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# ON STONE–TYPE EXTENSIONS FOR GROUP–VALUED MEASURES

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ABSTRACT. Let $X$ be any set, $\mathcal{A} \subset \mathcal{P}(X)$ any algebra, and $E$ the Stone space associated with $\mathcal{A}$.

Let $G$ be a Dedekind complete Abelian lattice group and $m\colon \mathcal{A} \to G$ a finitely additive positive measure and set $\mu \equiv m \circ \varphi$. We prove that $\mu$ has a $\sigma$-additive $G$-valued extension $\nu$, defined on the $\sigma$-algebra of all Borelian sets of $E$.

## 1. Introduction

Let $X$ be any set, and $\mathcal{A} \subset \mathcal{P}(X)$ any algebra. It is well known (see [13]) that there exists a compact totally disconnected topological space $E$ such that $\mathcal{A}$ is isomorphic to the field $\mathcal{F}$ of the clopen sets of $E$: we denote by $\varphi\colon \mathcal{F} \to \mathcal{A}$ such an isomorphism. $E$ will be called the *Stone space* associated with $\mathcal{A}$. In particular, if $X$ is endowed with the discrete topology and $\mathcal{A} = \mathcal{P}(X)$, then $E = \beta X$ (i.e. the Stone-Čech compactification of $X$).

Now, let $G$ be any $\sigma$-Dedekind complete Abelian lattice group (in short, $\sigma$-complete l-group) and assume that $m$ is a finitely additive positive measure, $m\colon \mathcal{A} \to G$, and put $\mu \equiv m \circ \varphi$. In this note, we will prove that $\mu$ has a $\sigma$-additive $G$-valued extension $\nu$, defined on the $\sigma$-algebra $\sigma(\mathcal{F})$ generated by $\mathcal{F}$.

To prove this, we will use the principle of transfinite induction. In general, it is impossible to obtain a result of this kind by a Carathéodory-type process; in fact, our assertion is not true if we assume that $\mathcal{F}$ is *any* algebra and $\mu$ is an *arbitrary* $G$-valued $\sigma$-additive positive measure. If $G$ is a vector lattice, the result is true if and only if $G$ is weakly $\sigma$-distributive (see [20]). We note that, if $G = \mathcal{C}(S) = \{f \in \mathbb{R}^S : f \text{ is continuous}\}$ and $S$ is a compact extremally disconnected topological space, then $\mathcal{C}(S)$ is weakly $\sigma$-distributive if and only if every $\sigma$-meager subset of $S$ is nowhere dense in $S$ (a set is $\sigma$-meager if and only if it is a subset of the union of a countable family of closed nowhere dense Baire sets; see also [18] and [20]).

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We also note that there exist spaces of type $\mathcal{C}(S)$ which do not have any Hausdorff vector topology for which each bounded monotone increasing sequence converges to its least upper bound (see [19]): such spaces cannot be topological groups with respect to the order topology, because the order topology is $T_1$ (see [12]), and a topological group which is $T_1$ is $T_2$ too (see [11]). This means that our results are not contained in similar extension theorems stated for topological groups (see [14]).

In the literature, there are many studies about the problem of extending a $\sigma$-additive group-valued (or vector-valued) set function from an algebra $\mathcal{A}$ to a suitable $\sigma$-algebra containing $\mathcal{A}$. Among the authors, together with J. D. M. Wright ([20], [21]), we recall Celada ([5]), Fremlin ([7]), Kats ([9]), Sion ([14]), Šipoš ([15], [16]), Volauf ([17]).

Finally, we will prove that, if $G$ is a Dedekind complete l-group, then $\mu$ can be extended to a $\sigma$-additive measure $\nu_1$, defined on the whole $\sigma$-algebra of Borel sets of $E$. We do not know if this extension can be obtained also when $G$ is just $\sigma$-Dedekind complete. Furthermore, we will see that, if $\mathcal{A} = \mathcal{P}(X)$ and $m$ is invariant with respect to an amenable semigroup $H \subset X^X$ of transformations, then $\nu$ and $\nu_1$ are invariant with respect to the semigroup $H'$ "corresponding" to $H$.

