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Mathematica Slovaca, Vol. 46 (1996), No. 1, 111--126

Persistent URL: http://dml.cz/dmlcz/128817

Math. Slovaca, **46** (1996), No. 1, 111–126

Mathematica Slovaca

QUADRATIC REGRESSION MODELS[1]

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(*Communicated by Anatolij Dvurečenskij*)

ABSTRACT. A quadratic vector function is used as an approximation of the mean value of the observation vector in nonlinear regression models. In the quadratic model, considered explicit formulae for estimators and their covariance matrices are given.

Introduction

Estimation procedures in nonlinear regression models, cf. [3], in comparison with procedures in linear models, see, e.g., [4], are relatively complicated. However, when the effects of nonlinearity are limited up to the second power of differences between the actual values of the parameters and a priori (by a statistician chosen) values, then the estimation procedure can be developed with similar features as the linear procedures have, e.g., nonrecursive calculation, the explicit formulae for the covariance matrix of the estimator, etc.

The aim of the paper is to contribute to a development of such procedures.

1. Notations and auxiliary statements

Let $\boldsymbol{Y} \sim N_n\big(\mathbf{F}\boldsymbol{\beta}+\frac{1}{2}\boldsymbol{\kappa}, \boldsymbol{\Sigma}\big)$, $\boldsymbol{\beta} \in \mathbb{R}^k$, i.e., the n-dimensional random vector $\boldsymbol{Y}$ is normally distributed with the mean value $E(\boldsymbol{Y} \mid \boldsymbol{\beta}) = \mathbf{F}\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\kappa}$ and the covariance matrix $\operatorname{Var}(\boldsymbol{Y} \mid \boldsymbol{\Sigma}) = \boldsymbol{\Sigma}$. Here the $n \times k$ matrix $\mathbf{F}$ with the rank $r(\mathbf{F}) = k$ is known, $\boldsymbol{\kappa} = (\kappa_1, \dots, \kappa_n)'$ ($'$ denotes the transposition), $\kappa_i = \boldsymbol{\beta}'\mathbf{H}_i\boldsymbol{\beta}$, $i = 1, \dots, n$, the $k \times k$ matrices $\mathbf{H}_1, \dots, \mathbf{H}_n$ are symmetric and they are known,

AMS Subject Classification (1991): Primary 62F10, 62J02.
Key words: nonlinear regression model, quadratic approximation, linear and quadratic estimation.

[1] Supported by the Alexander von Humboldt-Stiftung and the grant No. 366 of the Slovak Academy of Sciences.

and the $n \times n$ matrix $\boldsymbol{\Sigma}$ is positively definite (p.d.) and also known. The unknown vector $\boldsymbol{\beta}$ can attain any value from k-dimensional Euclidean space $\mathbb{R}^k$. The aim is to estimate the value of $\boldsymbol{\beta}$ or a function of $\boldsymbol{\beta}$ on the basis of the vector $\boldsymbol{Y}$ by a linear or by a quadratic statistic.

In the following, $\mathcal{M}(\mathbf{A})$ denotes the column space of the matrix $\mathbf{A}$, and $\operatorname{vec}(\mathbf{A})$ denotes the vector created by the columns of the matrix $\mathbf{A}$. Let $\mathbf{A}$ be any $n \times s$ matrix and $\mathbf{W}$ an $n \times n$ positively semi-definite matrix with the property $\mathcal{M}(\mathbf{A}) \subset \mathcal{M}(\mathbf{W})$; then $\mathbf{P}_{\mathbf{A}}^{\mathbf{W}} = \mathbf{A}(\mathbf{A}'\mathbf{W}\mathbf{A})^{-}\mathbf{A}'\mathbf{W}$ (in this case, it is a projection matrix on $\mathcal{M}(\mathbf{A})$; $^{-}$ denotes a generalized inverse, cf. [5]) and $\mathbf{M}_{\mathbf{A}}^{\mathbf{W}} = \mathbf{I} - \mathbf{P}_{\mathbf{A}}^{\mathbf{W}}$. The symbol $\mathbf{I}$ denotes the identity matrix.

If $\mathbf{W} = \mathbf{I}$, then $\mathbf{P}_{\mathbf{A}} = \mathbf{A}(\mathbf{A}'\mathbf{A})^{-}\mathbf{A}'$ and $\mathbf{M}_{\mathbf{A}} = \mathbf{I} - \mathbf{P}_{\mathbf{A}}$. If $\mathcal{M}(\mathbf{A}) \not\subset \mathcal{M}(\mathbf{W})$, then $\mathbf{P}_{\mathbf{A}}^{\mathbf{W}} = \mathbf{A}(\mathbf{A}'\mathbf{W}\mathbf{A})^{+}\mathbf{A}'\mathbf{W}$ ($^{+}$ denotes the Moore-Penrose generalized inverse, cf. [5]) is correctly defined, idempotent, but it is a projection matrix on $\mathcal{M}\left[\mathbf{A}(\mathbf{A}'\mathbf{W}\mathbf{A})^{+}\right]$ only and not on $\mathcal{M}(\mathbf{A})$.

2. Linear and quadratic unbiased estimability

LEMMA 2.1. *In the considered model, the function $h(\boldsymbol{\beta}) = \boldsymbol{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathbb{R}^k$, is an unbiasedly estimable by a linear statistic if and only if $\boldsymbol{h} \in \mathcal{M}(\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}})$, where*

$$\tilde{\mathbf{H}} = \begin{pmatrix} (\operatorname{vec}(\mathbf{H}_1))' \\ \vdots \\ (\operatorname{vec}(\mathbf{H}_n))' \end{pmatrix}.$$

Proof. The statistic $\boldsymbol{L}'\boldsymbol{Y}$ is unbiased estimator of the function $h(\cdot)$ if and only if there exists a vector $\boldsymbol{L} \in \mathbb{R}^n$ such that

$$\forall\{\boldsymbol{\beta} \in \mathbb{R}^k\} \quad \left(E(\boldsymbol{L}'\boldsymbol{Y} \mid \boldsymbol{\beta}) = \right) \boldsymbol{L}'\mathbf{F}\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{L}'\tilde{\mathbf{H}}\boldsymbol{\beta}^{2\otimes} = \boldsymbol{h}'\boldsymbol{\beta}$$

$$\iff \begin{pmatrix} \mathbf{F}' \\ \tilde{\mathbf{H}}' \end{pmatrix} \boldsymbol{L} = \begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{O} \end{pmatrix} \iff \boldsymbol{h} \in \mathcal{M}(\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}}).$$

□

COROLLARY 2.2. *If $\mathcal{M}(\mathbf{F}) \subset \mathcal{M}(\tilde{\mathbf{H}})$, then no linear function of the parameter $\boldsymbol{\beta}$ can be linearly and unbiasedly estimated.*

Proof.

$$\begin{aligned} \mathcal{M}(\mathbf{F}) \subset \mathcal{M}(\tilde{\mathbf{H}}) &\iff \mathcal{M}(\mathbf{M}_{\tilde{\mathbf{H}}}) = \operatorname{Ker}(\tilde{\mathbf{H}}') \subset \operatorname{Ker}(\mathbf{F}') \\ &\implies \mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}} = \mathbf{O}. \end{aligned}$$

□

Corollary 2.3. *If* $\mathcal{M}(\tilde{\mathbf{H}}) \subset \mathcal{M}(\mathbf{F})$, *then*

$$\mathcal{M}(\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}}) \subset \mathcal{M}(\mathbf{F}') \quad \& \quad \mathcal{M}(\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}}) \neq \mathcal{M}(\mathbf{F}'),$$

i.e., there exist linear unbiasedly estimable functions in the linearized model, i.e., $\boldsymbol{Y} \sim N_n(\mathbf{F}\boldsymbol{\beta}, \boldsymbol{\Sigma})$, *which are not linearly unbiasedly estimable in the quadratic model.*

P r o o f.

$$\mathcal{M}(\tilde{\mathbf{H}}) \subset \mathcal{M}(\mathbf{F}) \iff \mathcal{M}(\mathbf{P}_{\tilde{\mathbf{H}}}) \subset \mathcal{M}(\mathbf{P}_{\mathbf{F}}).$$

Let $\mathbf{Q}_1$, $\mathbf{Q}_2$ be matrices such that $\mathbf{Q}_1'\mathbf{Q}_1 = \mathbf{I}$, $\mathbf{Q}_1'\mathbf{Q}_2 = \mathbf{O}$, $\mathbf{Q}_2'\mathbf{Q}_2 = \mathbf{I}$, $\mathbf{P}_{\tilde{\mathbf{H}}} = \mathbf{Q}_1\mathbf{Q}_1'$, and $\mathbf{P}_{\mathbf{F}} = \mathbf{Q}_1\mathbf{Q}_1' + \mathbf{Q}_2\mathbf{Q}_2'$. Then $\mathbf{M}_{\tilde{\mathbf{H}}} = \mathbf{I} - \mathbf{Q}_1\mathbf{Q}_1'$, and there exists a regular matrix $\begin{pmatrix}\mathbf{R}_1\\ \mathbf{R}_2\end{pmatrix}$ such that $\mathbf{F} = (\mathbf{Q}_1, \mathbf{Q}_2)\begin{pmatrix}\mathbf{R}_1\\ \mathbf{R}_2\end{pmatrix}$.

Thus

$$\mathcal{M}(\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}}) = \mathcal{M}\big[(\mathbf{R}_1'\mathbf{Q}_1' + \mathbf{R}_2'\mathbf{Q}_2')(\mathbf{I} - \mathbf{Q}_1\mathbf{Q}_1')\big] = \mathcal{M}(\mathbf{R}_2'\mathbf{Q}_2') = \mathcal{M}(\mathbf{R}_2'),$$

$$\mathcal{M}(\mathbf{F}') = \mathcal{M}\left[(\mathbf{R}_1', \mathbf{R}_2')\begin{pmatrix}\mathbf{Q}_1'\\ \mathbf{Q}_2'\end{pmatrix}\right] = \mathcal{M}(\mathbf{R}_1', \mathbf{R}_2').$$

As $\mathcal{M}(\mathbf{R}_2') \subset \mathcal{M}(\mathbf{R}_1', \mathbf{R}_2')$ & $\mathcal{M}(\mathbf{R}_2') \neq \mathcal{M}(\mathbf{R}_1', \mathbf{R}_2')$, the proof is finished. □

Corollary 2.4. *In the model, the whole parameter* $\boldsymbol{\beta}$ *is linearly unbiasedly estimable if and only if there exist matrices* $\mathbf{X}$ *and* $\mathbf{B}$ *such that* $\mathbf{F} = (\mathbf{U}_1\mathbf{X} + \mathbf{U}_2)\mathbf{B}$, *where* $\mathbf{U}_1$ *and* $\mathbf{U}_2$ *are matrices with orthonormal columns such that*

$$\mathbf{U}_1'\mathbf{U}_2 = \mathbf{O}, \quad \mathbf{P}_{\tilde{\mathbf{H}}} = \mathbf{U}_1\mathbf{U}_1' \quad \text{and} \quad \mathbf{U}_1\mathbf{U}_1' + \mathbf{U}_2\mathbf{U}_2' = \mathbf{I}.$$

P r o o f. There exist matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{F} = \mathbf{U}_1\mathbf{A} + \mathbf{U}_2\mathbf{B}$. Then

$$\mathcal{M}(\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}}) = \mathcal{M}\big[(\mathbf{A}'\mathbf{U}_1' + \mathbf{B}'\mathbf{U}_2')\mathbf{U}_2\mathbf{U}_2'\big] = \mathcal{M}(\mathbf{B}'\mathbf{U}_2') = \mathcal{M}(\mathbf{B}'),$$

and

$$\mathcal{M}(\mathbf{F}') = \mathcal{M}\left[(\mathbf{A}', \mathbf{B}')\begin{pmatrix}\mathbf{U}_1'\\ \mathbf{U}_2'\end{pmatrix}\right] = \mathcal{M}(\mathbf{A}', \mathbf{B}').$$

If $\mathcal{M}(\mathbf{F}') = \mathcal{M}(\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}})$, then $\mathcal{M}(\mathbf{A}') \subset \mathcal{M}(\mathbf{B}') \iff \exists\{\mathbf{X} :\ \mathbf{A} = \mathbf{X}\mathbf{B}\} \implies \mathbf{F} = (\mathbf{U}_1\mathbf{X} + \mathbf{U}_2)\mathbf{B}$.

