Math. Slovaca 36, 1986, No. 2, 137—144

ELIMINATION OF NUISANCE PARAMETERS IN A REGRESSION MODEL

LUBOMÍR KUBÁČEK

Introduction

In a regression model $(\xi, \mathbf{X}\boldsymbol{\beta}, \boldsymbol{\Sigma})$, where ξ is a random vector with the mean value $\mathbf{X}\boldsymbol{\beta}$ and with the covariance matrix $\boldsymbol{\Sigma}$, a structure of the design matrix $\mathbf{X}$ is considered in the form $\mathbf{X}=(\mathbf{A}, \mathbf{S})$ and the vector of unknown parameters $\boldsymbol{\beta}$ in the form $\boldsymbol{\beta}=(\boldsymbol{\Theta}', \boldsymbol{\vartheta}')'$ (′ denotes transposition). The matrices $\mathbf{X}$ and $\boldsymbol{\Sigma}$ are known.

The aim of the experiment is an estimation of the vector $\boldsymbol{\Theta}$ (necessary parameters) from a realization of the vector ξ; $\boldsymbol{\vartheta}$ is the vector of nuisance parameters. (This problem is arisen, e.g. in metrology when systematic influence is to be eliminated from results of measurement.)

If $\mathbf{T}$ is a matrix from the class $\mathcal{T}=\{\mathbf{T}: \mathbf{TA}=\mathbf{A}, \mathbf{TS}=\mathbf{0}\}$, then the nuisance parameters $\boldsymbol{\vartheta}$ are eliminated from the transformed regression model $(\mathbf{T}\xi, \mathbf{A}\boldsymbol{\Theta}, \mathbf{T}\boldsymbol{\Sigma}\mathbf{T}')$; however, it is not clear if the vector $\mathbf{T}\xi$ has the same information on the parameter $\boldsymbol{\Theta}$ as the vector ξ.

The problem is to find such eliminating transformations $\mathbf{T}$ from the class $\mathcal{T}$ which do not cause a loss of information on the parameter $\boldsymbol{\Theta}$.

1. Definitions and auxiliary statements

The following notations will be used:

$\mathbf{B}^-$ ... g-inverse of the matrix $\mathbf{B}$ (i.e. $\mathbf{BB}^-\mathbf{B}=\mathbf{B}$; for more detail see [2]),

$\mathbf{B}_{m(\mathbf{N})}$ ... minimum $\mathbf{N}$-seminorm g-inverse of the matrix $\mathbf{B}$ (i.e. $\mathbf{BB}^-\mathbf{B}=\mathbf{B}$ $(\mathbf{B}_{m(\mathbf{N})}\mathbf{B}'\mathbf{N}=\mathbf{NB}^-_{m(\mathbf{N})}\mathbf{B}$; the symmetric matrix $\mathbf{N}$ has to be at least positive semi-definite (p.s.d.)),

$\mathcal{M}(\mathbf{B})$ ... column space of the matrix $\mathbf{B}$,

$E(\xi)$ ... mean value of the random vector ξ,

$\mathrm{Var}(\xi)$ ... covariance matrix of the random vector ξ,

$\mathrm{cov}(\xi, \eta)$ ... cross covariance matrix of the random vectors ξ and $\boldsymbol{\eta}$ $(\mathrm{cov}(\xi, \boldsymbol{\eta}) = E\{[\xi-E(\xi)][\boldsymbol{\eta}-E(\boldsymbol{\eta})]'\}$,

$\mathcal{R}^n$ ... n-dimensional real linear space.

If there is given in $\mathcal{R}^n$ an inner product $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}'\mathbf{R}\mathbf{y}$, $\mathbf{x}, \mathbf{y} \in \mathcal{R}^n$, where $\mathbf{R}$ is a symmetric and positive definite (p.d.) matrix, then the symbol $\mathbf{P}_\mathbf{A}^\mathbf{R}$ means the projection matrix onto the subspace $\mathcal{M}(\mathbf{A})$ with respect to the norm generated by the considered inner product; $\mathbf{P}_\mathbf{A}^\mathbf{R} = \mathbf{A}(\mathbf{A}'\mathbf{R}\mathbf{A})^-\mathbf{A}'\mathbf{R}$. If an inner product in $\mathcal{R}^n$ is not specified, then the generalization of projection matrix following [5] and [6] is used.