## 2. The extensions

**2.1.** *Let $(G, +, \leq)$ be a $\sigma$-Dedekind complete Abelian group lattice ($\sigma$-complete l-group). Then (see [2]) $G$ is Archimedean, and hence (see [3], [6]) there exists a compact Stonian topological space $S$, unique up to homeomorphisms, such that $G$ is a subgroup of $\mathcal{C}_\infty(S) = \{f \in \tilde{\mathbb{R}}^S : f$ is continuous, and $\{s : |f(s)| = +\infty\}$ is nowhere dense in $S\}$.*

In the sequel, we will often use the following result (see [3], [6]).

**2.2. Theorem.** *Let $G$ and $S$ be as in 2.1. If $\{a_\lambda\}_{\lambda\in\Lambda}$ is any net such that $\forall\lambda\ a_\lambda \in G$ and $a = \sup_\lambda a_\lambda \in G$ (where the supremum is with respect to $G$), then $a = \sup_\lambda a_\lambda$ with respect to $\mathcal{C}_\infty(S)$, and the set $\{s \in S : (\sup_\lambda a_\lambda)(s) \neq \sup_\lambda a_\lambda(s)\}$ is meager in $S$.*

**2.3. Definition.** Let $E$ be any set, assume that $G$ is a $\sigma$-complete l-group, and let $\mathcal{A} \subset \mathcal{P}(E)$ be such that $\emptyset, E \in \mathcal{A}$. We say that a $G$-valued map $P$, defined on $\mathcal{A}$, is a *$\sigma$-additive measure* if it is monotone, finitely additive (i.e. $P(A \cup B) = P(A) + P(B)$, whenever $A, B, A \cup B \in \mathcal{A}$ and $A \cap B = \emptyset$) and if it satisfies the following properties:

(2.3.1) If $A_n \uparrow A$, $A_n, A \in \mathcal{A}$, then $P(A) = \sup_n P(A_n)$.

(2.3.2) If $A_n \downarrow A$, $A_n, A \in \mathcal{A}$, then $P(A) = \inf_n P(A_n)$.

(If $\mathcal{A}$ is an algebra, then (2.3.1) and (2.3.2) are equivalent.)

Now, we note the following fact.

**2.4.** Remark. Let $X$ be any set, $\mathcal{A} \subset \mathcal{P}(X)$ an algebra, and assume that $E$ and $\mathcal{F}$ are as in the introduction. If $m\colon \mathcal{A} \to G$ is a finitely additive positive measure, then $\mu \equiv m \circ \varphi\colon \mathcal{F} \to G$ is $\sigma$-additive. Moreover, we note that there exists a nowhere dense set $N \subset S$ such that, $\forall s \in S \setminus N$ and $\forall A \in \mathcal{F}$, $m_s(A) \equiv m(A)(s)$ is a finitely additive positive real-valued measure. Thus, by virtue of classical results, the map $A \mapsto \mu_s(A) \equiv (m_s \circ \varphi)(A) = \mu(A)(s)$ is $\sigma$-additive, for each $s \in S \setminus N$; and so, it has a (unique) extension, $\nu_s$, defined on the whole $\sigma$-algebra $\mathcal{B}$ of Borelian sets of $E$, where $E$ is as in the introduction (see [4]).

To prove this, we have essentially used perfectness of $\mathcal{F}$ ([13]). In the sequel, these facts will play a fundamental role, in the construction of the required extension.

Now, we state the main result.

**2.5. THEOREM.** *Let $G$ be a $\sigma$-complete $l$-group, and $\mu$, $\mathcal{F}$ be as in Theorem 2.4. Then $\mu$ has a $\sigma$-additive extension $\nu\colon \sigma(\mathcal{F}) \to G$.*

To prove this theorem, we proceed by transfinite induction (see [1], [10]) and use the fact that every countable union of meager sets is meager.

Let $\mathcal{F}_0 \equiv \mathcal{F}$. If $\alpha$ is an ordinal of first kind, we put

$$\mathcal{F}_{\alpha-1,\sigma} = \Big\{F:\ F = \bigcup_n F_n\,,\ \ F_n \in \mathcal{F}_{\alpha-1}\,,\ \ F_n \uparrow\Big\}\,,$$

$$\mathcal{F}_{\alpha-1,\sigma\delta} = \Big\{F:\ F = \bigcap_n F_n\,,\ \ F_n \in \mathcal{F}_{\alpha-1,\sigma}\,,\ \ F_n \downarrow\Big\}\,,$$

$$\mathcal{F}_\alpha = \Big\{F:\ F = \limsup_n F_n = \liminf_n F_n\,,\ \ F_n \in \mathcal{F}_{\alpha-1}\Big\} \subset \mathcal{F}_{\alpha-1,\sigma\delta}\,.$$

If $\alpha$ is an ordinal of second kind, we set: $\mathcal{F}_\alpha = \bigcup\limits_{\beta<\alpha} \mathcal{F}_\beta$. Then $\sigma(\mathcal{F}) = \mathcal{F}_\Omega$, where $\Omega$ is the first uncountable ordinal (see [10]).

