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ON DETERMINING SETS FOR CERTAIN GENERALIZATIONS OF CONTINUITY

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Introduction

Let X, Y be sets. Let $F_{X,Y}$ be a class of functions $f\colon X \to Y$. A set $D \subset X$ is called a determining set for $F_{X,Y}$ if each two members of $F_{X,Y}$ which agree on this set must agree on all of X. Denote by $\mathscr{D}(F_{X,Y})$ the family of all determining sets for $F_{X,Y}$.

For the basic properties of determining sets see [1]. A survey of results of determining sets for derivatives is in [2].

The purpose of this paper is to investigate determining sets for certain classes of functions (quasi-continuous functions and somewhat continuous functions). We show that from this point of view is sufficient to research for each such class its subclass of characteristic functions of sets, and on this account we give a complete description of determining sets for each such class.

Let X, Y be two topological spaces. The function $f\colon X \to Y$ is said to be quasi-continuous at the point $x_0 \in X$ if for each neighbourhood $U(x_0)$ of the point x_0(in X) and each neighbourhood $V(f(x_0))$ of the point $f(x_0)$ (in Y) there exists a nonempty open set $U \subset U(x_0)$ such that $f(U) \subset V(f(x_0))$. The function f is said to be quasi-continuous on X if it is quasi-continuous at each point $x \in X$. (See [5] and [12].)

The function $f\colon X \to Y$ is said to be somewhat continuous if for each set $V \subset Y$ open in Y such that $f^{-1}(V) \neq \emptyset$ there exists a nonempty open set $U \subset X$ so that $U \subset f^{-1}(V)$. (See [4].)

In the sequel $Q_{X,Y}$ and $S_{X,Y}$ denote the sets of all functions $f\colon X \to Y$ which are quasi-continuous on X and somewhat continuous, respectively.

Let e, $t \in Y$, $e \neq t$. If A be a subset of X, then the characteristic function of A is the function $\chi_A^{e,t}\colon X \to Y$,

$$(\chi_A^{e,t})(x) = \begin{cases} e & \text{for } x \in X - A, \\ t & \text{for } x \in A. \end{cases}$$

Denote by $\chi_{X,Y}^{e,t}$ the class of all characteristic functions of the form $\chi_A^{e,t}\colon X \to Y$.

We assume throughout this paper that the set Y has at least two elements.

1. Preliminaries

1.1. Definition. *Let A be a subset of a topological space X. The set A is regular open if $A = \operatorname{Int} \operatorname{Cl} A$* (see [11]).

1.2. Remarks. 1. Observe that for each open subset G of a topological space X we have $Cl\ G = Cl\ Int\ Cl\ G$.

2. If A, B are regular open subsets of X, then $A \cap B$ is regular open.

3. If H is an open subset *of* X, then $Int\ Cl\ H$ is regular open.

1.3. Definition. *Let A be a subset of a topological space X. The set A is semi-open if there exists an open set G in X such that $G \subset A \subset \operatorname{Cl} G$* (see [7]).

1.4. Lemma. *The set $A \subset X$ is semi-open if and only if $A \subset \operatorname{Cl} \operatorname{Int} A$* (see [7]).

1.5. Lemma. *Let A be a semi-open subset of X. If $\operatorname{Int} A \subset B \subset \operatorname{Cl} A$, then B is semi-open* (see [7]).

1.6. Lemma. *Let S be an open subset of X. Then S is regular open if and only if $X - S$ is semi-open.*

1.7. Definition. *A function $f: X \to Y$ is said to be semi-continuous if $f^{-1}(V)$ is a semi-open set (in X) for every open subset V of Y.* (See [7].)

1.8. Lemma. *A function $f: X \to Y$ is semi-continuous if and only if f is quasi--continuous on X.* (See [8].)

1.9. Definition. *A space X is said to be hyperconnected if every nonempty open set is dense in X.* (See [9].)

1.10. Lemma. *Let X, Y be topological spaces. Then X is hyperconnected if and only if each somewhat continuous function $f: X \to Y$ is constant on X.* (See [3].)

1.11. Definition. *A space X is called a Urysohn space if for every pair of distinct points x and y in X there exist open sets U and V such that $x \in U$, $y \in V$, and* $\operatorname{Cl} U \cap \operatorname{Cl} V = \emptyset$.

1.12. Definition. *A topological space X is said to be extremally disconnected if the closure of every open set in X is open in X.* (See [10].)

2. Determining sets for the class of quasi-continuous functions

The following theorem shows that $\mathscr{D}(Q_{X,Y}) = \mathscr{D}(\chi^{c,l}_{X,Y} \cap Q_{X,Y})$ and gives a characterization of the family $\mathscr{D}(Q_{X,Y})$.