Definition 1.1. *Let $\mathcal{M}_1$, $\mathcal{M}_2$, $\mathcal{M}_3$ be subspaces of $\mathcal{R}^n$ with properties: $\mathcal{M}_1 \cap \mathcal{M}_2 = \{\mathbf{0}\}$, $\mathcal{M}_1 \cap \mathcal{M}_3 = \{\mathbf{0}\}$, $\mathcal{M}_2 \cap \mathcal{M}_3 = \{\mathbf{0}\}$, $\mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3 = \mathcal{R}^n$ ($\oplus$ denotes the direct sum). A matrix $\mathbf{P}_{1.(1)}$ with properties*

$$\forall \{\mathbf{x} \in \mathcal{M}_1\} \mathbf{P}_{1.(1)}\mathbf{x} = \mathbf{x}, \forall \{\mathbf{y} \in \mathcal{M}_2 \oplus \mathcal{M}_3\} \mathbf{P}_{1.(1)}\mathbf{y} = \mathbf{0},$$

is a projection matrix onto $\mathcal{M}_1$ along $\mathcal{M}_2 \oplus \mathcal{M}_3$. The symbols $\mathbf{P}_{2.(2)}$, $\mathbf{P}_{3.(3)}$ have an analogical meaning.

Remark 1.1. The notation $\mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3$ implies $\mathcal{M}_1 \cap \mathcal{M}_2 = \{\mathbf{0}\}$, $\mathcal{M}_1 \cap \mathcal{M}_3 = \{\mathbf{0}\}$, $\mathcal{M}_2 \cap \mathcal{M}_3 = \{\mathbf{0}\}$.

Lemma 1.1. *The matrices $\mathbf{P}_{1.(1)}$, $\mathbf{P}_{2.(2)}$, $\mathbf{P}_{3.(3)}$ from Definition 1.1 are unique and $\mathbf{I}$ (identical matrix) $= \mathbf{P}_{1.(1)} + \mathbf{P}_{2.(2)} + \mathbf{P}_{3.(3)}$.*

Proof is elementary.

Lemma 1.2. *Let $\mathbf{A}_{m,n}$, $\mathbf{B}_{p,q}$, $\mathbf{C}_{m,q}$ be given matrices with indicated dimension. A matrix $\mathbf{X}_{n,p}$ with property $\mathbf{AXB} = \mathbf{C}$ exists iff*

$$(+) \qquad \mathbf{A}\mathbf{A}^-\mathbf{C}\mathbf{B}^-\mathbf{B} = \mathbf{C}.$$

If the condition $(+)$ is fulfilled, then the class of matrices $\mathbf{X}$ is $\{\mathbf{A}^-\mathbf{C}\mathbf{B}^- + \mathbf{Z}_\mathbf{X} - \mathbf{A}^-\mathbf{A}\mathbf{Z}_\mathbf{X}\mathbf{B}\mathbf{B}^-$: $\mathbf{Z}_\mathbf{X}$ is arbitrary matrix with proper dimension$\}$.

Proof. See Theorem 2.3.2 in [2].

Lemma 1.3. *Let $\mathbf{P}$ be an idempotent $n \times n$ matrix and $\mathcal{M}_1 = \mathcal{M}(\mathbf{P})$, $\mathcal{M}_2 = \mathcal{M}(\mathbf{I} - \mathbf{P})$. Then $\mathcal{R}^n = \mathcal{M}_1 \oplus \mathcal{M}_2$, $\mathbf{P}$ is the projection matrix onto $\mathcal{M}_1$ along $\mathcal{M}_2$ and $\mathbf{I} - \mathbf{P}$ is the projection matrix onto $\mathcal{M}_2$ along $\mathcal{M}_1$.*

Proof is elementary.

Corollary 1.1. *If $\mathbf{A}$ is arbitrary $n \times r$ matrix, then $\mathbf{A}\mathbf{A}^-$ is the projection matrix onto $\mathcal{M}(\mathbf{A})$ along $\mathcal{M}(\mathbf{I} - \mathbf{A}\mathbf{A}^-)$.*

2. Class of eliminating transformations

Lemma 2.1. *If the condition $\mathcal{M}(\mathbf{A}) \cap \mathcal{M}(\mathbf{S}) = \{\mathbf{0}\}$ is fulfilled in the regression model $(\boldsymbol{\xi}, (\mathbf{A}, \mathbf{S})(\boldsymbol{\Theta}', \boldsymbol{\vartheta}')', \boldsymbol{\Sigma})$, then there exists an eliminating transformation.*

Proof. If $\mathcal{M}(\mathbf{A}) \cap \mathcal{M}(\mathbf{S}) = \{\mathbf{0}\}$, then $\mathcal{R}^n = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3$, where $\mathcal{M}_1 = \mathcal{M}(\mathbf{A})$, $\mathcal{M}_2 = \mathcal{M}(\mathbf{S})$ and $\mathcal{M}_3$ is an arbitrary subspace provided $\mathcal{R}^n = \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_3$. Projection matrices $\mathbf{P}_{1.(1)}$, $\mathbf{I} - \mathbf{P}_{2.(2)}$ from Definition 1.1 are obviously the eliminating transformations.

