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PAIRS OF MULTILATTICES DEFINED ON THE SAME SET

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1. Preliminaries

Specific pairs of lattices defined on the same set were investigated by J. Jakubík and M. Kolibiar [2, 3, 5]. In the present paper we shall investigate some properties of pairs of distributive multilattices analogous to those that have been dealt with in papers [2, 3, 5].

A multilattice [1] is a poset M in which the conditions (i) and its dual (ii) are satisfied: (i) If $a, b, h \in M$ and $a \leqq h$, $b \leqq h$, then there exists $v \in M$ such that (a) $v \leqq h$, $a \leqq v$, $b \leqq v$, and (b) $z \in M$, $z \leqq v$, $a \leqq z$, $b \leqq z$ implies $z = v$. $(a \vee b)_h$ designates the set of all elements $v \in M$ satisfying (i); the symbol $(a \wedge b)_d$ has a dual meaning. We denote $a \vee b = \bigcup (a \vee b)_h$, $a \wedge b = \bigcup (a \wedge b)_d$, where $h(d)$ runs over the set of all upper (lower) bounds of the set $\{a, b\}$. Let A and B be nonvoid subsets of M; then we define $A \vee B = \bigcup (a \vee b)$, $A \wedge B = \bigcup (a \wedge b)$, where $a \in A$ and $b \in B$. Throughout the paper we denote $a \vee b = x$, resp. $a \vee P = x$ ($a \wedge b = x$, resp. $a \wedge P = x$) if $a, b, x \in M$, P is a nonvoid subset of M and $a \vee b = \{x\}$, resp. $a \vee P = \{x\}$ ($a \wedge b = \{x\}$, resp. $a \wedge P = \{x\}$). If $a, b, c, d \in M$, $a \vee b = x$, $c \vee d = y$ and $x \leqq y$, then we write also $a \vee b \leqq c \vee d$ (and analogously for $a \wedge b$, $c \wedge d$).

A multilattice M is distributive [1] iff for every $a, b, b', d, h \in M$ satisfying the conditions $d \leqq a, b, b' \leqq h$, $(a \vee b)_h = (a \vee b')_h = h$, $(a \wedge b)_d = (a \wedge b')_d = d$ we have $b = b'$.

Multilattices M_1 and M_2 are said to be isomorphic [6] (denoted as $M_1 \sim M_2$) if there exists a bijection f of M_1 onto M_2 satisfying: $x \leqq y$ iff $f(x) \leqq f(y)$ $(x, y \in M_1)$.

Let M be a cardinal product of two posets M_1, M_2. M is a multilattice iff M_1 and M_2 are multilattices [6]. M is a distributive multilattice iff M_1 and M_2 are distributive multilattices [6].

Suppose that a multilattice M has a least element O and a greatest element e. By a complement [1] of an element a in the multilattice M we mean an element $a' \in M$ such that $a \wedge a' = O$, $a \vee a' = e$. Let $a, b \in M$, $a \leqq b$. The interval $\langle a, b \rangle$ is the set $\{x \in M\colon a \leqq x \leqq b\}$.

We need the following results:

Lemma A ([4, Lemma 11]). *Let M be a distributive multilattice, $a, b \in M$, $u \in a \wedge b$, $v \in a \vee b$; then there exists an isomorphism m of $\langle u, v \rangle$ onto $\langle a, v \rangle \times \langle b, v \rangle$ and an isomorphism n of $\langle a, v \rangle \times \langle b, v \rangle$ onto $\langle u, v \rangle$ such that $m(x) = ((a \vee x)_v, (b \vee x)_v)$ for each $x \in \langle u, v \rangle$, $n(x_1, x_2) - (x_1 \wedge x_2)_u$ for each $x_1 \in \langle a, v \rangle$ and each $x_2 \in \langle b, v \rangle$.*

Obviously n is an inverse of m and conversely.

Let M be a multilattice. A subset $\{a, b, u, v\}$ of M is called a quadruple if $u \in a \wedge b$, $v \in a \vee b$ and we denote it (a, b, u, v).