If $\mathbf{F} = (\mathbf{U}_1\mathbf{X} + \mathbf{U}_2)\mathbf{B}$, then

$$\begin{aligned}\mathcal{M}(\mathbf{F}') &= \mathcal{M}\big[\mathbf{B}'(\mathbf{X}'\mathbf{U}_1' + \mathbf{U}_2')\big] = \mathcal{M}\left[\mathbf{B}'(\mathbf{X}', \mathbf{I})\begin{pmatrix}\mathbf{U}_1'\\ \mathbf{U}_2'\end{pmatrix}\right]\\ &= \mathcal{M}\big[\mathbf{B}'(\mathbf{X}', \mathbf{I})\big] = \mathcal{M}(\mathbf{B}'\mathbf{X}', \mathbf{B}') = \mathcal{M}(\mathbf{B}') = \mathcal{M}(\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}}).\end{aligned}$$

□

Remark 2.5. The statement given in Corollary 2.4 can be reformulated in the following way.

$$\mathcal{M}(\mathbf{F}) \cap \mathcal{M}(\mathbf{U}_1) = \{\boldsymbol{O}\} \quad \Longleftrightarrow \quad \mathbf{F} = \mathbf{U}_1\mathbf{X}\mathbf{B} + \mathbf{U}_2\mathbf{B}\,.$$

Proof. Let $\mathcal{M}(\mathbf{F}) \cap \mathcal{M}(\mathbf{U}_1) = \{\boldsymbol{O}\}$. Then there exist matrices $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{F} = \mathbf{U}_1\mathbf{A} + \mathbf{U}_2\mathbf{B}$. Further

$$r\begin{pmatrix} \mathbf{F}' \\ \mathbf{U}_1' \end{pmatrix} = r(\mathbf{F}'\mathbf{U}_2) + r(\mathbf{U}_1) = r(\mathbf{F}') + r(\mathbf{U}_1)$$

$$\Longrightarrow r(\mathbf{F}'\mathbf{U}_2) = r(\mathbf{F}) \Longleftrightarrow r(\mathbf{U}_2'\mathbf{F}) = r(\mathbf{F})\,,$$

$$\left.\begin{array}{l} r(\mathbf{U}_2'\mathbf{F}) = r(\mathbf{B}) = r(\mathbf{F}) \\ r(\mathbf{U}_1'\mathbf{F}) = r(\mathbf{A}) \le r(\mathbf{F}) \end{array}\right. \Longrightarrow \mathcal{M}(\mathbf{A}') \subset \mathcal{M}(\mathbf{B}') \Longrightarrow \mathbf{A} = \mathbf{X}\mathbf{B}\,.$$

Let $\mathbf{F} = \mathbf{U}_1\mathbf{X}\mathbf{B} + \mathbf{U}_2\mathbf{B}$. Let $\boldsymbol{O} \neq \boldsymbol{x} \in \mathcal{M}(\mathbf{F}) \cap \mathcal{M}(\mathbf{U}_1)$. Then there exists a vector $\boldsymbol{u}$ such that $\boldsymbol{x} = \mathbf{U}_1\mathbf{X}\mathbf{B}\boldsymbol{u} + \mathbf{U}_2\mathbf{B}\boldsymbol{u}$. As $\mathbf{U}_2'\boldsymbol{x} = \boldsymbol{O} = \mathbf{B}\boldsymbol{u}$, one has $\boldsymbol{x} = \boldsymbol{O}$, that is a contradiction. □

THEOREM 2.6. *If, for the function* $p(\boldsymbol{\beta}) = \boldsymbol{p}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathbb{R}^k$, *it is valid* $\boldsymbol{p} \in \mathcal{M}(\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}})$, *then the BLUE (best linear unbiased estimator) of the function* $p(\cdot)$ *is*

$$\widehat{\boldsymbol{p}'\boldsymbol{\beta}} = \left\{\begin{array}{l} \boldsymbol{p}'\big[\mathbf{F}'(\mathbf{M}_{\tilde{\mathbf{H}}}\Sigma\mathbf{M}_{\tilde{\mathbf{H}}})^+\mathbf{F}\big]^+\mathbf{F}'(\mathbf{M}_{\tilde{\mathbf{H}}}\Sigma\mathbf{M}_{\tilde{\mathbf{H}}})^+\boldsymbol{Y} \\ \boldsymbol{p}'\mathbf{C}^{-1}\mathbf{F}'\Sigma^{-1}\mathbf{M}_{\tilde{\mathbf{H}}}^{(\mathbf{M}_\mathbf{F}\Sigma\mathbf{M}_\mathbf{F})^+}\boldsymbol{Y}\,, \end{array}\right.$$

and its dispersion is

$$\operatorname{Var}\big(\widehat{\boldsymbol{p}'\boldsymbol{\beta}} \mid \Sigma\big) = \boldsymbol{p}'\big[\mathbf{F}'(\mathbf{M}_{\tilde{\mathbf{H}}}\Sigma\mathbf{M}_{\tilde{\mathbf{H}}})^+\mathbf{F}\big]^+\boldsymbol{p}\,.$$

Proof. A statistic $\boldsymbol{L}'\boldsymbol{Y}$ is an unbiased estimator of the function $p(\cdot)$ if and only if $\mathbf{F}'\boldsymbol{L} = \boldsymbol{p}$ and $\tilde{\mathbf{H}}'\boldsymbol{L} = \boldsymbol{O}$. The BLUE can be found by the Lagrange procedure.

$$\Phi(\boldsymbol{L}) = \boldsymbol{L}'\Sigma\boldsymbol{L} - 2\boldsymbol{\lambda}_1'(\mathbf{F}'\boldsymbol{L} - \boldsymbol{p}) - 2\boldsymbol{\lambda}_2'\tilde{\mathbf{H}}'\boldsymbol{L}\,,$$

$$\frac{1}{2}\frac{\partial\Phi(\boldsymbol{L})}{\partial\boldsymbol{L}}\bigg|_{\boldsymbol{L}=\boldsymbol{L}^*} = \Sigma\boldsymbol{L}^* - \mathbf{F}\boldsymbol{\lambda}_1 - \tilde{\mathbf{H}}\boldsymbol{\lambda}_2 = \boldsymbol{O} \Longrightarrow \boldsymbol{L}^* = \Sigma^{-1}\mathbf{F}\boldsymbol{\lambda}_1 + \Sigma^{-1}\tilde{\mathbf{H}}\boldsymbol{\lambda}_2\,.$$

$$\left.\begin{array}{l} (\,\mathbf{F}'\boldsymbol{L}^* =)\ \mathbf{F}'\Sigma^{-1}\mathbf{F}\boldsymbol{\lambda}_1 + \mathbf{F}'\Sigma^{-1}\tilde{\mathbf{H}}\boldsymbol{\lambda}_2 \ = \boldsymbol{p} \\ (\,\tilde{\mathbf{H}}\boldsymbol{L}^* =)\ \tilde{\mathbf{H}}'\Sigma^{-1}\mathbf{F}\boldsymbol{\lambda}_1 + \tilde{\mathbf{H}}'\Sigma^{-1}\tilde{\mathbf{H}}'\boldsymbol{\lambda}_2 = \boldsymbol{O} \end{array}\right\}$$

$$\Longrightarrow \begin{pmatrix} \boldsymbol{\lambda}_1 \\ \boldsymbol{\lambda}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{F}'\Sigma^{-1}\mathbf{F}, & \mathbf{F}'\Sigma^{-1}\tilde{\mathbf{H}} \\ \tilde{\mathbf{H}}'\Sigma^{-1}\mathbf{F}, & \tilde{\mathbf{H}}'\Sigma^{-1}\tilde{\mathbf{H}}' \end{pmatrix}^+ \begin{pmatrix} \boldsymbol{p} \\ \boldsymbol{O} \end{pmatrix}$$

is one of the Lagrange vector multipliers $\boldsymbol{\lambda}_1$, $\boldsymbol{\lambda}_2$ (however the estimator is unique). The Moore-Penrose g-inverse of the p.s.d. matrix of the system gives

$$\begin{aligned}\boldsymbol{\lambda}_1 &= \left[\mathbf{F}'\left(\mathbf{M}_{\tilde{\mathbf{H}}}\Sigma\mathbf{M}_{\tilde{\mathbf{H}}}\right)^{+}\mathbf{F}\right]^{+}\boldsymbol{p}\\ &= \mathbf{C}^{-1}\boldsymbol{p} + \mathbf{C}^{-1}\mathbf{F}'\Sigma^{-1}\tilde{\mathbf{H}}\left[\tilde{\mathbf{H}}'(\mathbf{M}_{\mathbf{F}}\Sigma\mathbf{M}_{\mathbf{F}})^{+}\tilde{\mathbf{H}}\right]^{+}\tilde{\mathbf{H}}\Sigma^{-1}\mathbf{F}\mathbf{C}^{-1}\boldsymbol{p},\\ \boldsymbol{\lambda}_2 &= -(\tilde{\mathbf{H}}'\Sigma^{-1}\tilde{\mathbf{H}})^{+}\tilde{\mathbf{H}}'\Sigma^{-1}\mathbf{F}\left[\mathbf{F}'\left(\mathbf{M}_{\tilde{\mathbf{H}}}\Sigma\mathbf{M}_{\tilde{\mathbf{H}}}\right)^{+}\mathbf{F}\right]^{+}\boldsymbol{p}\\ &= -\left[\tilde{\mathbf{H}}'(\mathbf{M}_{\mathbf{F}}\Sigma\mathbf{M}_{\mathbf{F}})^{+}\tilde{\mathbf{H}}\right]^{+}\tilde{\mathbf{H}}\Sigma^{-1}\mathbf{F}\mathbf{C}^{-1}\boldsymbol{p},\end{aligned}$$

where $\mathbf{C}^{-1} := (\mathbf{F}'\Sigma^{-1}\mathbf{F})^{-1}$.

