b \in P_1 \implies a \vee^{P_1} b$ exists;
3. a *Boolean subalgebra* if it is a distributive sub-OML.

Note that if P_1 is a sub-DP of P, then a pair of elements of P_1 is orthogonal in P_1 if and only if it is orthogonal in P. A subset X of P is called *jointly orthogonal* if it is pairwise orthogonal and is contained in a Boolean subalgebra B of P. We define

$$J(P) := \{X \subseteq P : \ X \text{ is jointly orthogonal}\}\,.$$

Recall that a sub-OML L_1 of an OML L is called a SIP-sub-OML ([9], [2]) if and only if it satisfies the *Subsequential Interpolation Property*:

For every orthogonal sequence $(a_i)_{i\in\omega} \subseteq L_1$ and for every $N \in \mathcal{I}(\omega)$, there exist $M \in \mathcal{I}(N)$ and $b \in L_1$ such that

$$a_i \leq b \quad \forall i \in M\,, \qquad a_i \leq b' \quad \forall i \in \omega \setminus M\,.$$

L_1 is called a SCP-sub-OML if and only if it satisfies the *Subsequential Completeness Property*:

For every orthogonal sequence $(a_i)_{i\in\omega} \subseteq L_1$ there exists $M \in \mathcal{I}(\omega)$ such that the supremum $\bigvee\limits_{i\in M}^{L_1} a_i$ exists in L_1.

Take $L_1 = L$ in the above definitions to get the definition of a SIP- (resp., SCP-) OML.

3.2. DEFINITION.

(i) A sub-DP P_1 of P is called a WSIP-sub-DP (resp., WSCP-sub-DP) if and only if it satisfies the *Weak Subsequential Interpolation* (resp., *Weak Subequential Completeness*) *Property*:

For every sequence $(a_i)_{i\in\omega} \in J(P_1)$, there exist a subsequence $(a_{i_k})_{k\in\omega}$ of $(a_i)_{i\in\omega}$ and a SIP-sub-OML (resp., SCP-sub-OML) Q of P_1 that contains $(a_{i_k})_{k\in\omega}$.

Take $P_1 = P$ in the above definitions to get the definition of a WSIP- (resp., WSCP-) DP. WSIP-orthoalgebras and WSCP-orthoalgebras are defined similarly ([9]).

(ii) A DP P is called an *orthosummable difference poset* (resp., a *σ-difference poset*) if for every (resp., for every countable) $X \in J(P)$, the supremum

$$\bigoplus X := \bigvee_{F\in\mathcal{F}(X)} \bigoplus F$$

exists in P. If P is also an orthoalgebra, we say that P is an *orthosummable orthoalgebra* (resp., a *σ-orthoalgebra*) ([11]). For more about orthosummable orthoalgebras, we refer the reader to ([11]).

3.3. Remarks.

(1) Evidently, every SIP-OML is a WSIP-DP, but not conversely as can be seen from the Wright triangle example [4].

(2) For a DP P, WSCP implies WSIP, but not conversely as can be seen from F. J. Freniche's example [6; Theorem 7].

(3) Evidently, every WSIP-orthoalgebra is a WSIP-difference poset, but not conversely as the interval difference poset $\mathbb{R}^+[0,1]$ of Example 2.2 shows. In fact, $\mathbb{R}^+[0,1]$ is orthosummable, but not even an orthoalgebra.

(4) It is not difficult to show that a σ-difference poset is a WSCP-DP (and, hence, a WSIP-DP). However, the converse need not be true as can be seen from Example 3.9 of [10].

4. Results

Before we state and prove the main result (Theorem 4.4), which may be considered as both a Brooks-Jewett theorem (see [9], [2], [7]) and a Vitali-Hahn-Saks theorem (see [16], [6]) for k-triangular and s-bounded functions defined on a WSIP-difference poset with values in a triangular semigroup, we need to establish a few more definitions.

Let S be a commutative semigroup. Recall that a nonnegative functional $f\colon S \to [0,\infty)$ is called *triangular* ([16]) if it satisfies the following conditions:

(T0) $f(0) = 0$,

(T) $|f(x+y) - f(x)| \le f(y) \quad \forall x, y \in S$.