Lemma B ([4, Lemma 13]). *Let M be a distributive multilattice, $a, b, c, d, e, f \in M$. Let (c, b, a, d), (e, d, c, f) and $a \in e \wedge b$ ($f \in e \vee b$); then $f \in e \vee b$ ($a \in e \wedge b$).*

2. Properties of pairs of multilattices

Let $M = (A; \vee, \wedge)$ be a distributive multilattice with a least element O and a greatest element e, defined on the set A. Suppose that A possesses elements t, t' such that t' is a complement of t in M. The partial order in M will be denoted by $\leqq$. Let $a, b \in A$. Since M has the least element and the greatest element it follows that $a \vee b$, $a \wedge b$ are nonempty sets.

Lemma 1. *Let $a, b, c \in A$, $a \vee (b \vee c) = x$. Then $x \in (a \vee b) \vee c$.*

Proof. Let $a, b, c \in A$, $a \vee (b \vee c) = x$. Obviously there exist elements $y \in (a \vee b) \vee c$, $z \in a \vee (b \vee c)$ such that $y \leqq x$ and $z \leqq y$. Since $z = x$, we have $x = y$.

In a dual way we obtain

Lemma 1'. *Let $a, b, c \in A$, $a \wedge (b \wedge c) = x$. Then $x \in (a \wedge b) \wedge c$.*

Lemma 2. *Let $a \in A$. Then $a \vee t$, $a \vee t'$, $a \wedge t$, $a \wedge t'$ are one-element sets.*

Proof. Clearly $(a \vee t)_e = a \vee t$, $(a \vee t')_e = a \vee t'$ and hence according to Lemma A, $a \vee t$ and $a \vee t'$ are one-element sets. The dual assertion can be proved analogously.

Let $a, b \in A$. Put

$$(1) \qquad a \cup b = (a \vee b) \wedge [(a \vee t) \wedge (b \vee t)],$$

$$(2) \qquad a \cap b = (a \vee b) \wedge [(a \vee t') \wedge (b \vee t')].$$

Lemma 3. *Let $a, b \in A$. If $a \cup b = b$, then $a \vee t \geqq b \vee t$ and $a \vee t' \leqq b \vee t'$.*

Proof. Let $a, b \in A$, $a \cup b = b$, $r \in a \vee b$. Denote $a \vee t = v$, $b \vee t = w$, $r \vee t = u$, $a \vee t' = v'$, $b \vee t' = w'$, $r \vee t' = u'$ (see Lemma 2). Obviously $v \leqq u$, $w \leqq u$, $v' \leqq u'$, $w' \leqq u'$. From $a \cup b = b$ by Lemma 1' we get

$$(3) \qquad b = r \wedge (v \wedge w) \in (r \wedge w) \wedge v.$$

From (3) it follows that $b \leqq v$, consequently $w \leqq v$, hence $a \vee t \geqq b \vee t$. From this and from (3) we get $b \in r \wedge w$. From $u = r \vee t$ we obtain that $u \in r \vee w$ and have a quadruple (r, w, b, u). Since $t \vee t' = e$, $t \wedge t' = O$ by Lemma A, we have $v \wedge v' = a$, $w \wedge w' = b$, $u \wedge u' = r$. Because (u, u', r, e) is also a quadruple and $e \in u' \vee w$ by Lemma B we have $b \in u' \wedge w$. Hence (u', w, b, e) is a quadruple. Since $w' \vee w = e = u' \vee w$ and $(w' \wedge w)_b = b = (u' \wedge w)_b$ we have $u' = w'$ by the distributivity of M. Consequently $v' \leqq w'$, hence $a \vee t' \leqq b \vee t'$.