If $\alpha$ is an ordinal of first kind, let $\tilde{\mu}_\alpha\colon \mathcal{F}_{\alpha-1,\sigma} \to G$ be defined by setting

$\tilde{\mu}_\alpha(A) = \sup\limits_n \mu_{\alpha-1}(B_n)$, whenever $A = \bigcup\limits_n B_n$, $B_n \in \mathcal{F}_{\alpha-1}$, $B_n \uparrow$,

and define

$\mu^*_\alpha\colon \mathcal{F}_{\alpha-1,\sigma\delta} \to G$ by putting $\mu^*_\alpha(A) = \inf\limits_n \tilde{\mu}_\alpha(B_n)$,

whenever $A = \bigcap\limits_n B_n$, $B_n \in \mathcal{F}_{\alpha-1,\sigma}$, $B_n \downarrow$.

We will denote by $\mu_\alpha$ the restriction of $\mu^*_\alpha$ to $\mathcal{F}_\alpha$.

If $\alpha$ is an ordinal of second kind, let $\mu_\alpha\colon \mathcal{F}_\alpha \to G$ be defined in the following way: $\mu_\alpha(F) = \mu_\beta(F)$, whenever $F \in \mathcal{F}_\beta$, with $\beta < \alpha$.

To prove Theorem 2.5, it will be enough to prove the following two assertions:

**2.6. THEOREM.** *Let $G$ be a $\sigma$-complete $l$-group, and $\mu$, $\mathcal{F}$ be as in 2.4. Then the map $\mu_\alpha \colon \mathcal{F}_\alpha \to G$ is $\sigma$-additive, for all $\alpha \leq \Omega$.*

**2.7. LEMMA.** *Let $S$, $N$, $\mu$, $\mu_s$, $\nu_s$ be as above. Then for each fixed set $B \in \mathcal{F}_\alpha$, there exists a meager set $L_B$ such that $\nu_s(B) = \mu_\alpha(B)(s)$, $\forall s \in S \setminus L_B$, for each ordinal $\alpha$ such that $\alpha \leq \Omega$.*

To prove Lemma 2.7 and Theorem 2.6, we apply the principle of transfinite induction.

Firstly, suppose that $\alpha$ is an ordinal of first kind. By hypothesis of transfinite induction, assume that the assertions hold for $\alpha - 1$. It will be enough to prove Lemma 2.7 and Theorem 2.6 in the case in which one has $\mathcal{F}_{\alpha-1,\sigma}$ and $\tilde{\mu}_\alpha$ instead of $\mathcal{F}_\alpha$ and $\mu_\alpha$ respectively.

First of all, we prove that $\tilde{\mu}_\alpha$ is well-defined.

Let $B_n, C_n \in \mathcal{F}_{\alpha-1}$, $B_n \uparrow A$, $C_n \uparrow A$. Then there exists a meager set $M$ depending on $\{B_n\}$ and $\{C_n\}$ such that $\forall s \in S \setminus M$:

$$\begin{aligned}\Big[\sup_n \mu_{\alpha-1}(B_n)\Big](s) &= \sup_n \big[\mu_{\alpha-1}(B_n)(s)\big] = \nu_s(A)\\ &= \sup_n \big[\mu_{\alpha-1}(C_n)(s)\big] = \Big[\sup_n \mu_{\alpha-1}(C_n)\Big](s).\end{aligned}$$

As the complement of a meager set is a dense set in $S$, we have: $\sup_n \mu_{\alpha-1}(B_n) = \sup_n \mu_{\alpha-1}(C_n)$. So, our definition makes sense.

Now, let $A \in \mathcal{F}_{\alpha-1,\sigma}$, $A_n \uparrow A$, $A_n \in \mathcal{F}_{\alpha-1}$. We have: $\tilde{\mu}_\alpha(A)(s) = \Big[\sup_n \mu_{\alpha-1}(A_n)\Big](s) = \sup_n \big[\mu_{\alpha-1}(A_n)(s)\big] = \sup_n \nu_s(A_n) = \nu_s(A)$ up to the complement of a meager set: thus, Lemma 2.7. is proved.