4.1. DEFINITION. A pair (S, f) where S is a commutative semigroup endowed with a nonnegative triangular functional f is called a *triangular semigroup.* A sequence $(x_i)_{i\in\omega} \subseteq S$ is said to *converge in* S if the corresponding nonnegative sequence $\big(f(x_i)\big)_{i\in\omega}$ converges in $[0,\infty)$.

We now consider examples of triangular semigroups. Consider the commutative semigroup $[0,\infty]$ (or $[0,\infty)$) and define a functional $f\colon [0,\infty] \to [0,\infty)$ by $f(x) := x$ for all $x \in [0,\infty]$. Evidently, $[0,\infty]$ endowed with this f forms a triangular semigroup. More generally, let S be a commutative semigroup and let d be a semi-invariant pseudometric on S, namely a pseudometric satisfying the inequality

$$d(x+z, y+z) \le d(x,y) \quad \forall x, y, z \in S,$$

or, equivalently, the inequality

$$d(x+x', y+y') \le d(x,y) + d(x', y') \quad \forall x, x', y, y' \in S.$$

Define $f\colon S \to [0,\infty)$ by

$$f(x) := d(x, 0) \quad \forall x \in S. \tag{3}$$

One easily verifies that f satisfies (T0) and (T), and therefore (S, f) is a triangular semigroup.

Finally, if S is a commutative uniform semigroup, then it is known (see [15]) that the uniformity of S can be generated by a set $\mathcal{D}$ of continuous semi-invariant pseudometrics d on S. Thus, for each $d \in \mathcal{D}$, (3) defines a triangular functional f on S, and therefore (S, f) is a triangular semigroup.

4.2. DEFINITION. Let P be a DP and let $\phi\colon P \to [0,\infty)$. Following [16], we say that ϕ is *k-triangular* ($k \in (0,\infty)$) if it satisfies

(T0) $\phi(0) = 0$,

(kT) $|\phi(a \oplus b) - \phi(a)| \le k\phi(b)$ whenever $a, b \in P$ and $a \perp b$.

It is easy to check that a function $\phi\colon P \to [0,\infty)$ with $\phi(0) = 0$ is k-triangular if and only if

$$|\phi(b) - \phi(a)| \le k\phi(b \ominus a)$$

whenever $a, b \in P$ with $a \le b$. Moreover, if ϕ is k-triangular with $k \in (0,1)$, then ϕ is identically zero on P. Henceforth, we shall consider k-triangular functions with $k \ge 1$.

4.3. DEFINITION. Let (S, f) be a triangular semigroup, P a DP, and $\phi\colon P \to S$. We say that ϕ is *k-triangular* if the composite functional $f \circ \phi\colon P \to [0, \infty)$ is k-triangular. We say that ϕ is *s-bounded* (or *exhaustive*) if for every sequence $(a_i)_{i\in\omega} \in J(P)$, we have

$$\lim_{i\to\infty} \phi(a_i) = 0 .$$

Recall that the convergence of the sequence $(\phi(a_i))_{i\in\omega}$ to 0 in S means that the corresponding nonnegative sequence $(f(\phi(a_i)))_{i\in\omega}$ converges to 0 in $[0, \infty)$. We say that ϕ is *additive* if

(i) $\phi(0) = 0$, and

(ii) $\phi\left(\bigoplus_{i=0}^{n} a_i\right) = \sum_{i=0}^{n} \phi(a_i)$ for every finite $\{a_i : i = 0, \dots, n\} \in J(P)$.

Since any pair of orthogonal elements in P is jointly orthogonal, then, as a consequence of (ii), we have

(ii)′ $a, b \in P$ and $a \perp b \implies \phi(a \oplus b) = \phi(a) \oplus \phi(b)$.

A family Φ of s-bounded functions $\phi\colon P \to S$ is called *uniformly s-bounded* if for every sequence $(a_i)_{i\in\omega} \in J(P)$, we have

$$\lim_{i\to\infty} \phi(a_i) = 0 \qquad \text{uniformly in} \quad \phi \in \Phi .$$

Henceforth, unless otherwise stated, we assume that P is a *difference poset*, (S, f) is a *triangular semigroup*, the symbols $kt(P, S)$, $s(P, S)$, and $sa(P, S)$ denote, respectively, the *set of all k-triangular*, all *s-bounded*, and *all additive* and *s-bounded functions* $\phi\colon P \to S$.