Lemma 4. *If $a \vee t \geqq b \vee t$ and $a \vee t' \leqq b \vee t'$, then $a \cup b = b$.*

Proof. Let $a, b \in A$, $a \vee t = v$, $b \vee t = w$, $a \vee t' = v'$, $b \vee t' = w'$, $w \leqq v$, $v' \leqq w'$ and $r \in a \vee b$. By Lemma A we have

$$\langle O, e \rangle \sim \langle t, e \rangle \times \langle t', e \rangle, \tag{4}$$

where $a \mapsto (v, v')$, $b \mapsto (w, w')$, $r \mapsto (u, u')$, $u = r \vee t$, $u' = r \vee t'$. According to the isomorphism (4) we get $u \in v \vee w = v$, $u' \in v' \vee w' = w'$, hence $u = v$, $u' = w'$, consequently $a \vee b = r$. Thus we obtain

$$a \cup b = (a \vee b) \wedge [(a \vee t) \wedge (b \vee t)] = r \wedge w. \tag{5}$$

Further by (4) we get

$$(v, w') \wedge (w, e) = (v \wedge w, w' \wedge e) = (w, w').$$

From this and from (5) it follows that $a \cup b = b$.

Lemma 5. *Let $a, b \in A$. We define a relation R on A as follows: aRb iff $a \cup b = b$. The relation R is a partial order on the set A.*

Proof. Let $a \in A$, $v = a \vee t$; then

$$a \cup a = (a \vee a) \wedge [(a \vee t) \wedge (a \vee t)] = a \wedge v = a.$$

Hence aRa holds and R is reflexive.

Let $a, b \in A$ and aRb, bRa. Consequently

$$b = a \cup b = (a \vee b) \wedge [(a \vee t) \wedge (b \vee t)] = (b \vee a) \wedge [(b \wedge t) \wedge (a \vee t)] = b \cup a = a.$$

Therefore R is antisymmetric.

Let $a, b, c \in A$ and aRb, bRc. Using Lemma 3 we get $a \vee t \geqq b \vee t$, $a \vee t' \leqq b \vee t'$, $b \vee t \geqq c \vee t$, $b \vee t' \leqq c \vee t'$, hence $a \vee t \geqq c \vee t$, $a \vee t' \leqq c \vee t'$. From this by Lemma 4 we have $a \cup c = c$, hence aRc and the relation R is transitive.

Next we denote the relation R by $\subseteq$.

From Lemma A, Lemma 3 and Lemma 4 it follows:

Lemma 6. *The poset $(A, \subseteq)$ is isomorphic to the direct product of the intervals $\langle t, e \rangle^{\sim}$, $\langle t', e \rangle$, where $\langle t, e \rangle^{\sim}$ is the interval dual to $\langle t, e \rangle$.*

Corollary. *The poset $(A, \subseteq)$ is a distributive multilattice with the greatest element t and the least element t'.*

Lemma 7. *Let $M=(A\,;\ \vee,\wedge)$, $N=(A\,;\ \cup,\cap)$ be multilattices defined on the same set. Suppose that M is distributive with a greatest element e and a least element O. Let $M_1=(A_1;\ \vee,\wedge)$, $M_2=(A_2;\ \vee,\wedge)$ be multilattices. Let φ be an isomorphism of both M onto $M_1\times M_2$ and N onto $M_1^{\sim}\times M_2$ ($M_1^{\sim}$ is the multilattice dual to M_1). Then the multilattice operations of the multilattice N are given by (1) and (2).*

Proof. Consider the isomorphisms

$$\varphi\colon M\to M_1\times M_2,\quad \varphi\colon N\to M_1^{\sim}\times M_2$$

and denote $\varphi(O)=(O_1,O_2)$, $\varphi(e)=(e_1,e_2)$. Evidently $O_1(O_2,(O_1,O_2))$ is a least element of $M_1(M_2,M_1\times M_2)$ and $e_1(e_2,(e_1,e_2))$ is a greatest element of $M_1(M_2, M_1\times M_2)$. Denote $s=\varphi^{-1}(O_1,e_2)$ and $s'=\varphi^{-1}(e_1,O_2)$. Obviously (e_1,O_2) is the least element of $M_1^{\sim}\times M_2$ and (O_1,e_2) is the greatest element of $M_1^{\sim}\times M_2$, consequently $s(s')$ is the greatest (least) element of N. From

$$\varphi(s\vee s')=(O_1,e_2)\vee(e_1,O_2)=(O_1\vee e_1,e_2\vee O_2)=(e_1,e_2)=\varphi(e),$$
$$\varphi(s\wedge s')=(O_1,e_2)\wedge(e_1,O_2)=(O_1\wedge e_1,e_2\wedge O_2)=(O_1,O_2)=\varphi(O)$$