Moreover, if $A_n \uparrow A$, $A_n$, $A \in \mathcal{F}_{\alpha-1,\sigma}$, we have: $\Big[\sup_n \tilde{\mu}_\alpha(A_n)\Big](s) = \sup_n \big[\tilde{\mu}_\alpha(A_n)(s)\big] = \sup_n \nu_s(A_n) = \nu_s(A) = \tilde{\mu}_\alpha(A)(s)$ up to the complement of a meager set, and hence $\sup_n \tilde{\mu}_\alpha(A_n) = \tilde{\mu}_\alpha(A)$. Analogously, one can check the other required properties. So, 2.6 is proved at least in the case in which $\alpha$ is an ordinal of first kind.

Now, let $\alpha$ be an ordinal of second kind.

Fix $B \in \mathcal{F}_\alpha$. Then $B \in \mathcal{F}_\beta$ for some $\beta < \alpha$, and thus, by the hypothesis of transfinite induction, there exists a meager set $L_B$ such that $\forall s \in S \setminus L_B$, $\nu_s(B) = \mu_\beta(B)(s) = \mu_\alpha(B)(s)$ by the definition of $\mu_\alpha$. So, Lemma 2.7 is proved.

Now, pick $A_n, A \in \mathcal{F}_\alpha$. By virtue of 2.2, 2.4 and 2.7, we have

$$\Big[\sup_n \ \mu_\alpha(A_n)\Big](s) = \sup_n \big[\mu_\alpha(A_n)(s)\big] = \sup_n \big[\nu_s(A_n)\big] = \nu_s(A) = \mu_\alpha(A)(s)$$

up to the complement of a meager set. Thus, $\sup_n \mu_\alpha(A_n) = \mu_\alpha(A)$.

Moreover, it is easy to check the other required properties. So, Theorem 2.6, and hence Theorem 2.5, are completely proved. □

By the same technique, we will prove the following:

**2.8. THEOREM.** *Let $G$ be a Dedekind complete $l$-group, and let $\mu$ and $\mathcal{B}$ be the same as in 2.4. Then $\mu$ has a $\sigma$-additive extension $\nu_1 \colon \mathcal{B} \to G$, where $\mathcal{B}$ is the $\sigma$-algebra of the Borel sets of $E$, and $E$ is as in the introduction.*

Let $\mathcal{L}$ be the family of all open sets of $E$. For every $A \in \mathcal{L}$ put:

(2.8.1) $\mu_0(A) = \sup_{F\in\mathcal{F},\, F\subset A} \mu(A)$,

(2.8.2) $\lambda_s(A) = \sup_{F\in\mathcal{F},\, F\subset A} [\mu(A)(s)]$,

$\forall s \in S \setminus N$, where $N$ is the same as in 2.4.

We begin with a lemma.

**2.9. LEMMA.** *For each fixed $A \in \mathcal{L}$ there exists a meager set $M_A$ such that $\mu_0(A)(s) = \lambda_s(A)$, $\forall s \in S \setminus M_A$.*

Proof. We have:

$$\mu_0(A)(s) = \Big[\sup_{F\in\mathcal{F},\, F\subset A} \mu(A)\Big](s) = \sup_{F\in\mathcal{F},\, F\subset A} [\mu(A)(s)] = \lambda_s(A)$$

up to the complement of a meager set. From this, the assertion follows. □

As a consequence of Lemma 2.9, we prove the following:

**2.10. PROPOSITION.** *The map $\mu_0 \colon \mathcal{L} \to G$ defined in* (2.8.1) *is $\sigma$-additive.*

Proof. Firstly, we prove the finite additivity of $\mu_0$. Pick $A, B \in \mathcal{L}$ with $A \cap B = \emptyset$, and let $\lambda_s$ be as in (2.8.2). By Lemma 2.9, there exists a meager set $L_{A,B}$ such that $\lambda_s(A) = \mu_0(A)(s)$, $\lambda_s(B) = \mu_0(B)(s)$, $\lambda_s(A \cup B) = \mu_0(A \cup B)(s)$, $\forall s \in S \setminus L_{A,B}$. So we have: $\mu_0(A)(s) + \mu_0(B)(s) = \mu_0(A \cup B)(s)$, $\forall s \in S \setminus L_{A,B}$, and thus $\mu_0$ is finitely additive.