4.4. THEOREM. (Brooks-Jewett) *Let P be a WSIP-difference poset, and let $(\phi_n)_{n\in\omega\setminus\{0\}} \subseteq kt(P, S) \cap s(P, S)$ be such that*

$$\lim_{n\to\infty} f(\phi_n(a)) =: \gamma_0(a) \ \textit{exists} \quad \forall\, a \in P .$$

Then γ_0 is k-triangular. Moreover, γ_0 is s-bounded if and only if $(\phi_n)_{n\in\omega\setminus\{0\}}$ is uniformly s-bounded.

P r o o f . We first show that γ_0 is k-triangular. Evidently, $\gamma_0(0) = 0$. Let $a, b \in P$ with $a \perp b$. By the k-triangularity of each ϕ_n, we have for every $n \in \omega \setminus \{0\}$ that

$$\begin{aligned}
&|\gamma_0(a \oplus b) - \gamma_0(a)| \\
&\le |\gamma_0(a \oplus b) - f(\phi_n(a \oplus b))| + |f(\phi_n(a \oplus b)) - f(\phi_n(a))| + |f(\phi_n(a)) - \gamma_0(a)| \\
&\le |\gamma_0(a \oplus b) - f(\phi_n(a \oplus b))| + k f(\phi_n(b)) + |f(\phi_n(a)) - \gamma_0(a)| .
\end{aligned}$$

Since γ_0 is the pointwise limit of $(f \circ \phi_n)_{n \in \omega \setminus \{0\}}$, we can find for every $\varepsilon > 0$ an $n_0 \in \omega \setminus \{0\}$ such that

$$\left|\gamma_0(a \oplus b) - f\big(\phi_{n_0}(a \oplus b)\big)\right| < \frac{\varepsilon}{3}, \qquad \left|f\big(\phi_{n_0}(a)\big) - \gamma_0(a)\right| < \frac{\varepsilon}{3},$$

and

$$f\big(\phi_{n_0}(b)\big) \le \gamma_0(b) + \frac{\varepsilon}{3k}.$$

Hence, for every $\varepsilon > 0$, we have

$$|\gamma_0(a \oplus b) - \gamma_0(a)| < \frac{\varepsilon}{3} + k\gamma_0(b) + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \le k\gamma_0(b) + \varepsilon,$$

which implies that

$$|\gamma_0(a \oplus b) - \gamma_0(a)| \le k\gamma_0(b),$$

and therefore γ_0 is k-triangular.

Next, assume that $(\phi_n)_{n \in \omega \setminus \{0\}}$ is uniformly s-bounded. To show that γ_0 is s-bounded, let $(a_i)_{i \in \omega} \in J(P)$ and $\varepsilon > 0$ be given. By the uniform s-boundedness of $(\phi_n)_{n \in \omega \setminus \{0\}}$, there exists $i_0 \in \omega$ such that for every $i \ge i_0$ and every $n \in \omega \setminus \{0\}$, we have

$$f\big(\phi_n(a_i)\big) < \frac{\varepsilon}{2}.$$

Moreover, the hypothesis that $\lim_{n \to \infty} f\big(\phi_n(a)\big) = \gamma_0(a) \quad \forall a \in P$ implies that for every $a \in P$ there exists $n(a) \in \omega \setminus \{0\}$ such that

$$\left|f\big(\phi_{n(a)}(a)\big) - \gamma_0(a)\right| < \frac{\varepsilon}{2}.$$

Hence, for every $i \ge i_0$, we have

$$\gamma_0(a_i) \le \left|\gamma_0(a_i) - f\big(\phi_{n(a_i)}(a_i)\big)\right| + f\big(\phi_{n(a_i)}(a_i)\big) < \varepsilon,$$

which shows that γ_0 is s-bounded.