For $(\boldsymbol{L}^*)'\boldsymbol{Y}$ we thus obtain

$$\begin{aligned}\widehat{\boldsymbol{p}'\boldsymbol{\beta}} = (\boldsymbol{L}^*)'\boldsymbol{Y} &= \boldsymbol{p}'\left[\mathbf{F}'\left(\mathbf{M}_{\tilde{\mathbf{H}}}\Sigma\mathbf{M}_{\tilde{\mathbf{H}}}\right)^{+}\mathbf{F}\right]^{+}\mathbf{F}'\left(\mathbf{M}_{\tilde{\mathbf{H}}}\Sigma\mathbf{M}_{\tilde{\mathbf{H}}}\right)^{+}\boldsymbol{Y}\\ &= \boldsymbol{p}'\mathbf{C}^{-1}\mathbf{F}'\Sigma^{-1}\mathbf{M}_{\tilde{\mathbf{H}}}^{(\mathbf{M}_{\mathbf{F}}\Sigma\mathbf{M}_{\mathbf{F}})^{+}}\boldsymbol{Y}.\end{aligned}$$

Further

$$\operatorname{Var}\left(\widehat{\boldsymbol{p}'\boldsymbol{\beta}} \mid \Sigma\right) = \boldsymbol{p}'\left[\mathbf{F}'\left(\mathbf{M}_{\tilde{\mathbf{H}}}\Sigma\mathbf{M}_{\tilde{\mathbf{H}}}\right)^{+}\mathbf{F}\right]^{+}\boldsymbol{p},$$

since $\left(\mathbf{M}_{\tilde{\mathbf{H}}}\Sigma\mathbf{M}_{\tilde{\mathbf{H}}}\right)^{+}\Sigma\left(\mathbf{M}_{\tilde{\mathbf{H}}}\Sigma\mathbf{M}_{\tilde{\mathbf{H}}}\right)^{+} = \left(\mathbf{M}_{\tilde{\mathbf{H}}}\Sigma\mathbf{M}_{\tilde{\mathbf{H}}}\right)^{+}$. □

LEMMA 2.7. *A linear function $h(\boldsymbol{\beta}) = \boldsymbol{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathbb{R}^k$, is unbiasedly estimable by the quadratic statistic $l + \boldsymbol{L}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}$ if and only if*

$$\begin{pmatrix}\mathbf{F}', & \mathbf{O}\\ \frac{1}{2}\tilde{\mathbf{H}}', & \mathbf{F}'\otimes\mathbf{F}'\\ \mathbf{O}, & \mathbf{F}'\otimes\tilde{\mathbf{H}}'\\ \mathbf{O}, & \tilde{\mathbf{H}}'\otimes\tilde{\mathbf{H}}'\end{pmatrix}\begin{pmatrix}\boldsymbol{L}\\ \operatorname{vec}(\mathbf{A})\end{pmatrix} = \begin{pmatrix}\boldsymbol{p}\\ \boldsymbol{O}\\ \boldsymbol{O}\\ \boldsymbol{O}\end{pmatrix}.$$

Proof.

$$\begin{aligned}&\forall\{\boldsymbol{\beta}\in\mathbb{R}^k\}\\ &E(l + \boldsymbol{L}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y} \mid \boldsymbol{\beta})\\ &\quad = l + \boldsymbol{L}'\mathbf{F}\boldsymbol{\beta} + \boldsymbol{L}'\frac{1}{2}\tilde{\mathbf{H}}\boldsymbol{\beta}^{2\otimes} + \operatorname{Tr}(\mathbf{A}\Sigma) + \left(\mathbf{F}\boldsymbol{\beta} + \tfrac{1}{2}\tilde{\mathbf{H}}\boldsymbol{\beta}^{2\otimes}\right)'\mathbf{A}\left(\mathbf{F}\boldsymbol{\beta} + \tfrac{1}{2}\tilde{\mathbf{H}}\boldsymbol{\beta}^{2\otimes}\right)\\ &\quad = l + \operatorname{Tr}(\mathbf{A}\Sigma) + \boldsymbol{L}'\mathbf{F}\boldsymbol{\beta} + \left\{\boldsymbol{L}'\frac{1}{2}\tilde{\mathbf{H}} + \left(\operatorname{vec}(\mathbf{A})\right)'(\mathbf{F}\otimes\mathbf{F})\right\}\boldsymbol{\beta}^{2\otimes}\\ &\qquad + \left(\operatorname{vec}(\mathbf{A})\right)'(\mathbf{F}\otimes\tilde{\mathbf{H}})\boldsymbol{\beta}^{3\otimes} + \frac{1}{4}\left(\operatorname{vec}(\mathbf{A})\right)'(\tilde{\mathbf{H}}\otimes\tilde{\mathbf{H}})\boldsymbol{\beta}^{4\otimes}\\ &\quad = \boldsymbol{p}'\boldsymbol{\beta}\\ \iff\ & l = -\operatorname{Tr}(\mathbf{A}\Sigma) \quad \& \quad \boldsymbol{L}'\mathbf{F} = \boldsymbol{p}' \quad \& \quad \frac{1}{2}\tilde{\mathbf{H}}'\boldsymbol{L} + (\mathbf{F}'\otimes\mathbf{F}')\operatorname{vec}(\mathbf{A}) = \boldsymbol{O}\\ &\quad \& \quad (\mathbf{F}'\otimes\tilde{\mathbf{H}}')\operatorname{vec}(\mathbf{A}) = \boldsymbol{O} \quad \& \quad (\tilde{\mathbf{H}}'\otimes\tilde{\mathbf{H}}')\operatorname{vec}(\mathbf{A}) = \boldsymbol{O}.\end{aligned}$$

□

COROLLARY 2.8. *The function* $h(\boldsymbol{\beta}) = \boldsymbol{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathbb{R}^k$, *is unbiasedly estimable by a quadratic statistic* $l + \boldsymbol{L}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}$ *if and only if*

$$\boldsymbol{h} \in \mathcal{M}\Big\{(\mathbf{F}', \mathbf{O})\,\mathrm{Ker}\Big[\frac{1}{2}\tilde{\mathbf{H}}',\ \mathbf{F}'\otimes(\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}})\Big]\Big\}.$$

P r o o f. By Lemma 2.6, the function $h(\cdot)$ is unbiasedly estimable by a quadratic statistic $l + \boldsymbol{L}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}$ if and only if

$$l = -\,\mathrm{Tr}(\mathbf{A}\boldsymbol{\Sigma}) \quad \& \quad \boldsymbol{L}'\mathbf{F} = \boldsymbol{p}' \quad \& \quad \frac{1}{2}\tilde{\mathbf{H}}'\boldsymbol{L} + (\mathbf{F}'\otimes\mathbf{F}')\,\mathrm{vec}(\mathbf{A}) = \boldsymbol{O}$$
$$\& \quad (\mathbf{F}'\otimes\tilde{\mathbf{H}}')\,\mathrm{vec}(\mathbf{A}) = \boldsymbol{O} \quad \& \quad (\tilde{\mathbf{H}}'\otimes\tilde{\mathbf{H}}')\,\mathrm{vec}(\mathbf{A}) = \boldsymbol{O}.$$

The last two conditions are satisfied by the matrix $\mathbf{A}$ of the form

$$\mathbf{A} = \mathbf{Z}_{\mathbf{A}} - (\tilde{\mathbf{H}}')^{-}\tilde{\mathbf{H}}'\mathbf{Z}_{\mathbf{A}}(\mathbf{F},\tilde{\mathbf{H}}')(\mathbf{F},\tilde{\mathbf{H}}')^{-}.$$

Thus the condition $\frac{1}{2}\tilde{\mathbf{H}}'\boldsymbol{L}+(\mathbf{F}'\otimes\mathbf{F}')\,\mathrm{vec}(\mathbf{A}) = \boldsymbol{O}$ can be equivalently reformulated as $-\frac{1}{2}\tilde{\mathbf{H}}'\boldsymbol{L} = \mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}}\mathbf{Z}_{\mathbf{A}}\mathbf{F}$, since $(\mathbf{F},\tilde{\mathbf{H}}')(\mathbf{F},\tilde{\mathbf{H}}')^{-}\mathbf{F} = \mathbf{F}$, and the matrix $(\tilde{\mathbf{H}}')^{-}$ can be chosen in such a form that $\mathbf{I} - (\tilde{\mathbf{H}}')^{-}\tilde{\mathbf{H}}' = \mathbf{M}_{\tilde{\mathbf{H}}}$. Thus

$$\begin{pmatrix} \mathbf{F}', & \mathbf{O} \\ \frac{1}{2}\tilde{\mathbf{H}}, & \mathbf{F}'\otimes(\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}}) \end{pmatrix}\begin{pmatrix} \boldsymbol{L} \\ \mathrm{vec}(\mathbf{Z}_{\mathbf{A}}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{h} \\ \boldsymbol{O} \end{pmatrix}$$
$$\Longrightarrow \boldsymbol{h} \in \mathcal{M}\Big\{(\mathbf{F}', \mathbf{O})\,\mathrm{Ker}\Big[\frac{1}{2}\tilde{\mathbf{H}}',\ \mathbf{F}'\otimes(\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}})\Big]\Big\}.$$

□

COROLLARY 2.9. *Let* $\mathbf{H}_i^* = \begin{pmatrix} \boldsymbol{e}_i'\mathbf{H}_1 \\ \vdots \\ \boldsymbol{e}_i'\mathbf{H}_n \end{pmatrix}$, $i = 1,\dots,k$, *where* $\boldsymbol{e}_i = (0_1,\dots,0_{i-1}, 1_i, 0_{i+1},\dots,0_n)'$. *The function* $h(\boldsymbol{\beta}) = \boldsymbol{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta}\in\mathbb{R}^k$, *is unbiasedly estimable by a quadratic statistic* $l + \boldsymbol{L}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}$ *if and only if*

$$\boldsymbol{h} \in \mathcal{M}(\mathbf{F}'\mathbf{M}_{\mathbf{R}}), \qquad \mathbf{R} = \Big(\mathbf{H}_1^*\mathbf{M}_{\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}}},\dots,\mathbf{H}_k^*\mathbf{M}_{\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}}}\Big).$$

P r o o f. By Lemma 3.2 in [1], for any matrices $\mathbf{B}_{t,s}$, $\mathbf{C}_{t,u}$

$$\mathrm{Ker}(\mathbf{B},\mathbf{C}) = \mathcal{M}\begin{pmatrix} \mathbf{M}_{\mathbf{B}'\mathbf{M}_{\mathbf{C}}} \\ \mathbf{M}_{\mathbf{C}'\mathbf{M}_{\mathbf{B}}} \end{pmatrix}.$$