2.1. Theorem. *Let X be a topological and Y a Urysohn space. Let $e, t \in Y$, $e \neq t$. Let A be a nonempty subset of X. Then the following statements are equivalent*:

$$A \in \mathscr{D}(Q_{X,Y}), \tag{1}$$

$$A \in \mathscr{D}(\chi^{e,t}_{X,Y} \cap Q_{X,Y}), \tag{2}$$

(3) *for each $L \subset K \subset X$, $\emptyset \neq K - L \subset X - A$, some of the following sets is not semi-open*: K, L, $X - K$, $X - L$.

Proof. That (1) implies (2) is obvious.
(2) $\Rightarrow$ (3): Deny. Suppose that there exist sets $L \subset K \subset X$, $\emptyset \neq K - L \subset X - A$, such that the sets K, L, $X - K$, $X - L$, are semi-open. Put $f = \chi^{e,t}_K$ and $g = \chi^{e,t}_L$. It is not difficult to verify that f, g are semi-continuous functions which agree on the set A such that $f \neq g$. Thus $A \notin \mathscr{D}(\chi^{e,t}_{X,Y} \cap Q_{X,Y})$.
(3) $\Rightarrow$ (1): By contradiction. Suppose that there exist functions $f, g \in Q_{X,Y}$, $f \neq g$, such that $f(x) = g(x)$ for each $x \in A$. Choose $a \in X$ such that

$$f(a) \neq g(a). \tag{4}$$

First we shall prove that for each nonempty open set G in X we have

$$\text{if } G \text{ is regular open, then } A \cap G \neq \emptyset. \tag{5}$$

Deny. Suppose that $G \subset X - A$. Putting in (3) $K = G$, $L = \emptyset$, we obtain that $X - G$ is not semi-open. Therefore by Lemma 1.6 the set G is not regular open. Choose U and V open neighbourhoods of the points $f(a)$ and $g(a)$, respectively, such that

$$\mathrm{Cl}\, U \cap \mathrm{Cl}\, V = \emptyset. \tag{6}$$

Put

$$W = \mathrm{Int}\, \mathrm{Cl}\, \mathrm{Int}\, f^{-1}(U) \cap \mathrm{Int}\, \mathrm{Cl}\, \mathrm{Int}\, g^{-1}(V).$$

We shall prove that

$$W = \emptyset. \tag{7}$$

Let $w \in W$. Let H be a neighbourhood of the point $f(w)$. Since f is quasi-continuous at the point w, there exists a nonempty open set $S \subset W$ such that $f(S) \subset H$. Snce S is nonempty, open, and $S \subset \mathrm{Cl}\, \mathrm{Int} f^{-1}(U)$, we obtain $S \cap \mathrm{Int} f^{-1}(U) \neq \emptyset$. Choose a point s in this intersection. Then $f(s) \in H$, $f(s) \in U$, hence $H \cap U \neq \emptyset$. This shows that each neighbourhood of $f(w)$ intersects U, i.e. $f(w) \in \mathrm{Cl}\, U$. Analogously, we have $g(w) \in Cl\ V$. By (6) we obtain $f(w) \neq g(w)$, hence $w \in X - A$. This shows that $W \subset X - A$. Since the set W is regular open, by (5) we have $W = \emptyset$.
Now, we shall prove that

$$(8) \qquad a \in \operatorname{Cl} \operatorname{Int} f^{-1}(U).$$

Since f is semi-continuous, the set $f^{-1}(U)$ is semi-open. Then by 1.4 we have $a \in f^{-1}(U) \subset \operatorname{Cl} \operatorname{Int} f^{-1}(U)$.
Analogously, we obtain

$$(9) \qquad a \in \operatorname{Cl} \operatorname{Int} g^{-1}(V).$$

Put

$$E = \operatorname{Int} \operatorname{Cl} \operatorname{Int} f^{-1}(U).$$

Now, we shall prove that

$$(10) \qquad a \notin E.$$

By contradiction. Suppose that $a \in E$. By (9) we have that each neighbourhood of a intersects the set $\operatorname{Int} g^{-1}(V)$, hence $\emptyset \neq E \cap \operatorname{Int} g^{-1}(V) \subset W$. This is contrary to (7).
Putting in (3) $K = E \cup \{a\}$, $L = E$, we obtain that some of the following sets is not semi-open: $E \cup \{a\}$, E, $X - (E \cup \{a\})$, $X - E$. The proof will be complete when we show that this is not true. The set E is open. By (8) we have