Conversely, assume that γ_0 is s-bounded. To show that $(\phi_n)_{n \in \omega \setminus \{0\}}$ is uniformly s-bounded, suppose the contrary. Then, by passing to a subsequence if necessary, we may assume that there exist a sequence $(a_i)_{i \in \omega} \in J(P)$ and an $\varepsilon > 0$ such that

$$f\big(\phi_i(a_i)\big) \ge \varepsilon \quad \forall i \in \omega \setminus \{0\}. \tag{4}$$

Now, using WSIP, pick a subsequence $(a_{i_j})_{j \in \omega}$ of $(a_i)_{i \in \omega}$ and a SIP-sub-OML Q of P containing $(a_{i_j})_{j \in \omega}$. Then, by [7; 3.3], $(\phi_n|_Q)_{n \in \omega \setminus \{0\}}$ is uniformly s-bounded. Hence, there exists $j_0 \in \omega$ such that

$$f\big(\phi_n(a_{i_{j_0}})\big) < \varepsilon \quad \forall n \in \omega \setminus \{0\},$$

which contradicts (4). □

If we take $S = [0, \infty)$ in Theorem 4.4, which is clearly a triangular semigroup, we obtain the following theorem.

4.5. THEOREM. (Brooks-Jewett) *Let P be a WSIP-difference poset, and let $(\phi_n)_{n\in\omega\setminus\{0\}} \subseteq kt(P,[0,\infty)) \cap s(P,[0,\infty))$ be such that*

$$\lim_{n\to\infty} \phi_n(a) =: \phi_0(a) \ \textit{exists} \quad \forall\, a \in P\,.$$

Then ϕ_0 is k-triangular. Moreover, ϕ_0 is s-bounded if and only if $(\phi_n)_{n\in\omega\setminus\{0\}}$ is uniformly s-bounded.

Remark. Note that Theorem 4.4 (resp., Theorem 4.5) contains the result (3.3) (resp., the result (3.2)) of [7].

Let P_1 be a subdifference poset of P. A function $\phi\colon P \to S$ is called *P_1-s-bounded* (or *P_1-exhaustive*) if for every sequence $(a_i)_{i\in\omega} \in J(P_1)$, we have $\lim_{i\to\infty} \phi(a_i) = 0$. A family Φ of P_1-s-bounded functions is called *uniformly P_1-s-bounded* if for every sequence $(a_i)_{i\in\omega} \in J(P_1)$, we have

$$\lim_{i\to\infty} \phi(a_i) = 0 \ \text{uniformly in} \ \phi \in \Phi\,.$$

Here is another consequence of Theorem 4.4.

4.6. THEOREM. *Let P_1 be a WSIP-subdifference poset of P, and let $(\phi_n)_{n\in\omega\setminus\{0\}}$ be a sequence of k-triangular and P_1-s-bounded functions from P to S (resp., to $[0,\infty)$) such that*

$$\lim_{n\to\infty} f\big(\phi_n(a)\big) =: \gamma_0(a) \quad (\textit{resp.,} \ \lim_{n\to\infty} \phi_n(a) =: \gamma_0(a)\,) \quad \textit{exists} \ \forall\, a \in P_1\,.$$

Then γ_0 is k-triangular. Moreover, γ_0 is P_1-s-bounded if and only if $(\phi_n)_{n\in\omega\setminus\{0\}}$ is uniformly P_1-s-bounded.

The following result is a consequence of Theorem 4.6.

4.7. THEOREM. *Let P_1 be a WSIP-subdifference poset of P, S a commutative uniform semigroup, and $(\mu_n)_{n\in\omega}$ a sequence of additive and P_1-s-bounded functions from P to S such that*

$$\lim_{n\to\infty} \mu_n(a) = \mu_0(a) \quad \forall\, a \in P_1\,.$$

Then $(\mu_n)_{n\in\omega}$ is uniformly P_1-s-bounded.