Regarding Corollary 2.8 let $\mathbf{B} = \frac{1}{2}\tilde{\mathbf{H}}'$, $\mathbf{C} = \mathbf{F}'\otimes(\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}})$; thus

$$\mathbf{B}'\mathbf{M}_{\mathbf{C}} = \frac{1}{2}\tilde{\mathbf{H}}\Big(\mathbf{I} - \mathbf{P}_{\mathbf{F}'\otimes(\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}})}\Big) = \frac{1}{2}\tilde{\mathbf{H}}\Big(\mathbf{I}\otimes\mathbf{I} - \mathbf{P}_{\mathbf{F}'}\otimes\mathbf{P}_{\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}}}\Big).$$

Since

$$\mathbf{I} \otimes \mathbf{I} - \mathbf{P}_{\mathbf{F}'} \otimes \mathbf{P}_{\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}}} = \mathbf{M}_{\mathbf{F}'} \otimes \mathbf{P}_{\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}}} + \mathbf{I} \otimes \mathbf{M}_{\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}}}$$

and

$$\mathcal{M}(\mathbf{F}') = \mathbb{R}^k \implies \mathbf{M}_{\mathbf{F}'} = \mathbf{O},$$

we obtain

$$\mathbf{B}'\mathbf{M}_{\mathbf{C}} = \frac{1}{2}\tilde{\mathbf{H}}\Big(\mathbf{I} \otimes \mathbf{M}_{\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}}}\Big) = \frac{1}{2}\Big(\mathbf{H}_1^*\mathbf{M}_{\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}}}, \dots, \mathbf{H}_k^*\mathbf{M}_{\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}}}\Big).$$

Thus

$$\mathcal{M}(\mathbf{B}'\mathbf{M}_{\mathbf{C}}) = \mathcal{M}\Big(\mathbf{H}_1^*\mathbf{M}_{\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}}}, \dots, \mathbf{H}_k^*\mathbf{M}_{\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}}}\Big) = \mathcal{M}(\mathbf{R})$$

and

$$\mathcal{M}\Big\{(\mathbf{F}', \mathbf{O}) \operatorname{Ker}\Big[\frac{1}{2}\tilde{\mathbf{H}}', \mathbf{F}' \otimes (\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}})\Big]\Big\} = \mathcal{M}(\mathbf{F}'\mathbf{M}_{\mathbf{R}}).$$

□

Lemma 2.10. *If $\mathcal{M}(\mathbf{F}) \subset \mathcal{M}(\tilde{\mathbf{H}})$, then there does not exist a linear function of the parameter $\boldsymbol{\beta}$ which can be unbiasedly estimated by a quadratic statistic $l + \boldsymbol{L}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}$.*

Proof.

$$\begin{aligned}
\mathcal{M}(\mathbf{F}) \subset \mathcal{M}(\tilde{\mathbf{H}}) &\iff \exists\{\mathbf{E}: \ \mathbf{F} = \tilde{\mathbf{H}}\mathbf{E}\} \\
&\implies \quad \mathcal{M}\Big\{(\mathbf{F}', \mathbf{O}) \operatorname{Ker}\Big[\frac{1}{2}\tilde{\mathbf{H}}', \mathbf{F}' \otimes (\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}})\Big]\Big\} \\
&\qquad = \mathcal{M}\Big\{(\mathbf{E}'\tilde{\mathbf{H}}', \mathbf{O}) \operatorname{Ker}\Big[\frac{1}{2}\tilde{\mathbf{H}}, \mathbf{E}'\tilde{\mathbf{H}}' \otimes (\mathbf{E}'\tilde{\mathbf{H}}'\mathbf{M}_{\tilde{\mathbf{H}}})\Big]\Big\} \\
&\qquad = \mathcal{M}\Big\{(\mathbf{E}'\tilde{\mathbf{H}}', \mathbf{O}) \operatorname{Ker}\Big(\frac{1}{2}\tilde{\mathbf{H}}', \mathbf{O}\Big)\Big\} = \{\boldsymbol{O}\}.
\end{aligned}$$

□

Lemma 2.11. *The BLUE of the mean value of the observation vector $\boldsymbol{Y}$ is*

$$\begin{aligned}
\widehat{E(\boldsymbol{Y} \mid \boldsymbol{\beta})} &= \Big(\mathbf{P}_{\mathbf{F}}^{(\mathbf{M}_{\tilde{\mathbf{H}}}\Sigma\mathbf{M}_{\tilde{\mathbf{H}}})^+} + \mathbf{P}_{\tilde{\mathbf{H}}}^{\Sigma^{-1}}\mathbf{M}_{\mathbf{F}}^{(\mathbf{M}_{\tilde{\mathbf{H}}}\Sigma\mathbf{M}_{\tilde{\mathbf{H}}})^+}\Big)\boldsymbol{Y} \\
&= \Big(\mathbf{P}_{\tilde{\mathbf{H}}}^{(\mathbf{M}_{\mathbf{F}}\Sigma\mathbf{M}_{\mathbf{F}})^+} + \mathbf{P}_{\mathbf{F}}^{\Sigma^{-1}}\mathbf{M}_{\tilde{\mathbf{H}}}^{(\mathbf{M}_{\mathbf{F}}\Sigma\mathbf{M}_{\tilde{\mathbf{H}}})^+}\Big)\boldsymbol{Y}.
\end{aligned}$$

Proof. Let $\boldsymbol{L}_0 \in \mathbb{R}^n$ be arbitrary, and $f(\boldsymbol{\beta}) = \boldsymbol{L}_0'\mathbf{F}\boldsymbol{\beta} + \boldsymbol{L}_0'\frac{1}{2}\boldsymbol{\kappa}$, $\boldsymbol{\beta} \in \mathbb{R}^k$. An unbiased linear estimator $\boldsymbol{L}'\boldsymbol{Y}$ of the function $f(\cdot)$ must fulfil the conditions

$$\boldsymbol{L}'\mathbf{F} = \boldsymbol{L}_0'\mathbf{F} \qquad \& \qquad \tilde{\mathbf{H}}'\boldsymbol{L} = \tilde{\mathbf{H}}'\boldsymbol{L}_0.$$

The vector $\boldsymbol{L}$ which minimizes the quantity $\boldsymbol{L}'\boldsymbol{\Sigma}\boldsymbol{L}$ under these conditions can be obtained by the standard Lagrange procedure; the auxiliary function is

$$\Phi(\boldsymbol{L}) = \boldsymbol{L}'\boldsymbol{\Sigma}\boldsymbol{L} - 2\boldsymbol{\lambda}_1'(\mathbf{F}'\boldsymbol{L} - \mathbf{F}'\boldsymbol{L}_0) - 2\boldsymbol{\lambda}_2'(\tilde{\mathbf{H}}'\boldsymbol{L} - \tilde{\mathbf{H}}'\boldsymbol{L}_0).$$

The equation

$$\frac{\partial\Phi(\boldsymbol{L})}{\partial\boldsymbol{L}} = \boldsymbol{O},$$

implies the following solution for the Lagrange vector coefficients $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$:

$$\begin{pmatrix}\boldsymbol{\lambda}_1\\ \boldsymbol{\lambda}_2\end{pmatrix} = \begin{pmatrix}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F}, & \mathbf{F}'\boldsymbol{\Sigma}^{-1}\tilde{\mathbf{H}}\\ \tilde{\mathbf{H}}\boldsymbol{\Sigma}^{-1}\mathbf{F}, & \tilde{\mathbf{H}}'\boldsymbol{\Sigma}^{-1}\tilde{\mathbf{H}}\end{pmatrix}^{+}\begin{pmatrix}\mathbf{F}'\\ \tilde{\mathbf{H}}'\end{pmatrix}\boldsymbol{L}_0.$$

Thus, we obtain the estimator of the function $f(\cdot)$ in the form

$$\boldsymbol{L}_0'(\mathbf{F},\tilde{\mathbf{H}})\begin{pmatrix}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F}, & \mathbf{F}'\boldsymbol{\Sigma}^{-1}\tilde{\mathbf{H}}\\ \tilde{\mathbf{H}}\boldsymbol{\Sigma}^{-1}\mathbf{F}, & \tilde{\mathbf{H}}'\boldsymbol{\Sigma}^{-1}\tilde{\mathbf{H}}\end{pmatrix}^{+}\begin{pmatrix}\mathbf{F}'\\ \tilde{\mathbf{H}}'\end{pmatrix}\boldsymbol{\Sigma}^{-1}\boldsymbol{Y} = \boldsymbol{L}_0'\mathbf{P}^{\boldsymbol{\Sigma}^{-1}}_{(\mathbf{F},\tilde{\mathbf{H}})}\boldsymbol{Y}.$$

Now it is sufficient to express the projection matrix $\mathbf{P}^{\boldsymbol{\Sigma}^{-1}}_{(\mathbf{F},\tilde{\mathbf{H}})}$ via the projection matrices given in the statement and to realize that the vector $\boldsymbol{L}_0$ is arbitrary. □

Let

$$\begin{pmatrix}\mathbf{F}'\\ \tilde{\mathbf{H}}'\end{pmatrix}^{-} = (\mathbf{U},\mathbf{V}) = \boldsymbol{\Sigma}^{-1}(\mathbf{F},\tilde{\mathbf{H}})\begin{pmatrix}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\mathbf{F}, & \mathbf{F}'\boldsymbol{\Sigma}^{-1}\tilde{\mathbf{H}}\\ \tilde{\mathbf{H}}'\boldsymbol{\Sigma}^{-1}\mathbf{F}, & \tilde{\mathbf{H}}'\boldsymbol{\Sigma}^{-1}\tilde{\mathbf{H}}\end{pmatrix}^{+},$$

i.e., $\mathbf{U} = (\mathbf{M}_{\tilde{\mathbf{H}}}\boldsymbol{\Sigma}\mathbf{M}_{\tilde{\mathbf{H}}})^{+}\mathbf{F}[\mathbf{F}'(\mathbf{M}_{\tilde{\mathbf{H}}}\boldsymbol{\Sigma}\mathbf{M}_{\tilde{\mathbf{H}}})^{+}\mathbf{F}]^{+}$.