$$(11) \qquad a \in \operatorname{Cl} \operatorname{Int} f^{-1}(U) = \operatorname{Cl} E,$$

hence $E \subset E \cup \{a\} \subset \operatorname{Cl} E$. Thus the set $E \cup \{a\}$ is semi-open. Since E is regular open, by (10) and 1.6 the set $X - E$ is semi-open. Put

$$Z = \operatorname{Int}(X - E).$$

By (11) we obtain $a \in \operatorname{Cl} E = X - Z$, hence $Z \subset X - \{a\}$.
Since by 1.2 we have $Z \subset (X - E) \cap (X - \{a\}) = X - (E \cup \{a\}) \subset X - E = \operatorname{Cl}(X - \operatorname{Cl} \operatorname{Int} f^{-1}(U)) = \operatorname{Cl}(X - \operatorname{Cl} E) = \operatorname{Cl} Z$, the set $X - (E \cup \{a\})$ is semi-open.
The proof is complete.

2.2. Theorem. *Let X be a T_1-space, which has at least two elements. Let Y be a Urysohn space. Let there exists for each accumulation point $a \in X$ a regular open set $A \subset X$ such that $a \in \operatorname{Cl} A - A$. Then $\mathscr{D}(Q_{X,Y}) = \{X\}$.*

Proof. Let $a \in X$. We shall prove that the set $X - \{a\}$ is not determining for the class $Q_{X,Y}$. Suppose that a is an accumulation point of X (the opposite case is trivial). By the assumption there exists a regular open set $A \subset X$ such that $a \in \operatorname{Cl} A - A$. Choose $e, t \in Y$ such that $e \neq t$. Put $f = \chi_{A \cup \{a\}}^{e,t}$, $g = \chi_A^{e,t}$. Then $f \neq g$ and $f(x) = g(x)$ for each $x \in X - \{a\}$. It is not difficult to verify (by 1.4 and 1.6) that $f, g \in Q_{X,Y}$. Thus $X - \{a\} \notin \mathscr{D}(Q_{X,Y})$.

2.3. Theorem. *Let X be a first countable Hausdorff space. Let Y be a Urysohn space. Then $\mathscr{D}(Q_{X,Y}) = \{X\}$.*

Proof. By 2.2. Let $a \in X$ be an accumulation point of the set X. It is not difficult to verify that there exists a countable base $\{U_n\}_{n=1}^{\infty}$ at the point a such that for each positive integer n we have

$$U_{n+1} \subset U_n, \tag{12}$$

$$\operatorname{Int}(U_n - U_{n+1}) \neq \emptyset. \tag{13}$$

For each positive integer n put $Z_n = \operatorname{Int}(U_n - U_{n+1})$. Put

$$B = \bigcup_{k=1}^{\infty} Z_{2k}, \ A = \operatorname{Int} Cl\, B.$$

Since B is open, by 1.2 the set A is regular open. It is easy to prove that

$$Z_{2n-1} \subset X - B \ (n = 1, 2, 3, \ldots). \tag{14}$$

We shall prove that

$$a \in \operatorname{Cl} A. \tag{15}$$

Let U be a neighbourhood of a. Let k be a positive integer such that $U_{2k} \subset U$. By (13) we have $\emptyset \neq Z_{2k} \subset Z_{2k} \cap U_{2k} \subset B \cap U$. This shows that each neighbourhood of a intersects the set B, i.e. $a \in \operatorname{Cl} B = \operatorname{Cl} A$.
We shall prove that

$$a \notin A. \tag{16}$$

By contradiction. Suppose that $a \in A$. Then A is a neighbourhood of a. Thus there exists a positive integer m such that $U_{2m} \subset A$. By (13) we have $\emptyset \neq Z_{2m+1} \subset U_{2m} \subset A$. By (14) we obtain $Z_{2m+1} \subset X - B$, hence $Z_{2m+1} \subset A - B \subset \operatorname{Cl} B$. Then $\emptyset \neq Z_{2m+1} \subset \operatorname{Int}(\operatorname{Cl} B - B) = \emptyset$, a contradiction.
By (15) and (16) we obtain $a \in Cl\, A - A$.
The proof is complete.

2.4. Lemma. *Let X be an extremally disconnected space and Y a regular space. If a function $f \colon X \to Y$ is quasi-continuous on X, then it is continuous on X.* (See [6; Theorem 2] and [10; Theorem 3.2].)

The following example shows that the assumption "first countable" in Theorem 2.3 cannot be omitted.

2.5. Example. Let X be the Čech-Stone compactification of the set of all positive integer numbers. Let Y be the real line with the Euclidean topology. Let

A be a dense subset of X such that $A \neq X$. By 2.4 the set A is determining for the class $Q_{X,Y}$. Thus $\mathscr{D}(Q_{X,Y}) \neq \{X\}$.