it follows that s, s' are complementary elements of M. Let $a,b\in A$ and $\varphi(a)=(a_1,a_2)$, $\varphi(b)=(b_1,b_2)$. Then

$$\varphi((a\vee b)\wedge[(a\vee s)\wedge(b\vee s)])=((a_1\vee b_1)\wedge[(a_1\vee O_1)\wedge(b_1\vee O_1)],$$
$$(a_2\vee b_2)\wedge[(a_2\vee e_2)\wedge(b_2\vee e_2)])=((a_1\vee b_1)\wedge(a_1\wedge b_1),$$
$$(a_2\vee b_2)\wedge e_2)=(a_1\wedge b_1,a_2\vee b_2)=\varphi(a\cup b),$$

hence

$$a\cup b=(a\vee b)\wedge[(a\vee s)\wedge(b\vee s)].$$

Analogously we get $a\cap b=(a\vee b)\wedge[(a\vee s')\wedge(b\vee s')]$.

Corollary. *The multilattice operations of the multilattice $(A,\subseteq)$ from Corollary of Lemma 6 are given by (1) and (2).*

Lemma 8. *The greatest element e and the least element O in the multilattice $(A\,;\ \vee,\wedge)$ are complementary elements in the multilattice $(A\,;\ \cup,\cap)$.*

Proof. We have

$$e\cup O=(e\vee O)\wedge[(e\vee t)\wedge(O\vee t)]=e\wedge(e\wedge t)=t,$$

and similarly $e\cap O=t'$.

From Lemma A, Lemma 6, Corollary of Lemma 6, Lemma 7, Corollary of Lemma 7 we now get

Theorem. *Let $M=(A\,;\ \vee,\wedge)$ be a distributive multilattice with a greatest and a least element. There is a one-one correspondence between couples (t,t') of complementary elements in M and multilattices $N=(A\,;\ \cup,\cap)$ such that there* *exist multilattices M_1 and M_2 and isomorphisms φ: $M \to M_1 \times M_2$, ψ: $N \to M_{\tilde{1}} \times M_2$ such that the following diagram commutes*

$$\begin{array}{ccc} M & \xrightarrow{\ i\ } & N \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ M_1 \times M_2 & \xrightarrow{\ j\ } & M_{\tilde{1}} \times M_2 \end{array}$$

where $i(x) = x$ for any $x \in M$ and $j(a, b) = (a, b)$ for each element (a, b) of $M_1 \times M_2$. Given a couple (t, t') the corresponding operations $\cup$ and $\cap$ are given by (1) and (2). Given a multilattice N the corresponding couple (t, t') consists of the greatest and the least element of N. Moreover the multilattices N are distributive.

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ПАРА МУЛЬТИСТРУКТУР ОПРЕДЕЛЕННЫХ НА ОДИНАКОВОМ МНОЖЕСТВЕ

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Резюме

Пусть $M = (A\,;\ \vee, \wedge)$ — дистрибутивная мультиструктура с найбольшим и наименьшим элементом. Пусть существуют элементы t, $t' \in M$, так, что t' является дополнением к t. Для каждого a, $b \in A$ мы определим множества

(1) $$a \cup b = (a \vee b) \wedge [(a \vee t) \wedge (b \vee t)], \quad a \cap b = (a \vee b) \wedge [(a \vee t') \wedge (b \vee t')].$$

В лемме 5 определяется отношение $a \subseteq b$, если $a \cup b = b$ для каждого $a, b \in A$ и показывается, что $(A\ ;\ \subseteq)$ является частично упорядоченным множеством. Далее доказывается утверждение: существует взаимно однозначное соответствие между парами (t, t') и мультиструктурами $N = (A\ ;\ \cup, \cap)$ так, что существуют мультиструктуры M_1, M_2 и изоморфизмы $\varphi : M \rightarrow M_1 \times M_2$, $\psi : N \rightarrow M_1^{\sim} \times M_2$. Данной паре (t, t') соответствуют операции $\cup$ и $\cap$ определенные равенствами (1).