THEOREM 2.12. *Let a function $p(\boldsymbol{\beta}) = \boldsymbol{p}'\boldsymbol{\beta}$, $\boldsymbol{\beta}\in\mathbb{R}^k$, be unbiasedly estimable by a quadratic statistic $-\operatorname{Tr}(\mathbf{A}\boldsymbol{\Sigma}) + \boldsymbol{L}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}$, i.e.,*

$$\boldsymbol{p}\in\mathcal{M}\Big\{(\mathbf{F}',\mathbf{O})\operatorname{Ker}\Big[\tfrac{1}{2}\tilde{\mathbf{H}}',\ \mathbf{F}'\otimes(\mathbf{F}'\mathbf{M}_{\tilde{\mathbf{H}}})\Big]\Big\}.$$

Then, at the point $\boldsymbol{\beta}=\boldsymbol{O},\boldsymbol{\Sigma}$, the best unbiased estimator is

$$\widehat{\boldsymbol{p}'\boldsymbol{\beta}} = \boldsymbol{p}'\Big[\mathbf{F}'\Big(\boldsymbol{\Sigma}+\tfrac{1}{2}\mathbf{S}_{\mathbf{U}'\boldsymbol{\Sigma}\mathbf{U}}\Big)^{-1}\mathbf{F}\Big]^{-1}\mathbf{F}'\Big(\boldsymbol{\Sigma}+\tfrac{1}{2}\mathbf{S}_{\mathbf{U}'\boldsymbol{\Sigma}\mathbf{U}}\Big)^{-1}$$
$$\cdot\left\{\boldsymbol{Y}+\frac{1}{2}\begin{pmatrix}\operatorname{Tr}\{[\mathbf{F}'(\mathbf{M}_{\tilde{\mathbf{H}}}\boldsymbol{\Sigma}\mathbf{M}_{\tilde{\mathbf{H}}})^{+}\mathbf{F}]^{+}\mathbf{H}_1\} - (\boldsymbol{\beta}^*)'\mathbf{H}_1\boldsymbol{\beta}^*\\ \vdots\\ \operatorname{Tr}\{[\mathbf{F}'(\mathbf{M}_{\tilde{\mathbf{H}}}\boldsymbol{\Sigma}\mathbf{M}_{\tilde{\mathbf{H}}})^{+}\mathbf{F}]^{+}\mathbf{H}_n\} - (\boldsymbol{\beta}^*)'\mathbf{H}_n\boldsymbol{\beta}^*\end{pmatrix}\right\},$$

where

$$\mathbf{U}'\boldsymbol{\Sigma}\mathbf{U} = [\mathbf{F}'(\mathbf{M}_{\tilde{\mathbf{H}}}\boldsymbol{\Sigma}\mathbf{M}_{\tilde{\mathbf{H}}})^{+}\mathbf{F}]^{+},$$
$$\{\mathbf{S}_{\mathbf{U}'\boldsymbol{\Sigma}\mathbf{U}}\}_{i,j} = \operatorname{Tr}(\mathbf{H}_i\mathbf{U}'\boldsymbol{\Sigma}\mathbf{U}\mathbf{H}_j\mathbf{U}'\boldsymbol{\Sigma}\mathbf{U}),\qquad i,j=1,\dots k,$$

and

$$\boldsymbol{\beta}^* = \left[\mathbf{F}'\left(\mathbf{M}_{\tilde{\mathbf{H}}}\boldsymbol{\Sigma}\mathbf{M}_{\tilde{\mathbf{H}}}\right)^+\mathbf{F}\right]^+\mathbf{F}'\left(\mathbf{M}_{\tilde{\mathbf{H}}}\boldsymbol{\Sigma}\mathbf{M}_{\tilde{\mathbf{H}}}\right)^+\boldsymbol{Y}.$$

P r o o f. Let $\mathbf{H}_{\boldsymbol{L}} = \sum_{i=1}^{n} L_i\mathbf{H}_i$. The matrix $\mathbf{A}$ satisfying the equality

$$\begin{pmatrix}\mathbf{F}'\\ \tilde{\mathbf{H}}'\end{pmatrix}\mathbf{A}(\mathbf{F},\tilde{\mathbf{H}}) = \begin{pmatrix}-\frac{1}{2}\mathbf{H}_{\boldsymbol{L}}, & \mathbf{O}\\ \mathbf{O}, & \mathbf{O}\end{pmatrix}$$

can be expressed as

$$\mathbf{A} = -\frac{1}{2}\mathbf{U}\mathbf{H}_{\boldsymbol{L}}\mathbf{U}' + \mathbf{Z}_{\mathbf{A}} - (\mathbf{U}\mathbf{F}' + \mathbf{V}\tilde{\mathbf{H}}')\mathbf{Z}_{\mathbf{A}}(\mathbf{F}\mathbf{U}' + \tilde{\mathbf{H}}\mathbf{V}').$$

Now we shall proceed by the standard Lagrange method. The auxiliary function is

$$\begin{aligned}
&\Phi(\boldsymbol{L},\mathbf{Z}_{\mathbf{A}})\\
&\quad= \boldsymbol{L}'\boldsymbol{\Sigma}\boldsymbol{L} + 2\operatorname{Tr}\Big\{\Big[-\frac{1}{2}\mathbf{U}\mathbf{H}_{\boldsymbol{L}}\mathbf{U}' + \mathbf{Z}_{\mathbf{A}} - (\mathbf{U}\mathbf{F}' + \mathbf{V}\tilde{\mathbf{H}}')\mathbf{Z}_{\mathbf{A}}(\mathbf{F}\mathbf{U}' + \tilde{\mathbf{H}}\mathbf{V}')\Big]\\
&\qquad\cdot\boldsymbol{\Sigma}\Big[-\frac{1}{2}\mathbf{U}\mathbf{H}_{\boldsymbol{L}}\mathbf{U}' + \mathbf{Z}_{\mathbf{A}} - (\mathbf{U}\mathbf{F}' + \mathbf{V}\tilde{\mathbf{H}}')\mathbf{Z}_{\mathbf{A}}(\mathbf{F}\mathbf{U}' + \tilde{\mathbf{H}}\mathbf{V}')\Big]\boldsymbol{\Sigma}\Big\} - 2\boldsymbol{\lambda}'(\mathbf{F}'\boldsymbol{L} - \boldsymbol{p}),\\
&\frac{\partial\Phi(\boldsymbol{L},\mathbf{Z}_{\mathbf{A}})}{\partial\boldsymbol{L}} = 2\boldsymbol{\Sigma}\boldsymbol{L} - 2\mathbf{F}\boldsymbol{\lambda} + \mathbf{S}_{\mathbf{U}'\boldsymbol{\Sigma}\mathbf{U}}\boldsymbol{L}\\
&\qquad - 2\begin{pmatrix}\operatorname{Tr}\{[\boldsymbol{\Sigma}\mathbf{Z}_{\mathbf{A}}\boldsymbol{\Sigma} - \boldsymbol{\Sigma}(\mathbf{U}\mathbf{F}' + \mathbf{V}\tilde{\mathbf{H}}')\mathbf{Z}_{A}(\mathbf{F}\mathbf{U}' + \tilde{\mathbf{H}}\mathbf{V}')\boldsymbol{\Sigma}]\mathbf{U}\mathbf{H}_1\mathbf{U}'\}\\ \vdots\\ \operatorname{Tr}\{[\boldsymbol{\Sigma}\mathbf{Z}_{\mathbf{A}}\boldsymbol{\Sigma} - \boldsymbol{\Sigma}(\mathbf{U}\mathbf{F}' + \mathbf{V}\tilde{\mathbf{H}}')\mathbf{Z}_{\mathbf{A}}(\mathbf{F}\mathbf{U}' + \tilde{\mathbf{H}}\mathbf{V}')\boldsymbol{\Sigma}]\mathbf{U}\mathbf{H}_n\mathbf{U}'\}\end{pmatrix},\\
&\frac{\partial\Phi(\boldsymbol{L},\mathbf{Z}_{\mathbf{A}})}{\partial\mathbf{Z}_{\mathbf{A}}} + \frac{1}{2}\operatorname{Diag}\Big(\frac{\partial\Phi(\boldsymbol{L},\mathbf{Z}_{\mathbf{A}})}{\partial\mathbf{Z}_{\mathbf{A}}}\Big)\\
&\qquad = -4\boldsymbol{\Sigma}\mathbf{U}\mathbf{H}_{\boldsymbol{L}}\mathbf{U}'\boldsymbol{\Sigma} + 4(\mathbf{F}\mathbf{U}' + \tilde{\mathbf{H}}\mathbf{V}')\boldsymbol{\Sigma}\mathbf{U}\mathbf{H}_{\boldsymbol{L}}\mathbf{U}'\boldsymbol{\Sigma}(\mathbf{U}\mathbf{F}' + \mathbf{V}\tilde{\mathbf{H}}')\\
&\qquad\quad + 8\boldsymbol{\Sigma}\mathbf{Z}_{\mathbf{A}}\boldsymbol{\Sigma} - 8(\tilde{\mathbf{H}}\mathbf{V}' + \mathbf{F}\mathbf{U}')\boldsymbol{\Sigma}\mathbf{Z}_{\mathbf{A}}\boldsymbol{\Sigma}(\mathbf{U}\mathbf{F}' + \mathbf{V}\tilde{\mathbf{H}}')\\
&\qquad\quad - 8\boldsymbol{\Sigma}(\mathbf{U}\mathbf{F}' + \mathbf{V}\tilde{\mathbf{H}}')\mathbf{Z}_{\mathbf{A}}(\mathbf{F}\mathbf{U}' + \tilde{\mathbf{H}}\mathbf{V}')\boldsymbol{\Sigma}\\
&\qquad\quad + 8(\mathbf{F}\mathbf{U}' + \tilde{\mathbf{H}}\mathbf{V}')\boldsymbol{\Sigma}(\mathbf{U}\mathbf{F}' + \mathbf{V}\tilde{\mathbf{H}}')\mathbf{Z}_{\mathbf{A}}(\mathbf{F}\mathbf{U}' + \tilde{\mathbf{H}}\mathbf{V}')\boldsymbol{\Sigma}(\mathbf{U}\mathbf{F}' + \mathbf{V}\tilde{\mathbf{H}}').
\end{aligned}$$