The following example shows that the assumption "Hausdorff space" in Theorem 2.3 cannot be replaced by the assumption "T_1-space".

2.6. Example. Let X be an infinite countable set with the cofinite topology. Let Y be the set of all real numbers with the Euclidean topology. Let $a \in X$. Since X is hyperconnected, by 1.10 the set $\{a\}$ is determining for the class $Q_{X,Y}$. Thus $\mathscr{D}(Q_{X,Y}) \neq \{X\}$.

3. Determining sets for the class of somewhat continuous functions

The following theorem shows that $\mathscr{D}(S_{X,Y}) = \mathscr{D}(\chi^{e,t}_{X,Y} \cap S_{X,Y})$ and gives a characterization of the family $\mathscr{D}(S_{X,Y})$.

3.1. Theorem. *Let X be a topological and Y a Urysohn space. Let $e, t \in Y$, $e \neq t$. Let A be a nonempty subset of X. Then the following statements are equivalent:*

(i) $$A \in \mathscr{D}(S_{X,Y}),$$

(ii) $$A \in \mathscr{D}(\chi^{e,t}_{X,Y} \cap S_{X,Y}),$$

(iii) *for each $L \subset K \subset X$, $\emptyset \neq K - L \subset X - A$, some of the following assertions holds:*

$$\mathit{Int}\ K = \emptyset, \tag{18}$$

$$L \text{ is dense in } X, \tag{19}$$

$$\mathit{Int}\ L = \emptyset \text{ and } L \neq \emptyset, \tag{20}$$

$$K \text{ is dense in } X \text{ and } K \neq X. \tag{21}$$

Proof. That (i) implies (ii) is obvious.
(ii) $\Rightarrow$ (iii): Deny. Suppose that there exist $L \subset K \subset X$ such that

$$\emptyset \neq K - L \subset X - A,$$
$$\mathrm{Int}\ K \neq \emptyset,$$
$$L \text{ is not dense in } X,$$
$$\mathrm{Int}\ L = \emptyset \text{ implies } L = \emptyset,$$
$$K \text{ is dense in } X \text{ implies } K = X.$$

Put $f = \chi^{e,t}_K$ and $g = \chi^{e,t}_L$. It is not difficult to verify that f, g are somewhat continuous functions which agree on the set A such that $f \neq g$. Thus $A \notin \mathscr{D}(\chi^{e,t}_{X,Y} \cap S_{X,Y})$.
(iii) $\Rightarrow$ (i): By contradiction. Suppose that (iii) holds and there exist two different

functions f, $g \in S_{X,Y}$ which agree on the set A. Choose a, $b \in X$ such that

(22) $$f(a) = g(a) \text{ and } f(b) \neq g(b).$$

First we shall prove that for each nonempty open set $G \subset X$ we have

(23) $$\text{if } G \subset X - A, \text{ then } G \text{ is dense in } X.$$

Let G be a nonempty open subset of X such that $G \subset X - A$. Putting in (iii) $K = G$ and $L = \emptyset$, we obtain that G is dense in X.
Now, we shall prove that for each nonempty open subset E in X we have

(24) $$E \subset Cl\{b\} \text{ implies } E \cap A \neq \emptyset.$$

By contradiction. Suppose that there exists a nonempty open subset E of X such that $E \subset \mathrm{Cl}\{b\}$ and $E \cap A = \emptyset$. Let D be an open neighbourhood of $f(a)$ in Y. Since $f \in S_{X,Y}$, we have $\mathrm{Int} f^{-1}(D) \neq \emptyset$. By (23) the set E is dense in X, therefore $\emptyset \neq E \cap \mathrm{Int} f^{-1}(D) \subset \mathrm{Cl}\{b\} \cap \mathrm{Inf} f^{-1}(D)$. Thus $b \in \mathrm{Int} f^{-1}(D)$. Hence $f(b) \in D$. This shows that the point $f(b)$ lies in every neighbourhood of $f(a)$. Thus $f(a) = f(b)$. Analogously, we obtain $g(a) = g(b)$. This is contrary to (22). Choose U and V open neighbourhoods of the points $f(b)$ and $g(b)$, respectively, such that