Definition 2.1. *Regression model $(\boldsymbol{\xi}, (\mathbf{A}, \mathbf{S})(\Theta', \boldsymbol{\vartheta}')', \boldsymbol{\Sigma})$ is regular if the ranks of the matrices $(\mathbf{A}, \mathbf{S})$, $\boldsymbol{\Sigma}$ are $R(\mathbf{A}_{n,k_1}, \mathbf{S}_{n,k_2}) = k_1 + k_2 \leq n$, $R(\boldsymbol{\Sigma}) = n$.*

Corollary 2.1. *An eliminating transformation exists in the regular regression model* $(\xi, (\mathbf{A}, \mathbf{S})(\boldsymbol{\Theta}', \boldsymbol{\vartheta}')', \Sigma)$.

Theorem 2.1. *In a regression model* $(\xi, (\mathbf{A}, \mathbf{S})(\boldsymbol{\Theta}', \boldsymbol{\vartheta}')', \Sigma)$, *where* $\mathcal{M}((\mathbf{A}) \cap \mathcal{M}(\mathbf{S}) = \{\mathbf{0}\}$, *the class of eliminating transformations is*

$$\mathcal{T} = \{\mathbf{T}: \mathbf{T} = \mathbf{P}_{1.(1)} + \mathbf{Z}_\mathbf{T}\mathbf{P}_{3.(3)}, \mathbf{Z}_\mathbf{T} \text{ is an arbitrary } n \times n \text{ matrix}\},$$

where $\mathbf{P}_{1.(1)}$ *and and* $\mathbf{P}_{3.(3)}$ *are projection matrices from Definition 1.1 onto* $\mathcal{M}_1 = \mathcal{M}(\mathbf{A})$ *and* $\mathcal{M}_3 = \mathcal{M}(\mathbf{I} - (\mathbf{A}, \mathbf{S})(\mathbf{A}, \mathbf{S})^-)$, *respectively. The g-inverse* $(\mathbf{A}, \mathbf{S})^-$ *is arbitrary however fixed.*

Proof. The equation $\mathbf{T}(\mathbf{A}, \mathbf{S}) = (\mathbf{A}, \mathbf{0})$ is solvable for $\mathbf{T}$ with respect to Lemma 2.1. Following Lemma 1.2 the class $\mathcal{T}$ is $\{\mathbf{T}: \mathbf{T} = (\mathbf{A}, \mathbf{0})(\mathbf{A}, \mathbf{S})^- + \mathbf{Z}_\mathbf{T}(\mathbf{I} - (\mathbf{A}, \mathbf{S})(\mathbf{A}, \mathbf{S})^-), \mathbf{Z}_\mathbf{T} \text{ arbitrary}\}$. From $(\mathbf{A}, \mathbf{0})(\mathbf{A}, \mathbf{S})^-[\mathbf{A}, \mathbf{S}, (\mathbf{I} - (\mathbf{A}, \mathbf{S})(\mathbf{A}, \mathbf{S})^-] = (\mathbf{A}, \mathbf{0}, \mathbf{0})$ there follows $(\mathbf{A}, \mathbf{0})(\mathbf{A}, \mathbf{S})^- = \mathbf{P}_{1.(1)}$. Since $\mathbf{I} - (\mathbf{A}, \mathbf{S})(\mathbf{A}, \mathbf{S})^-$ is idempotent and $[\mathbf{I} - (\mathbf{A}, \mathbf{S})(\mathbf{A}, \mathbf{S})^-](\mathbf{A}, \mathbf{S}) = (\mathbf{0}, \mathbf{0})$, it follows that $\mathbf{I} - (\mathbf{A}, \mathbf{S})(\mathbf{A}, \mathbf{S})^- = \mathbf{P}_{3.(3)}$ (Lemma 1.3, Corollary 1.1).

Lemma 2.2. *Let the condition* $\mathcal{M}(\mathbf{A}) \cap \mathcal{M}(\mathbf{S}) = \{\mathbf{0}\}$ *be fulfilled in a regression model* $(\xi, (\mathbf{A}, \mathbf{S})(\boldsymbol{\Theta}', \boldsymbol{\vartheta}')', \Sigma)$. *Then the following is true:*

a) *the* BLUE (*best linear unbiased estimator*) *of the vector* $\mathbf{A}\boldsymbol{\Theta}$ *is*

$$\widehat{\mathbf{A}\boldsymbol{\Theta}} = (\mathbf{A}, \mathbf{0})\left[\begin{pmatrix}\mathbf{A}' \\ \mathbf{S}'\end{pmatrix}^-_{m(\Sigma)}\right]' \xi,$$

b) *for an eliminating transformation* $\mathbf{T}$

$$\mathbf{A}[(\mathbf{A}')^-_{m(\mathbf{T}\Sigma\mathbf{T}')}]'\mathbf{T}\xi = \widehat{\mathbf{A}\boldsymbol{\Theta}}$$

iff

$$(++) \qquad \mathbf{P}_{3.(3)}\Sigma\mathbf{T}'(\mathbf{A}')^-_{m(\mathbf{T}\Sigma\mathbf{T}')}\mathbf{A}' = \mathbf{0}.$$

$\mathbf{P}_{3.(3)}$ *is a projection matrix from Theorem* 2.1.