Proof. Suppose contrariwise that $(\mu_n)_{n\in\omega}$ is not uniformly P_1-s-bounded. Then, by passing to a subsequence if necessary, we may assume that there exist a sequence $(a_i)_{i\in\omega} \in J(P_1)$, $d \in \mathcal{D}$ (where $\mathcal{D}$ is the set of continuous pseudometrics that generate the uniformity of S), and $\varepsilon > 0$ such that

$$d\big(\mu_i(a_i), 0\big) \geq \varepsilon \quad \forall\, i \in \omega\,. \tag{5}$$

Define, for every $i \in \omega$, a function $\phi_i \colon P \to [0, \infty)$ by

$$\phi_i(a) := d(\mu_i(a), 0) \qquad (a \in P_1).$$

Evidently, the sequence $(\phi_i)_{i\in\omega}$ is 1-triangular and P_1-s-bounded. Moreover, the hypothesis that $\lim_{i\to\infty} \mu_i(a) = \mu_0(a)$ $\forall\, a \in P_1$ implies that $\lim_{i\to\infty} \phi_i(a) = \phi_0(a)$ $\forall\, a \in P_1$. Now apply Theorem 4.6 to the sequence $(\phi_i)_{i\in\omega\setminus\{0\}}$ to get the desired contradiction to (5). □

Remarks.

(1) If we assume in Theorem 4.7 that $P_1 = P$ is an orthoalgbra, then we see that this theorem yields Theorem 4.1 of [9] as a special case.

(2) If we assume in Theorem 4.7 that $P_1 = P$ is an orthomodular lattice, then we see that this theorem yields the result (5.1) of [2].

(3) If an orthomodular sublattice G (as defined by [7]) of an orthomodular lattice L contains the largest element 1 of L, i.e., G is a subalgebra of L ([12]), then a SIP- (resp., SCP-) sublattice of L in the sense of Guariglia [7] is the same thing as a WSIP- (resp., WSCP-) sublattice of L in our sense. In this case, we note that Theorem 4.6 (resp., Theorem 4.7) contains the result (4.1) (resp., the result (4.2)) of [7].

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REFERENCES

[1] BIRKHOFF, G.—von NEUMANN, J.: *The logic of quantum mechanics*, Ann. of Math. **37** (1936), 823–843.

[2] D'ANDREA, A. B.—De LUCIA, P.: *The Brooks-Jewett theorem on an orthomodular lattice*, J. Math. Anal. Appl. **154** (1991), 507–522.

[3] DVUREČENSKIJ, A.—RIEČAN, B.: *Decomposition of measures on orthoalgebras and difference posets*, Internat. J. Theoret. Phys. **33** (1994), 1387–1402.

[4] FOULIS, D. J.—GREECHIE, R. J.—RÜTTIMANN, G. T.: *Filters and supports*, Internat. J. Theoret. Phys. **31** (1992), 789–807.

[5] FOULIS, D. J.—BENNETT, M. K.: *Effect algebras and unsharp quantum logics*, Found. Phys. **24** (1994), 1325–1346.

[6] FRENICHE, F. J.: *The Vitali-Hahn-Saks theorem for Boolean algebras with the subsequential interpolation property*, Proc. Amer. Math. Soc. **92** (1984), 362–366.

[7] GUARIGLIA, E.: *K-triangular functions on an orthomodular lattice and the Brooks-Jewett theorem*, Rad. Mat. **6** (1990), 241–251.

[8] GUDDER, S. P.: *Quantum Probability*, Academic Press, Boston, 1988.

[9] HABIL, E. D.: *Brooks-Jewett and Nikodym convergence theorems for orthoalgebras that have the weak subsequential interpolation property*, Intern. J. Theoret. Phys. **34** (1995), 465–491.

[10] HABIL, E. D.: *Morphisms and pasting of orthoalgebras*, Math. Slovaca **47** (1997), 405–416.

[11] HABIL, E. D.: *Orthosummable orthoalgebras*, Intern. J. Theoret. Phys. **33** (1994), 1957–1984.

[12] KALMBACH, G.: *Orthomodular Lattices*, Academic Press, London-New York, 1983.

[13] KÔPKA, F.—CHOVANEC, F.: *D-posets*, Math. Slovaca **44** (1994), 21–34.

[14] NAVARA, M.—PTÁK, P.: *Difference posets and orthoalgebras*, BUSEFAL **69** (1997), 64–69.

[15] PAGE, W.: *Topological Uniform Structures*, Wiley, New York, 1978.

[16] PAP, E.: *The Vitali-Hahn-Saks theorems for k-triangular set functions*, Atti. Sem. Mat. Fis. Univ. Modena **25** (1987), 21–32.

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