From the equation $\dfrac{\partial\Phi(\boldsymbol{L},\mathbf{Z}_{\mathbf{A}})}{\partial\boldsymbol{L}} = \boldsymbol{O}$, we obtain

$$\begin{aligned}
&\Big(\boldsymbol{\Sigma} + \frac{1}{2}\mathbf{S}_{\mathbf{U}'\boldsymbol{\Sigma}\mathbf{U}}\Big)\boldsymbol{L}\\
&\quad = \mathbf{F}\boldsymbol{\lambda} + \begin{pmatrix}\operatorname{Tr}\{[\boldsymbol{\Sigma}\mathbf{Z}_{\mathbf{A}}\boldsymbol{\Sigma} - \boldsymbol{\Sigma}(\mathbf{U}\mathbf{F}' + \mathbf{V}\tilde{\mathbf{H}}')\mathbf{Z}_{A}(\mathbf{F}\mathbf{U}' + \tilde{\mathbf{H}}\mathbf{V}')\boldsymbol{\Sigma}]\mathbf{U}\mathbf{H}_1\mathbf{U}'\}\\ \vdots\\ \operatorname{Tr}\{[\boldsymbol{\Sigma}\mathbf{Z}_{\mathbf{A}}\boldsymbol{\Sigma} - \boldsymbol{\Sigma}(\mathbf{U}\mathbf{F}' + \mathbf{V}\tilde{\mathbf{H}}')\mathbf{Z}_{\mathbf{A}}(\mathbf{F}\mathbf{U}' + \tilde{\mathbf{H}}\mathbf{V}')\boldsymbol{\Sigma}]\mathbf{U}\mathbf{H}_n\mathbf{U}'\}\end{pmatrix},
\end{aligned}$$

and, from the equation $\frac{\partial\Phi(\mathbf{L},\mathbf{Z_A})}{\partial\mathbf{Z_A}} + \frac{1}{2}\,\mathrm{Diag}\Big(\frac{\partial\Phi(\mathbf{L},\mathbf{Z_A})}{\partial\mathbf{Z_A}}\Big) = \mathbf{O}$, we obtain

$$\begin{aligned}\mathbf{A} &= \Sigma^{-1}(\mathbf{F},\tilde{\mathbf{H}})\begin{pmatrix}\mathbf{U}'\\ \mathbf{V}'\end{pmatrix}\Sigma\mathbf{A}\Sigma(\mathbf{U},\mathbf{V})\begin{pmatrix}\mathbf{F}'\\ \tilde{\mathbf{H}}'\end{pmatrix}\Sigma^{-1}\\ &= (\mathbf{U},\mathbf{V})\begin{pmatrix}-\frac{1}{2}\mathbf{H}_{\mathbf{L}}, & \mathbf{O}\\ \mathbf{O}, & \mathbf{O}\end{pmatrix}\begin{pmatrix}\mathbf{U}'\\ \mathbf{V}'\end{pmatrix},\end{aligned}$$

thus

$$\mathbf{A} = -\frac{1}{2}\mathbf{U}\mathbf{H}_{\mathbf{L}}\mathbf{U}'.$$

Since

$$\mathbf{A} + \frac{1}{2}\mathbf{U}\mathbf{H}_{\mathbf{L}}\mathbf{U}' = \mathbf{Z_A} - (\mathbf{U}\mathbf{F}' + \tilde{\mathbf{H}}')\mathbf{Z_A}(\mathbf{F}\mathbf{U}' + \tilde{\mathbf{H}}\mathbf{V}') = \mathbf{O},$$

we obtain

$$\mathbf{L} = \Big(\Sigma + \frac{1}{2}\mathbf{S}_{\mathbf{U}'\Sigma\mathbf{U}}\Big)^{-1}\mathbf{F}\Big[\mathbf{F}'\Big(\Sigma + \frac{1}{2}\mathbf{S}_{\mathbf{U}'\Sigma\mathbf{U}}\Big)^{-1}\mathbf{F}\Big]^{-1}\mathbf{p}.$$

Now it is easy to find

$$-\,\mathrm{Tr}(\mathbf{A}\Sigma) = \frac{1}{2}\mathbf{L}'\begin{pmatrix}\mathrm{Tr}\big\{\big[\mathbf{F}'(\mathbf{M}_{\tilde{\mathbf{H}}}\Sigma\mathbf{M}_{\tilde{\mathbf{H}}})^{+}\mathbf{F}\big]^{+}\mathbf{H}_1\big\}\\ \vdots\\ \mathrm{Tr}\big\{\big[\mathbf{F}'(\mathbf{M}_{\tilde{\mathbf{H}}}\Sigma\mathbf{M}_{\tilde{\mathbf{H}}})^{+}\mathbf{F}\big]^{+}\mathbf{H}_n\big\}\end{pmatrix}$$

and

$$\mathbf{Y}'\mathbf{A}\mathbf{Y} = -\frac{1}{2}\mathbf{L}'\begin{pmatrix}(\boldsymbol{\beta}^*)'\mathbf{H}_1\boldsymbol{\beta}^*\\ \vdots\\ (\boldsymbol{\beta}^*)'\mathbf{H}_n\boldsymbol{\beta}^*\end{pmatrix},$$

and to finish the proof. □

Remark 2.13. From Corollary 2.8 and Theorem 2.11, it can be seen that we have a relatively simple quadratic estimator for unbiasedly estimable functions; however the class of such functions need not be sufficiently large. Therefore, in the next section, another approach to the estimation is chosen.

3. Estimability of the second order

Definition 3.1. A statistic $l + \mathbf{L}'\mathbf{Y} + \mathbf{Y}'\mathbf{A}\mathbf{Y}$ is a (2)-*unbiased estimator of a function* $h(\boldsymbol{\beta}) = \mathbf{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathbb{R}^k$, if

$$l = -\,\mathrm{Tr}(\mathbf{A}\Sigma), \qquad \mathbf{F}'\mathbf{L} = \mathbf{h} \quad \text{and} \quad \mathbf{L}'\frac{1}{2}\tilde{\mathbf{H}} + \big[\mathrm{vec}(\mathbf{A})\big]'(\mathbf{F}\otimes\mathbf{F}) = \mathbf{O},$$

i.e., the bias caused by the third and fourth powers of the components of the parameter $\boldsymbol{\beta}$ is neglected.

LEMMA 3.2. *In the model considered here, the whole vector parameter $\boldsymbol{\beta}$ is (2)-unbiasedly estimable.*

Proof. As $r(\mathbf{F}) = k$, for any $\boldsymbol{h} \in \mathbb{R}^k$ there exists $\boldsymbol{L} \in \mathbb{R}^n$ such that $\mathbf{F}'\boldsymbol{L} = \boldsymbol{h}$. Now, it will be shown that for any $\boldsymbol{L} \in \mathbb{R}^n$ there exists a symmetric $n \times n$-matrix $\mathbf{A}$ such that $\frac{1}{2}\sum_{i=1}^{n}\{\boldsymbol{L}\}_i \mathbf{H}_i = \mathbf{F}'\mathbf{A}\mathbf{F}$.

Let $\mathbf{H}$ be an arbitrary $k \times k$ symmetric matrix. Then there exist two p.s.d. matrices $\mathbf{H}^+$ and $\mathbf{H}^-$ such that $\mathbf{H} = \mathbf{H}^+ - \mathbf{H}^-$. Let $\mathbf{H}^+ = \mathbf{J}_1\mathbf{J}_1'$ be the full rank factorization, and, similarly, $\mathbf{H}^- = \mathbf{J}_2\mathbf{J}_2'$. Then

$$\begin{aligned}\mathcal{M}(\mathbf{F}') = \mathbb{R}^k &\implies \mathcal{M}(\mathbf{J}_i) \subset \mathcal{M}(\mathbf{F}')\,, \quad i = 1,2\\ &\iff \exists\{\mathbf{S}_i :\ \mathbf{J}_i = \mathbf{F}'\mathbf{S}_i\,,\ i = 1,2\}\\ &\implies \mathbf{H} = \mathbf{F}'(\mathbf{S}_1\mathbf{S}_1' - \mathbf{S}_2\mathbf{S}_2')\mathbf{F}\,.\end{aligned}$$

Thus $\mathbf{A} = \mathbf{S}_1\mathbf{S}_1' - \mathbf{S}_2\mathbf{S}_2'$. □

THEOREM 3.3. *The (2)-unbiased estimator of the vector parameter $\boldsymbol{\beta}$ which has the minimal (in the Loewner sense) variance matrix among all (2)-unbiased estimators at the point $(\boldsymbol{\beta}{=}\boldsymbol{O}, \Sigma)$ is*

$$\hat{\hat{\boldsymbol{\beta}}} = \left[\mathbf{F}'\left(\Sigma + \tfrac{1}{2}\mathbf{S}_{\mathbf{C}^{-1}}\right)^{-1}\mathbf{F}\right]^{-1}\mathbf{F}'\left(\Sigma + \tfrac{1}{2}\mathbf{S}_{\mathbf{C}^{-1}}\right)^{-1} \cdot \left\{\boldsymbol{Y} + \frac{1}{2}\begin{pmatrix}\mathrm{Tr}(\mathbf{H}_1\mathbf{C}^{-1}) - \hat{\boldsymbol{\beta}}'\mathbf{H}_1\hat{\boldsymbol{\beta}}\\ \vdots\\ \mathrm{Tr}(\mathbf{H}_n\mathbf{C}^{-1}) - \hat{\boldsymbol{\beta}}'\mathbf{H}_n\hat{\boldsymbol{\beta}}\end{pmatrix}\right\},$$

where

$$\left\{\mathbf{S}_{\mathbf{C}^{-1}}\right\}_{i,j} = \mathrm{Tr}(\mathbf{H}_i\mathbf{C}^{-1}\mathbf{H}_j\mathbf{C}^{-1})\,, \quad i,j = 1,\dots,n\,, \qquad \hat{\boldsymbol{\beta}} = \mathbf{C}^{-1}\mathbf{F}'\Sigma^{-1}\boldsymbol{Y}\,.$$

The covariance matrix of this estimator at the point $(\boldsymbol{\beta}{=}\boldsymbol{O}, \Sigma)$ is

$$\mathrm{Var}(\hat{\hat{\boldsymbol{\beta}}} \mid \boldsymbol{\beta} = \boldsymbol{O}, \Sigma) = \left[\mathbf{F}'\left(\Sigma + \tfrac{1}{2}\mathbf{S}_{\mathbf{C}^{-1}}\right)^{-1}\mathbf{F}\right]^{-1}.$$

Proof. (2)-unbiased estimator of the function $h(\boldsymbol{\beta}) = \boldsymbol{h}'\boldsymbol{\beta}$, $\boldsymbol{\beta} \in \mathbb{R}^k$, is $-\mathrm{Tr}(\mathbf{A}\Sigma) + \boldsymbol{L}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y}$, where

$$\mathbf{F}'\boldsymbol{L} = \boldsymbol{h} \quad \& \quad \frac{1}{2}\mathbf{H}_{\boldsymbol{L}} + \mathbf{F}'\mathbf{A}\mathbf{F} = \mathbf{O}\,,$$

$(\mathbf{H}_{\boldsymbol{L}} = \sum_{i=1}^{n}\{\boldsymbol{L}\}_i\mathbf{H}_i)$. In the following, the symbol $=_{(2)}$ denotes the equivalence up to the second power of the components of the parameter $\boldsymbol{\beta}$. Thus

$$\begin{aligned}&\mathrm{Var}(\boldsymbol{L}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y} \mid \boldsymbol{\beta}, \Sigma)\\ &=_{(2)} \boldsymbol{L}'\Sigma\boldsymbol{L} + 4\boldsymbol{L}'\Sigma\mathbf{A}\mathbf{F}\boldsymbol{\beta} + 2\boldsymbol{L}'\Sigma\mathbf{A}\boldsymbol{\kappa} + 2\,\mathrm{Tr}(\mathbf{A}\Sigma\mathbf{A}\Sigma) + 4\boldsymbol{\beta}'\mathbf{F}'\mathbf{A}\Sigma\mathbf{A}\mathbf{F}\boldsymbol{\beta}\end{aligned}$$

and

$$\mathrm{Var}(\boldsymbol{L}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y} \mid \beta=\boldsymbol{O}, \Sigma) = \boldsymbol{L}'\boldsymbol{\Sigma}\boldsymbol{L} + 2\,\mathrm{Tr}(\mathbf{A}\Sigma\mathbf{A}\Sigma)\,.$$