Proof. The statement a) is a consequence of Theorem 1.2 and Theorem 3.1 in [3]. The statement b) is implied by the fact that the class of all linear unbiased estimators of the function $g(.,..): \mathcal{R}^{k_1} \times \mathcal{R}^{k_2} \to \{\mathbf{0}\}$, $(\boldsymbol{\Theta}, \boldsymbol{\vartheta}) \in \mathcal{R}^{k_1} \times \mathcal{R}^{k_2}$ is $\{\boldsymbol{\lambda}'\mathbf{P}_{3.(3)}\xi: \mathbf{L} \in \mathcal{R}^n\}$ and by the fundamental lemma of C. R. Rao [4] p. 257.

Definition 2.2. *The eliminating transformation which satisfies* $(++)$ *is optimal.*

Remark 2.1. In general the condition $R(\mathbf{A}) = k_1 \leq n$ need not be fulfilled, thus $\boldsymbol{\Theta}$ need not be unbiasedly estimable. The vector $\mathbf{A}\boldsymbol{\Theta}$ provided $\mathcal{M}(\mathbf{A}) \cap \mathcal{M}(\mathbf{S}) = \{\mathbf{0}\}$ represents the class of all linear unbiased estimable functions of the parameter $\boldsymbol{\Theta}$. That is why, the estimator of the vector $\mathbf{A}\boldsymbol{\Theta}$ is given in Lemma 2.2 instead of the estimator of the vector $\boldsymbol{\Theta}$.

Theorem 2.2. *Each eliminating transformation* $\mathbf{T}$ *which satisfies the condition*

$$(+++) \qquad \mathbf{P}_{3.(3)}(\mathbf{I} - \mathbf{T}) = \mathbf{0}$$

is optimal.

Proof. If the condition $(+++)$ is fulfilled, the $(++)$ implies: $\mathbf{P}_{3\,(3)}[\mathbf{T}+(\mathbf{I}-\mathbf{T})]\boldsymbol{\Sigma}\mathbf{T}'(\mathbf{A}')^-_{m(\mathbf{T}\Sigma\mathbf{T}')}\mathbf{A}' = \mathbf{P}_{3\,(3)}\mathbf{A}[(\mathbf{A}')^-_{m(\mathbf{T}\Sigma\mathbf{T}')}]'\mathbf{T}\boldsymbol{\Sigma}\mathbf{T}' = \mathbf{0}$, because $\mathbf{P}_{3\,(3)}\mathbf{A} = \mathbf{0}$.

Corollary 2.2. *If $\mathcal{M}_3$ is an arbitrary subspace of $\mathcal{R}^n$ (provided $\mathcal{R}^n = \mathcal{M}(\mathbf{A})\oplus\mathcal{M}(\mathbf{S})\oplus\mathcal{M}_3$), then $\mathbf{I}-\mathbf{P}_{2\,(2)}$ is an optimal eliminating transformation.*

Lemma 2.3. *A regression model $(\xi, (\mathbf{A}, \mathbf{S})(\boldsymbol{\Theta}', \boldsymbol{\vartheta}')', \boldsymbol{\Sigma})$ is equivalent to a model*

$$\left[\begin{pmatrix}\mathbf{T}\\ \mathbf{I}-\mathbf{T}\end{pmatrix}\xi,\ \begin{pmatrix}\mathbf{A}, & \mathbf{0}\\ \mathbf{0}, & \mathbf{S}\end{pmatrix}\begin{pmatrix}\boldsymbol{\Theta}\\ \boldsymbol{\vartheta}\end{pmatrix},\ \begin{pmatrix}\boldsymbol{\Sigma}_{1,1}, & \boldsymbol{\Sigma}_{1,2}\\ \boldsymbol{\Sigma}_{2,1}, & \boldsymbol{\Sigma}_{2,2}\end{pmatrix}\right]$$

where $\mathbf{T}$ is an arbitrary eliminating transformation and $\boldsymbol{\Sigma}_{1,1} = \mathrm{Var}(\mathbf{T}\xi)$, $\boldsymbol{\Sigma}_{1,2} = \mathrm{cov}(\mathbf{T}\xi, (\mathbf{I}-\mathbf{T})\xi) = \boldsymbol{\Sigma}_{2,1}'$, $\boldsymbol{\Sigma}_{2,2} = \mathrm{Var}[(\mathbf{I}-\mathbf{T})\xi]$.