(25) $$\mathrm{Cl}\, U \cap \mathrm{Cl}\, V = \emptyset.$$

Since f, $g \in S_{X,Y}$, we have

(26) $$\mathrm{Int} f^{-1}(U) \neq \emptyset \neq \mathrm{Int}\, g^{-1}(V).$$

Put

$$W = \mathrm{Int} f^{-1}(U) \cap \mathrm{Int}\, g^{-1}(V).$$

It is easy to prove that

(27) $$W \subset X - A.$$

Now, we shall prove that

(28) $$W = \emptyset.$$

By contradiction. Suppose that $W \neq \emptyset$. Let H be an open neighbourhood of the point $f(a)$. Since $f \in S_{X,Y}$, we have $\mathrm{Int} f^{-1}(H) \neq \emptyset$. By (27) and (23) the set W is dense in X, therefore we obtain $\emptyset \neq W \cap \mathrm{Int} f^{-1}(H) \subset f^{-1}(U \cap H)$. Thus we have $U \cap H \neq \emptyset$. This shows that each neighbourhood of $f(a)$ intersects the set U, i.e. $f(a) \in \mathrm{Cl}\, U$. Analogously, we obtain $g(a) \in \mathrm{Cl}\, V$. Then by (25) we have $f(a) \neq g(a)$. This is contrary to (22).
In the following we distinguish three cases.
a.) Suppose that $\mathrm{Int} f^{-1}(U) - \mathrm{Cl}\{b\} = \emptyset = \mathrm{Int}\, g^{-1}(V) - \mathrm{Cl}\{b\}$. Since the point

b lies in every nonempty open subset of $\mathrm{Cl}\{b\}$, by (26) we obtain $b \in \mathrm{Int} f^{-1}(U)$ and $b \in \mathrm{Int}\, g^{-1}(V)$. This is contrary to (28). This shows that the case a.) is not true.
b.) Suppose that $\mathrm{Int} f^{-1}(U) - \mathrm{Cl}\{b\} \neq \emptyset$. Putting in (iii) $K = \mathrm{Int} f^{-1}(U) \cup \{b\}$, $L = \mathrm{Int} f^{-1}(U) - \{b\}$, we obtain that

$$\mathrm{Int} f^{-1}(U) \cup \{b\} \text{ is dense in } X. \tag{29}$$

Then by (28) we have $\emptyset \neq (\mathrm{Int} f^{-1}(U) \cup \{b\}) \cap \mathrm{Int}\, g^{-1}(V) = \{b\} \cap \mathrm{Int}\, g^{-1}(V)$. Thus

$$b \in \mathrm{Int}\, g^{-1}(V). \tag{30}$$

We distinguish two cases. First, suppose that $\mathrm{Int}\, g^{-1}(V) - \mathrm{Cl}\{b\} = \emptyset$. Then by (26) and (24) we have $\mathrm{Int}\, g^{-1}(V) \cap A \neq \emptyset$. Choose a point z in this intersection. Then $f(z) = g(z) \in V$. Hence $f^{-1}(V) \neq \emptyset$. Since $f \in S_{X,Y}$, we have $\mathrm{Int} f^{-1}(V) \neq \emptyset$. Then by (29) we obtain $\emptyset \neq (\mathrm{Int} f^{-1}(U) \cup \{b\}) \cap \mathrm{Int} f^{-1}(V) \subset f^{-1}(U) \cap f^{-1}(V)$, which contradicts (25). Now, suppose that $\mathrm{Int}\, g^{-1}(V) - \mathrm{Cl}\{b\} \neq \emptyset$. Analogously as for (30) we obtain $b \in f^{-1}(U)$. Thus $b \in W$. This is contrary to (28). This shows that the case b.) is not true.
c.) Suppose that $\mathrm{Int}\, g^{-1}(V) - \mathrm{Cl}\{b\} \neq \emptyset$. Analogously as for b.) we obtain that the case c.) is not true.
The proof is complete.

The following theorem is obvious.

3.2. Theorem. *Let X be a Hausdorff space and Y a Urysohn space. Then*

$$\mathscr{D}(S_{X,Y}) = \{X\}.$$

Example 2.6 shows that the assumption "Hausdorff space" in Theorem 3.2 cannot be replaced by the assumption "T_1-space".

Question. *Can the assumption "Urysohn space" in Theorems 2.1 and 3.1 be replaced by the assumption "Hausdorff space"?*

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ОБ ОПРЕДЕЛЯЮЩИХ МНОЖЕСТВАХ ДЛЯ НЕКОТОРЫХ ОБОБЩЕНИЙ НЕПРЕРЫВНОСТИ

Jozef Doboš

Резюме

В настоящей работе изучаем системы определяющих множеств для классов квазинепрерывных функций и немножно-непрерывных функций. Показываем, что с точки зрения систем определяющих множеств для этих классов функций достаточно исследовать только их подклассы, элементамм которых являются характеристические функции множеств, а на основании этого даем характеризацию этих систем.