The last quantity must be minimized under the conditions $\mathbf{X}'\boldsymbol{L} = \boldsymbol{h}$, $\frac{1}{2}\mathbf{H}_{\boldsymbol{L}} + \mathbf{F}'\mathbf{A}\mathbf{F} = \mathbf{O}$. The Lagrange procedure gives

$$\Phi(\boldsymbol{L}, \mathbf{A}) \quad = \boldsymbol{L}'\boldsymbol{\Sigma}\boldsymbol{L} + 2\,\mathrm{Tr}(\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma}) - 2\boldsymbol{\lambda}'(\mathbf{F}'\boldsymbol{L} - \boldsymbol{h}) - \mathrm{Tr}\Big[\boldsymbol{\Omega}'\Big(\frac{1}{2}\mathbf{H}_{\boldsymbol{L}} + \mathbf{F}'\mathbf{A}\mathbf{F}\Big)\Big]\,.$$

$$\frac{\partial\Phi(\boldsymbol{L},\mathbf{A})}{\partial \boldsymbol{L}} = 2\boldsymbol{\Sigma}\boldsymbol{L} - 2\mathbf{F}\boldsymbol{\lambda} - \frac{1}{2}\begin{pmatrix}\mathrm{Tr}(\boldsymbol{\Omega}'\mathbf{H}_1)\\ \vdots \\ \mathrm{Tr}(\boldsymbol{\Omega}'\mathbf{H}_n)\end{pmatrix},$$

$$\frac{\partial\Phi(\boldsymbol{L},\mathbf{A})}{\partial \mathbf{A}} = 8\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma} - \mathbf{F}(\boldsymbol{\Omega} + \boldsymbol{\Omega}')\mathbf{F}' - \big[4\,\mathrm{Diag}(\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma}) - \mathrm{Diag}(\mathbf{F}\boldsymbol{\Omega}\mathbf{F}')\big]\,.$$

Thus, the following system of equations must be solved

$$\boldsymbol{\Sigma}\boldsymbol{L} - \mathbf{F}\boldsymbol{\lambda} - \frac{1}{4}\begin{pmatrix}\mathrm{Tr}(\boldsymbol{\Omega}'\mathbf{H}_1)\\ \vdots \\ \mathrm{Tr}(\boldsymbol{\Omega}'\mathbf{H}_n)\end{pmatrix} = \boldsymbol{O}\,, \qquad 8\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma} - \mathbf{F}(\boldsymbol{\Omega} + \boldsymbol{\Omega}')\mathbf{F}' = \mathbf{O}\,,$$

$$\mathbf{F}'\boldsymbol{L} = \boldsymbol{O}\,, \qquad \mathbf{F}'\mathbf{A}\mathbf{F} + \frac{1}{2}\mathbf{H}_{\boldsymbol{L}} = \mathbf{O}\,.$$

Obviously,

$$\boldsymbol{L} = \boldsymbol{\Sigma}^{-1}\mathbf{F}\boldsymbol{\lambda} + \frac{1}{4}\boldsymbol{\Sigma}^{-1}\begin{pmatrix}\mathrm{Tr}(\boldsymbol{\Omega}'\mathbf{H}_1)\\ \vdots \\ \mathrm{Tr}(\boldsymbol{\Omega}'\mathbf{H}_n)\end{pmatrix}, \qquad \mathbf{A} = \frac{1}{8}\boldsymbol{\Sigma}^{-1}\mathbf{F}(\boldsymbol{\Omega} + \boldsymbol{\Omega}')\mathbf{F}'\boldsymbol{\Sigma}^{-1}\,,$$

$$\begin{aligned}\mathbf{F}'\mathbf{A}\mathbf{F} + \frac{1}{2}\mathbf{H}_{\boldsymbol{L}} = \mathbf{O} &\implies \frac{1}{8}\mathbf{C}(\boldsymbol{\Omega} + \boldsymbol{\Omega}')\mathbf{C} + \frac{1}{2}\mathbf{H}_{\boldsymbol{L}} = \mathbf{O}\\ &\implies \frac{1}{2}(\boldsymbol{\Omega} + \boldsymbol{\Omega}') = -2\mathbf{C}^{-1}\mathbf{H}_{\boldsymbol{L}}\mathbf{C}^{-1}\,.\end{aligned}$$

Thus

$$\mathbf{A} = -\frac{1}{2}\boldsymbol{\Sigma}^{-1}\mathbf{F}\mathbf{C}^{-1}\mathbf{H}_{\boldsymbol{L}}\mathbf{C}^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}$$

and

$$\boldsymbol{L} = \boldsymbol{\Sigma}^{-1}\mathbf{F}\boldsymbol{\lambda} + \frac{1}{4}\boldsymbol{\Sigma}^{-1}(-2\mathbf{S}_{\mathbf{C}^{-1}})\boldsymbol{L} \implies \boldsymbol{L} = \Big(\boldsymbol{\Sigma} + \frac{1}{2}\mathbf{S}_{\mathbf{C}^{-1}}\Big)^{-1}\mathbf{F}\boldsymbol{\lambda}\,.$$

Further,

$$\begin{aligned}\mathbf{F}'\boldsymbol{L} = \boldsymbol{h} &\implies \boldsymbol{\lambda} = \Big[\mathbf{F}'\Big(\boldsymbol{\Sigma} + \frac{1}{2}\mathbf{S}_{\mathbf{C}^{-1}}\Big)^{-1}\mathbf{F}\Big]^{-1}\boldsymbol{h}\\ &\implies \boldsymbol{L} = \Big(\boldsymbol{\Sigma} + \frac{1}{2}\mathbf{S}_{\mathbf{C}^{-1}}\Big)^{-1}\mathbf{F}\Big[\mathbf{F}'\Big(\boldsymbol{\Sigma} + \frac{1}{2}\mathbf{S}_{\mathbf{C}^{-1}}\Big)^{-1}\mathbf{F}\Big]^{-1}\boldsymbol{h}\,.\end{aligned}$$

As $\hat{\boldsymbol{\beta}} = \mathbf{C}^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Y}$, we obtain

$$\boldsymbol{Y}'\mathbf{A}\boldsymbol{Y} = -\frac{1}{2}\hat{\boldsymbol{\beta}}'\mathbf{H}_{\boldsymbol{L}}\hat{\boldsymbol{\beta}} = -\boldsymbol{L}'\frac{1}{2}\begin{pmatrix} \hat{\boldsymbol{\beta}}'\mathbf{H}_1\hat{\boldsymbol{\beta}} \\ \vdots \\ \hat{\boldsymbol{\beta}}'\mathbf{H}_n\hat{\boldsymbol{\beta}} \end{pmatrix}$$

and

$$-\operatorname{Tr}(\mathbf{A}\boldsymbol{\Sigma}) = \boldsymbol{L}'\frac{1}{2}\begin{pmatrix} \operatorname{Tr}(\mathbf{H}_1\mathbf{C}^{-1}) \\ \vdots \\ \operatorname{Tr}(\mathbf{H}_n\mathbf{C}^{-1}) \end{pmatrix}.$$

Thus

$$\begin{aligned} &-\operatorname{Tr}(\mathbf{A}\boldsymbol{\Sigma}) + \boldsymbol{L}'\boldsymbol{Y} + \boldsymbol{Y}'\mathbf{A}\boldsymbol{Y} \\ &= \boldsymbol{h}'\left[\mathbf{F}'\left(\boldsymbol{\Sigma} + \tfrac{1}{2}\mathbf{S}_{\mathbf{C}^{-1}}\right)^{-1}\mathbf{F}\right]^{-1}\mathbf{F}'\left(\boldsymbol{\Sigma} + \tfrac{1}{2}\mathbf{S}_{\mathbf{C}^{-1}}\right)^{-1} \\ &\qquad\cdot\left\{\boldsymbol{Y} + \frac{1}{2}\begin{pmatrix} \operatorname{Tr}(\mathbf{H}_1\mathbf{C}^{-1}) - \hat{\boldsymbol{\beta}}'\mathbf{H}_1\hat{\boldsymbol{\beta}} \\ \vdots \\ \operatorname{Tr}(\mathbf{H}_n\mathbf{C}^{-1}) - \hat{\boldsymbol{\beta}}'\mathbf{H}_n\hat{\boldsymbol{\beta}} \end{pmatrix}\right\}. \end{aligned}$$

Since the vector $\boldsymbol{h} \in \mathbb{R}^k$ was arbitrary, the first statement is proved.

The relations

$$\begin{aligned} &\operatorname{cov}(\boldsymbol{Y}, \hat{\boldsymbol{\beta}}'\mathbf{H}_i\hat{\boldsymbol{\beta}} \mid \boldsymbol{\beta}, \boldsymbol{\Sigma}) \\ &= \operatorname{cov}(\boldsymbol{Y}, \boldsymbol{Y}'\boldsymbol{\Sigma}^{-1}\mathbf{F}\mathbf{C}^{-1}\mathbf{H}_i\mathbf{C}^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\boldsymbol{Y} \mid \boldsymbol{\beta}, \boldsymbol{\Sigma}) \\ &= 2\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{-1}\mathbf{F}\mathbf{C}^{-1}\mathbf{H}_i\mathbf{C}^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\left(\mathbf{F}\boldsymbol{\beta} + \tfrac{1}{2}\boldsymbol{\kappa}\right), \end{aligned}$$

and

$$\begin{aligned} &\operatorname{cov}\left(\hat{\boldsymbol{\beta}}'\mathbf{H}_i\hat{\boldsymbol{\beta}}, \hat{\boldsymbol{\beta}}'\mathbf{H}_j\hat{\boldsymbol{\beta}} \mid \boldsymbol{\beta}, \boldsymbol{\Sigma}\right) \\ &= 2\operatorname{Tr}(\mathbf{H}_i\mathbf{C}^{-1}\mathbf{H}_j\mathbf{C}^{-1}) + 4\left(\boldsymbol{\beta} + \mathbf{C}^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-}\tfrac{1}{2}\boldsymbol{\kappa}\right)'\mathbf{H}_i\mathbf{C}^{-1}\mathbf{H}_j\left(\boldsymbol{\beta} + \mathbf{C}^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-}\tfrac{1}{2}\boldsymbol{\kappa}\right) \end{aligned}$$

are necessary for the proof of the second statement. If we use them for $\boldsymbol{\beta} = \boldsymbol{O}$, then we finish the proof easily. □

Remark 3.4. The estimator

$$\tilde{\boldsymbol{\beta}} = \hat{\boldsymbol{\beta}} + \frac{1}{2}\mathbf{C}^{-1}\mathbf{F}'\boldsymbol{\Sigma}^{-1}\begin{pmatrix} \operatorname{Tr}(\mathbf{H}_1\mathbf{C}^{-1}) - \hat{\boldsymbol{\beta}}'\mathbf{H}_1\hat{\boldsymbol{\beta}} \\ \vdots \\ \operatorname{Tr}(\mathbf{H}_n\mathbf{C}^{-1}) - \hat{\boldsymbol{\beta}}'\mathbf{H}_n\hat{\boldsymbol{\beta}} \end{pmatrix}$$

is also a (2)-unbiased estimator of the parameter $\boldsymbol{\beta}$.