Proof. Implied by the fact that the matrix

$$\begin{pmatrix}\mathbf{T}\\ \mathbf{I}-\mathbf{T}\end{pmatrix} = \begin{pmatrix}\mathbf{I}, & -\mathbf{I}\\ \mathbf{0}, & \mathbf{I}\end{pmatrix}\begin{pmatrix}\mathbf{I}\\ \mathbf{I}-\mathbf{T}\end{pmatrix}$$

has full rank in columns.

Lemma 2.4. *One version of the matrix*

$$\begin{pmatrix}\mathbf{A}', & \mathbf{0}\\ \mathbf{0}, & \mathbf{S}'\end{pmatrix}^-_{m(\mathbf{D})},\quad \mathbf{D} = \begin{pmatrix}\boldsymbol{\Sigma}_{1,1}, & \boldsymbol{\Sigma}_{1,2}\\ \boldsymbol{\Sigma}_{2,1}, & \boldsymbol{\Sigma}_{2,2}\end{pmatrix}$$

(notations from Lemma 2.3) is

$$\begin{pmatrix}\mathbf{B}_{1,1}, & \mathbf{B}_{1,2}\\ \mathbf{B}_{2,1}, & \mathbf{B}_{2,2}\end{pmatrix},$$

where

$$\mathbf{B}_{1,1} = (\mathbf{A}')^-_{m(*)},\ \mathbf{B}_{1,2} = -\{\mathbf{I}-(\mathbf{A}')^-_{m(\boldsymbol{\Sigma}_{1,1})}\mathbf{A}'\}\boldsymbol{\Sigma}^-_{1,1}\boldsymbol{\Sigma}_{1,2}(\mathbf{S}')^-_{m(**)},$$

$$\mathbf{B}_{2,1} = -\{\mathbf{I}-(\mathbf{S}')^-_{m(\boldsymbol{\Sigma}_{2,2})}\mathbf{S}'\}\boldsymbol{\Sigma}^-_{2,2}\boldsymbol{\Sigma}_{2,1}(\mathbf{A}')^-_{m(*)},\ \mathbf{B}_{2,2} = (\mathbf{S}')^-_{m(**)},$$

$$(*) = \boldsymbol{\Sigma}_{1,1} - \boldsymbol{\Sigma}_{1,2}\boldsymbol{\Sigma}^-_{2,2}[\boldsymbol{\Sigma}_{2,2} - \boldsymbol{\Sigma}_{2,2}(\mathbf{S}')^-_{m(\boldsymbol{\Sigma}_{2,2})}\mathbf{S}']\boldsymbol{\Sigma}^-_{2,2}\boldsymbol{\Sigma}_{2,1},$$

$$(**) = \boldsymbol{\Sigma}_{2,2} - \boldsymbol{\Sigma}_{2,1}\boldsymbol{\Sigma}^-_{1,1}[\boldsymbol{\Sigma}_{1,1} - \boldsymbol{\Sigma}_{1,1}(\mathbf{A}')^-_{m(\boldsymbol{\Sigma}_{1,1})}\mathbf{A}']\boldsymbol{\Sigma}^-_{1,1}\boldsymbol{\Sigma}_{1,2}.$$

Proof. See in [1], Theorem 3.1.

Theorem 2.3. *The BLUE of the vector $\mathbf{A}\boldsymbol{\Theta}$ in the regression model $(\xi, (\mathbf{A}, \mathbf{S})(\boldsymbol{\Theta}, \boldsymbol{\vartheta}')', \boldsymbol{\Sigma})$, $\mathcal{M}(\mathbf{A})\cap\mathcal{M}(\mathbf{S}) = \{\mathbf{0}\}$, can be expressed in the form*

$$\widehat{\mathbf{A}\boldsymbol{\Theta}} = \mathbf{A}[(\mathbf{A}')^-_{m(*)}]'\{\mathbf{T}\xi - \boldsymbol{\Sigma}_{1,2}\boldsymbol{\Sigma}^-_{2,2}\{\mathbf{I}-\mathbf{S}[(\mathbf{S}')^-_{m(\boldsymbol{\Sigma}_{2,2})}]'\}\ (\mathbf{I}-\mathbf{T})\xi\},$$

where $(*)$ is the notation from Lemma 2.4 and the others are from Lemma 2.3.

Proof follows immediately from Lemma 2.2, Lemma 2.3 and Lemma 2.4.

Remark 2.3. The estimator from Theorem 2.3 can be also expressed in the form $\widehat{\mathbf{A}\boldsymbol{\Theta}} = \mathbf{A}[(\mathbf{A}')^-_{m(\mathbf{T}_0\boldsymbol{\Sigma}\mathbf{T}_0')}]'\mathbf{T}_0\boldsymbol{\xi}$, where

$$\mathbf{T}_0 = \mathbf{T} - \boldsymbol{\Sigma}_{1,2}\boldsymbol{\Sigma}_{2,2}^-\{\mathbf{I} - \mathbf{S}[(\mathbf{S}')^-_{m(\boldsymbol{\Sigma}_{2,2})}]'\}(\mathbf{I} - \mathbf{T})$$

($\mathbf{T}$ is an arbitrary eliminating transformation) since Var $(\mathbf{T}_0\boldsymbol{\xi}) = (*)$. It means that $\mathbf{T}_0$ is optimal with respect to Definition 2.2.