Now, let $A_n \uparrow A$, $A_n, A \in \mathcal{L}$, $\forall n \in \mathbb{N}$. One has:

$$\Big[\sup_n \mu_0(A_n)\Big](s) = \sup_n [\mu_0(A_n)](s) = \sup_n \lambda_s(A_n) = \lambda_s(A) = \mu_0(A)(s)$$

up to the complement of a meager set, and hence $\sup_n \mu_0(A_n) = \mu_0(A)$.

The proof of (2.3.2) is analogous. □

Now, let $\mathcal{D} = \{D = A \setminus B : \ A, B \in \mathcal{L},\ A \supset B\}$, $\mathcal{G} = \{F : F$ is a finite disjoint union of elements of $\mathcal{D}\}$. For all $D \in \mathcal{D}$, set

(2.10.1) $\hat{\mu}(D) = \mu_0(A) - \mu_0(B)$, whenever $D = A \setminus B$, with $A \supset B$.

For every $F \in \mathcal{G}$ put

(2.10.2) $\hat{\mu}(F) = \sum_{i=1}^{n} \hat{\mu}(F_i)$, whenever $F = \bigcup_{i=1}^{n} F_i$, $F_i \cap F_j = \emptyset$ if $i \neq j$.

We note that $\mathcal{G}$ is the algebra generated by $\mathcal{L}$ (see [8]).

Now, we claim the following:

**2.11. Proposition.** *The map $\hat{\mu}$ defined in* (2.10.1) *is well-defined.*

Proof. Let $\lambda_s$ be as in (2.8.2), $N$ and $\nu_s$ as in 2.4. If $D = A_1 \setminus B_1 = A_2 \setminus B_2$ with $A_j, B_j \in \mathcal{L}$ $(j = 1, 2)$, we have, up to the complement of a meager set $P$, depending on $A_j$ and $B_j$:

$$\begin{aligned} \mu_0(A_1)(s) - \mu_0(B_1)(s) &= \lambda_s(A_1) - \lambda_s(B_1) = \nu_s(A_1) - \nu_s(B_1) = \nu_s(D) \\ &= \nu_s(A_2) - \nu_s(B_2) = \lambda_s(A_2) - \lambda_s(B_2) \\ &= \mu_0(A_2)(s) - \mu_0(B_2)(s) . \end{aligned}$$

So, $\mu_0(A_1) - \mu_0(B_1) \equiv \mu_0(A_2) - \mu_0(B_2)$, and hence $\hat{\mu}$ is well-defined. □

Analogously as in Lemma 2.9 and Proposition 2.10, we can prove the following:

**2.12. Lemma.** *Let $\nu_s$ be as in the proof of Proposition* 2.11. *Then for every $D \in \mathcal{D}$, there exists a meager set $F_D$ such that $\nu_s(D) = \hat{\mu}(D)(s)$, $\forall s \in S \setminus F_D$.*

**2.13. Proposition.** *The map $\hat{\mu}$ defined in* (2.10.1) *and* (2.10.2) *is $\sigma$-additive.*

Similarly as above, we can check that $\hat{\hat{\mu}}$ is well-defined and $\sigma$-additive.

Though $\mathcal{G}$ is not perfect (in general), we can proceed as in 2.5, 2.6 and 2.7 (thanks to 2.9) and extend $\hat{\hat{\mu}}$ to a $\sigma$-additive $G$-valued measure $\nu_1$ defined on $\sigma(\mathcal{G}) = \mathcal{B}$ (starting with $\mathcal{F}_0 = \mathcal{G}$).

2.14. Remark. If $\mathcal{A} = \mathcal{P}(X)$, $G$ is a vector lattice, and $m \colon \mathcal{P}(X) \to G$ is invariant with respect to an amenable semigroup $H \subset X^X$, then $\mu \colon \mathcal{F} \to G$ is $H'$-invariant (where $H' \subset \beta X^{\beta X}$ is the semigroup "corresponding" to $H$: see [4]); moreover, $\mu_s \colon \mathcal{F} \to \mathbb{R}$ is $H'$-invariant too, for all $s \in S \setminus N$. As $\nu_s$ is $H'$-invariant ($\forall s \in S \setminus N$), it is easy to prove by using Lemma 2.7 (when $\alpha = \Omega$), that $\nu$ and $\nu_1$ are $H'$-invariant.

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