Its variance matrix at the point $\beta = \mathbf{O}, \Sigma$ is

$$\operatorname{Var}(\tilde{\beta} \mid \beta = \mathbf{O}, \Sigma) = \mathbf{C}^{-1} + \frac{1}{2}\mathbf{C}^{-1}\mathbf{F}'\Sigma^{-1}\mathbf{S}_{\mathbf{C}^{-1}}\Sigma^{-1}\mathbf{F}\mathbf{C}^{-1}.$$

P r o o f. It can be done in an analogous way as in Theorem 3.3. □

Remark 3.5. It may be of some interest to know something on the difference

$$\operatorname{Var}(\tilde{\beta} \mid \beta = \mathbf{O}, \Sigma) - \operatorname{Var}(\hat{\hat{\beta}} \mid \beta = \mathbf{O}, \Sigma),$$

even we know that it is p.s.d.

Let $\frac{1}{2}\mathbf{S}_{\mathbf{C}^{-1}} = \mathbf{J}\mathbf{J}'$ be the full rank factorization of the matrix $\frac{1}{2}\mathbf{S}_{\mathbf{C}^{-1}}$. Then it can be easily proved

$$\left[\mathbf{F}'\left(\Sigma + \frac{1}{2}\mathbf{S}_{\mathbf{C}^{-1}}\right)^{-1}\mathbf{F}\right]^{-1} = \mathbf{C}^{-1} + \mathbf{C}^{-1}\mathbf{F}'\Sigma^{-1}\mathbf{J}\left[\mathbf{I} + \mathbf{J}'(\mathbf{M}_{\mathbf{F}}\Sigma\mathbf{M}_{\mathbf{F}})^{+}\mathbf{J}\right]^{-1}\mathbf{J}'\Sigma^{-1}\mathbf{F}\mathbf{C}^{-1}.$$

Let $\sum_{i=1}^{r} \lambda_i \boldsymbol{f}_i \boldsymbol{f}_i'$ be the spectral decomposition of the matrix $\mathbf{J}'(\mathbf{M}_{\mathbf{F}}\Sigma\mathbf{M}_{\mathbf{F}})^{+}\mathbf{J}$, where r is the rank of this matrix and obviously $\lambda_i > 0$, $i = 1, \dots, r$. Then

$$\begin{aligned}
&\operatorname{Var}(\tilde{\beta} \mid \beta = \mathbf{O}, \Sigma) - \operatorname{Var}(\hat{\hat{\beta}} \mid \beta = \mathbf{O}, \Sigma) \\
&= \mathbf{C}^{-1}\mathbf{F}'\Sigma^{-1}\mathbf{J}\mathbf{J}'\Sigma^{-1}\mathbf{F}\mathbf{C}^{-1} - \mathbf{C}^{-1}\mathbf{F}'\Sigma^{-1}\mathbf{J}\left[\mathbf{I} + \mathbf{J}'(\mathbf{M}_{\mathbf{F}}\Sigma\mathbf{M}_{\mathbf{F}})^{+}\mathbf{J}\right]^{-1}\mathbf{J}'\Sigma^{-1}\mathbf{F}\mathbf{C}^{-1} \\
&= \mathbf{C}^{-1}\mathbf{F}'\Sigma^{-1}\mathbf{J}\{\mathbf{I} - [\mathbf{I} + \mathbf{J}'(\mathbf{M}_{\mathbf{F}}\Sigma\mathbf{M}_{\mathbf{F}})^{+}\mathbf{J}]^{-1}\}\mathbf{J}'\Sigma^{-1}\mathbf{F}\mathbf{C}^{-1} \\
&= \mathbf{C}^{-1}\mathbf{F}'\Sigma^{-1}\mathbf{J}\left(\sum_{i=1}^{r} \frac{\lambda_i}{1+\lambda_i}\boldsymbol{f}_i\boldsymbol{f}_i'\right)\mathbf{J}'\Sigma^{-1}\mathbf{F}\mathbf{C}^{-1} <_{\boldsymbol{L}} \mathbf{C}^{-1}\mathbf{F}'\Sigma^{-1}\frac{1}{2}\mathbf{S}_{\mathbf{C}^{-1}}\Sigma^{-1}\mathbf{F}\mathbf{C}^{-1},
\end{aligned}$$

since $\sum_{i=1}^{r} \frac{\lambda_i}{1+\lambda_i}\boldsymbol{f}_i\boldsymbol{f}_i' <_{\boldsymbol{L}} \mathbf{I}$ and $\mathbf{J}\mathbf{J}' = \frac{1}{2}\mathbf{S}_{\mathbf{C}^{-1}}$.

Remark 3.6. Let a quadratic model arise from the Taylor series of the mean value of an observation vector $\boldsymbol{Y}$, i.e.,

$$E(\boldsymbol{Y} \mid \boldsymbol{\beta}) = \boldsymbol{f}(\boldsymbol{\beta}^{(0)}) + \left.\frac{\partial \boldsymbol{f}(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}'}\right|_{\boldsymbol{\beta}=\boldsymbol{\beta}^{(0)}} \delta\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\kappa}(\delta\boldsymbol{\beta}),$$

where

$$\left.\frac{\partial \boldsymbol{f}(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}'}\right|_{\boldsymbol{\beta}=\boldsymbol{\beta}^{(0)}} = \mathbf{F}, \qquad \delta\boldsymbol{\beta} = \boldsymbol{\beta} - \boldsymbol{\beta}^{(0)},$$

$$\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) = (\kappa_1, \dots, \kappa_n)', \qquad \kappa_i = (\delta\boldsymbol{\beta})'\left.\frac{\partial^2 f_i(\boldsymbol{\beta})}{\partial\boldsymbol{\beta}\partial\boldsymbol{\beta}'}\right|_{\boldsymbol{\beta}=\boldsymbol{\beta}^{(0)}} \delta\boldsymbol{\beta}, \quad i = 1, \dots, n,$$

and $\boldsymbol{\beta}^{(0)}$ is a value chosen by a statistician. Then the problem, if $\boldsymbol{\beta}^{(0)}$ is chosen properly, i.e., if the third and higher powers of the components of $\delta\boldsymbol{\beta}$ can be neglected, can be solved on the basis of the following consideration.

Let a function $h(\delta\boldsymbol{\beta}) = \boldsymbol{h}'\delta\boldsymbol{\beta}$, $\delta\boldsymbol{\beta} \in \mathbb{R}^k$, be considered, and let

$$\boldsymbol{L}'_h = \boldsymbol{h}'\Big[\mathbf{F}'\Big(\Sigma + \frac{1}{2}\mathbf{S}_{\mathbf{C}^{-1}}\Big)^{-1}\mathbf{F}\Big]^{-1}\mathbf{F}'\Big(\Sigma + \frac{1}{2}\mathbf{S}_{\mathbf{C}^{-1}}\Big)^{-1} = \boldsymbol{h}'\mathbf{T}.$$

Then, from Theorem 3.3, we obtain

$$\begin{aligned}&E\big(\boldsymbol{h}'\hat{\hat{\boldsymbol{\beta}}} \mid \boldsymbol{\beta}^{(0)}+\delta\boldsymbol{\beta}\big)\\&= \boldsymbol{h}'\delta\boldsymbol{\beta} + \sum_{i=1}^{n}\{\boldsymbol{L}_{\boldsymbol{h}}\}_i\Big[-\frac{1}{2}\boldsymbol{\kappa}'(\delta\boldsymbol{\beta})\mathbf{T}'\mathbf{H}_i\delta\boldsymbol{\beta} - \frac{1}{8}\boldsymbol{\kappa}'(\delta\boldsymbol{\beta})\mathbf{T}'\mathbf{H}_i\mathbf{T}\boldsymbol{\kappa}(\delta\boldsymbol{\beta})\Big].\end{aligned}$$

Let us choose, in the parameter space, an arbitrary vector $\boldsymbol{s}$ with the Euclidean norm one, i.e., $\boldsymbol{s}'\boldsymbol{s} = 1$, and let $\delta\boldsymbol{\beta} = \boldsymbol{s}t$, $t \in \mathbb{R}^1$. Then

$$-\frac{1}{2}\boldsymbol{\kappa}'(\delta\boldsymbol{\beta})\mathbf{T}'\mathbf{H}_i\delta\boldsymbol{\beta} = \mathbf{C}_3(i,\boldsymbol{s})t^3$$

and

$$-\frac{1}{8}\boldsymbol{\kappa}'(\delta\boldsymbol{\beta})\mathbf{T}'\mathbf{H}_i\mathbf{T}\boldsymbol{\kappa}(\delta\boldsymbol{\beta}) = \mathbf{C}_4(i,\boldsymbol{s})t^4.$$

Let c_b be a constant such that the value $c_b\sqrt{\boldsymbol{h}'\big[\mathbf{F}'\big(\Sigma + \frac{1}{2}\mathbf{S}_{\mathbf{C}^{-1}}\big)^{-1}\mathbf{F}\big]^{-1}\boldsymbol{h}}$ is tolerable in the bias of the estimator of the function $h(\cdot)$. Now, if t is increasing from $t = 0$, then, in $t = t_{\boldsymbol{s}}$, the quantity

$$\sum_{i=1}^{n}\{\boldsymbol{L}_{\boldsymbol{h}}\}_i\big(\mathbf{C}_3(i,\boldsymbol{s})t_{\boldsymbol{s}}^3 + \mathbf{C}_4(i,\boldsymbol{s})t_{\boldsymbol{s}}^4\big),$$

attains the value $c_b\sqrt{\boldsymbol{h}'\big[\mathbf{F}'\big(\Sigma + \frac{1}{2}\mathbf{S}_{\mathbf{C}^{-1}}\big)^{-1}\mathbf{F}\big]^{-1}\boldsymbol{h}}$.

If a statistician is sure that the actual value of $\boldsymbol{\beta}$ is located in the set

$$(*) = \big\{\boldsymbol{\beta}^{(0)} + \boldsymbol{s}t_{\boldsymbol{s}} :\ \boldsymbol{s} \in \mathbb{R}^k\big\},$$

then the approximation $E(\boldsymbol{Y} \mid \boldsymbol{\beta}) = \boldsymbol{f}_0 + \mathbf{F}\delta\boldsymbol{\beta} + \frac{1}{2}\boldsymbol{\kappa}(\delta\boldsymbol{\beta})$ of the model $E(\boldsymbol{Y} \mid \boldsymbol{\beta}) = f(\boldsymbol{\beta})$ can be used.

The quadratic approach gives a chance to obtain the set $(*)$ larger than the analogous set in case of a linearization of the model (in more detail, cf. [2]).

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Received September 16, 1993

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