Corollary 2.3. *If $\mathbf{T}\boldsymbol{\Sigma}(\mathbf{I}-\mathbf{T})' = \mathbf{0}$ for an eliminating transformation $\mathbf{T}$, then $\mathbf{T}$ is optimal. The equality $\mathbf{T}\boldsymbol{\Sigma}(\mathbf{I}-\mathbf{T})' = \boldsymbol{\Sigma}_{1,2}$ implies $(*) = \boldsymbol{\Sigma}_{1,1} = \mathbf{T}\boldsymbol{\Sigma}\mathbf{T}'$ and $\mathbf{T}_0 = \mathbf{T}$.*

Corollary 2.4. *If $\mathbf{I} - \mathbf{T} = \mathbf{SC}$ holds true for an eliminating transformation $\mathbf{T}$, then the estimator $\widehat{\mathbf{A}\boldsymbol{\Theta}}$ from Theorem 2.3 is $\widehat{\mathbf{A}\boldsymbol{\Theta}} = \mathbf{A}[(\mathbf{A}')^-_{m(\mathbf{T}\boldsymbol{\Sigma}\mathbf{T}')}]'\mathbf{T}\boldsymbol{\xi}$. In this case there is $\{\mathbf{I} - \mathbf{S}[(\mathbf{S}')^-_{m(\boldsymbol{\Sigma}_{2,2})}]'\}$ $(\mathbf{I}-\mathbf{T}) = \mathbf{0}$ and $(*) =$ Var $(\mathbf{T}\boldsymbol{\Sigma}\mathbf{T}')$. (See Corollary 2.2.)*

Lemma 2.5. *One of the matrices $(\mathbf{A}, \mathbf{S})^-$ is $\left[\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}^-_{m(\boldsymbol{\Sigma})}\right]' = \mathbf{Q}$ and one of the matrices $\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}^-_{m(\boldsymbol{\Sigma})}$ is $[(\mathbf{AA}' + \mathbf{SS}')^-_{m(\boldsymbol{\Sigma})}\mathbf{A}, (\mathbf{AA}' + \mathbf{SS}')^-_{m(\boldsymbol{\Sigma})}\mathbf{S}]$.*

Proof. It suffices to verify the relations $(\mathbf{A}, \mathbf{S})\mathbf{Q}(\mathbf{A}, \mathbf{S}) = (\mathbf{A}, \mathbf{S})$, $\boldsymbol{\Sigma}[(\mathbf{AA}' + \mathbf{SS}')^-_{m(\boldsymbol{\Sigma})}\mathbf{A}, (\mathbf{AA}' + \mathbf{SS}')^-_{m(\boldsymbol{\Sigma})}\mathbf{S}]\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix} = (\mathbf{A}, \mathbf{S})[(\mathbf{AA}' + \mathbf{SS}')^-_{m(\boldsymbol{\Sigma})}\mathbf{A}, (\mathbf{AA}' + \mathbf{SS}')^-_{m(\boldsymbol{\Sigma})}\mathbf{S}]'\boldsymbol{\Sigma}$ and $(\mathbf{A}, \mathbf{S})[(\mathbf{AA}' + \mathbf{SS}')^-_{m(\boldsymbol{\Sigma})}\mathbf{A}, (\mathbf{AA}' + \mathbf{SS}')^-_{m(\boldsymbol{\Sigma})}\mathbf{S}]'(\mathbf{A}, \mathbf{S}) = (\mathbf{A}, \mathbf{S})$. The relations $\mathcal{M}(\mathbf{A}) \subset \mathcal{M}(\mathbf{AA}' + \mathbf{SS}') = \mathcal{M}(\mathbf{A}, \mathbf{S})$, $\mathcal{M}(\mathbf{S}) \subset \mathcal{M}(\mathbf{AA}' + \mathbf{SS}') = \mathcal{M}(\mathbf{A}, \mathbf{S})$, implied by Theorem 6.2.5 in [2], have to be utilized.

Lemma 2.6. *Let $\boldsymbol{\eta}$ be an n-dimensional random vector with $E(\boldsymbol{\eta}) = \boldsymbol{\mu}$ and Var $(\boldsymbol{\eta}) = \boldsymbol{\Sigma}$. Then $P\{\boldsymbol{\eta} - \boldsymbol{\mu} \in \mathcal{M}(\boldsymbol{\Sigma})\} = 1$.*

Proof. Let $R(\boldsymbol{\Sigma}) = r \leq n$. Then there exists an $n \times r$ matrix $\mathbf{J}$ with property $\boldsymbol{\Sigma} = \mathbf{JJ}'$. If $\mathbf{F}'$ is such a matrix that $\mathbf{F}'\mathbf{J} = \mathbf{I}$, then $E[\mathbf{F}'(\boldsymbol{\eta} - \boldsymbol{\mu})] = \mathbf{0}$ and Var $[\mathbf{F}'(\boldsymbol{\eta} - \boldsymbol{\mu})] = \mathbf{I}$. For a vector $\boldsymbol{\eta}_0 = \boldsymbol{\mu} + \mathbf{JF}'(\boldsymbol{\eta} - \boldsymbol{\mu})$ there holds: $E(\boldsymbol{\eta} - \boldsymbol{\eta}_0) = \mathbf{0}$ and Var $(\boldsymbol{\eta} - \boldsymbol{\eta}_0) =$ Var $[(\mathbf{I} - \mathbf{JF}')(\boldsymbol{\eta} - \boldsymbol{\mu})] = (\mathbf{I} - \mathbf{JF}')\mathbf{JJ}'(\mathbf{I} - \mathbf{FJ}') = \mathbf{0} \Rightarrow P\{\boldsymbol{\eta} - \boldsymbol{\mu} = \mathbf{JF}'(\boldsymbol{\eta} - \boldsymbol{\mu})\} = 1$. Since $\mathcal{M}(\mathbf{J}) = \mathcal{M}(\boldsymbol{\Sigma})$, the statement is obvious.

Theorem 2.4. $\mathbf{T} = \mathbf{AA}'[(\mathbf{AA}' + \mathbf{SS}')^-_{m(\boldsymbol{\Sigma})}]'$ *is a unique optimal eliminating transformation with the property $P\{\mathbf{T}\boldsymbol{\xi} = \widehat{\mathbf{A}\boldsymbol{\Theta}}\} = 1$, where $\widehat{\mathbf{A}\boldsymbol{\Theta}}$ is the estimator from Theorem 2.3 and Lemma 2.2a), respectively.*

Proof. $\mathbf{P}_{3.(3)}$ from Lemma 2.2 is fixed as $\mathbf{P}_{3.(3)} = \mathbf{I} - (\mathbf{A}, \mathbf{S})\left[\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}^-_{m(\boldsymbol{\Sigma})}\right]'$ and the same is done in Theorem 2.1. An arbitrary eliminating transformation can be thus written as

$$\mathbf{T} = (\mathbf{A}, \mathbf{0})\left[\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}^-_{m(\boldsymbol{\Sigma})}\right]' + \mathbf{Z}_T\left\{\mathbf{I} - (\mathbf{A}, \mathbf{S})\left[\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}^-_{m(\boldsymbol{\Sigma})}\right]'\right\}$$

and the condition $(++)$ can be rewritten as

$$\left[\left\{\mathbf{I}-(\mathbf{A},\mathbf{S})\left[\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}^{-}_{m(\boldsymbol{\Sigma})}\right]'\right\}\boldsymbol{\Sigma}\left[\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}^{-}_{n(\boldsymbol{\Sigma})}\begin{pmatrix}\mathbf{A}'\\ \mathbf{0}\end{pmatrix}+\left\{\mathbf{I}-\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}^{-}_{m(\boldsymbol{\Sigma})}\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}\right\}\mathbf{Z}'_{\mathbf{T}}\right]\right]=$$

$$=\boldsymbol{\Sigma}\left\{\mathbf{I}-\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}^{-}_{m(\boldsymbol{\Sigma})}\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}\right\}\mathbf{Z}'_{\mathbf{T}}=\mathbf{0},$$

because

$$(\mathbf{A},\mathbf{S})\left[\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}^{-}_{m(\boldsymbol{\Sigma})}\right]'\boldsymbol{\Sigma}=\boldsymbol{\Sigma}\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}^{-}_{m(\boldsymbol{\Sigma})}\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}$$

and

$$\left\{\mathbf{I}-(\mathbf{A},\mathbf{S})\left[\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}^{-}_{m(\boldsymbol{\Sigma})}\right]'\right\}\boldsymbol{\Sigma}\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}^{-}_{m(\boldsymbol{\Sigma})}\begin{pmatrix}\mathbf{A}'\\ \mathbf{0}\end{pmatrix}=\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\end{pmatrix}.$$

With respect to Lemma 2.5 and Lemma 2.6 there holds

$$P\{\mathbf{T}\xi=\mathbf{AA}'[(\mathbf{AA}'+\mathbf{SS}')^{-}_{m(\boldsymbol{\Sigma})}]'\xi\}=1,$$

because

$$P\left\{\mathbf{Z}_{\mathbf{T}}\left\{\mathbf{I}-(\mathbf{A},\mathbf{S})\left[\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}^{-}_{m(\boldsymbol{\Sigma})}\right]'\right\}\xi=\boldsymbol{0}\right\}=1.$$

(Lemma 2.6 implies that almost each realization of the vector ξ is $\boldsymbol{x}=(\mathbf{A},\mathbf{S})(\boldsymbol{\Theta}',\boldsymbol{\vartheta}')'+\boldsymbol{\Sigma}\boldsymbol{u}$ and from the rewritten condition $(++)$ it follows that

$$\mathbf{Z}_{\mathbf{T}}\left\{\mathbf{I}-(\mathbf{A},\mathbf{S})\left[\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}^{-}_{m(\boldsymbol{\Sigma})}\right]'\right\}\boldsymbol{x}=\mathbf{Z}_{\mathbf{T}}\left\{\mathbf{I}-(\mathbf{A},\mathbf{S})\left[\begin{pmatrix}\mathbf{A}'\\ \mathbf{S}'\end{pmatrix}^{-}_{m(\boldsymbol{\Sigma})}\right]'\right\}\boldsymbol{\Sigma}\boldsymbol{u}=\boldsymbol{0}.$$

Theorem 2.5. *In the regular regression model* (ξ, $(\mathbf{A},\mathbf{S})(\boldsymbol{\Theta}',\boldsymbol{\vartheta}')'$, $\boldsymbol{\Sigma}$) *the* BLUEs *of the vectors* $\mathbf{A}\boldsymbol{\Theta}$ *and* $\mathbf{S}\boldsymbol{\vartheta}$ *are*:

$$\mathbf{A}\hat{\boldsymbol{\Theta}}=\mathbf{P}_{\mathbf{A}}\xi,\quad \mathbf{S}\hat{\boldsymbol{\vartheta}}=\mathbf{P}_{\mathbf{S}}\xi,$$

where

$$\mathbf{P}_{\mathbf{A}}=\mathbf{A}(\mathbf{A}'\mathbf{LA})^{-1}\mathbf{A}'\mathbf{L},\quad \mathbf{L}=\boldsymbol{\Sigma}^{-1}(\mathbf{I}-\mathbf{P}_{\mathbf{S}}^{\boldsymbol{\Sigma}^{-1}}),$$

$$\mathbf{P}_{\mathbf{S}}=\mathbf{S}(\mathbf{S}'\mathbf{KS})^{-1}\mathbf{S}'\mathbf{K},\quad \mathbf{K}=\boldsymbol{\Sigma}^{-1}(\mathbf{I}-\mathbf{P}_{\mathbf{A}}^{\boldsymbol{\Sigma}^{-1}}).$$

The matrix $\mathbf{P}_{\mathbf{A}}$ *is* $\mathbf{P}_{1.(1)}$*(Definition* 1.1) *onto* $\mathcal{M}_1=\mathcal{M}(\mathbf{A})$ *and* $\mathbf{P}_{\mathbf{S}}$ *is* $\mathbf{P}_{2.(2)}$ *onto* $\mathcal{M}_2=\mathcal{M}(\mathbf{S})$, *where* $\mathcal{M}_3=\mathcal{M}(\mathbf{I}-\mathbf{P}^{\boldsymbol{\Sigma}^{-1}}_{(\mathbf{A},\mathbf{S})})$ *is an orthogonal complement of the subspace* $\mathcal{M}_1\oplus\mathcal{M}_2$ *with respect to Mahalanobis inner product* $\langle\boldsymbol{x},\boldsymbol{y}\rangle=\boldsymbol{x}'\boldsymbol{\Sigma}^{-1}\boldsymbol{y}$, $\boldsymbol{x},\boldsymbol{y}\in\mathcal{R}^n$.

Proof. If the notations

$$(\mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A})^{11}=[\mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A}-\mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{S}(\mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{S}')^{-1}\mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{A}]^{-1},$$

$$(\mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{S})^{22}=[\mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{S}-\mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{A}(\mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A})^{-1}\mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{S}]^{-1},$$

(the existence of the inversion of the indicated matrices is ensured by the assumption of the regularity of the considered regression model) are used, then the BLUE of the vector $(\boldsymbol{\Theta}', \boldsymbol{\vartheta}')'$ is

$$\begin{pmatrix} \hat{\boldsymbol{\Theta}} \\ \hat{\boldsymbol{\vartheta}} \end{pmatrix} = \left[\begin{pmatrix} \mathbf{A}' \\ \mathbf{S}' \end{pmatrix} \boldsymbol{\Sigma}^{-1} (\mathbf{A}, \mathbf{S}) \right]^{-1} \begin{pmatrix} \mathbf{A}' \\ \mathbf{S}' \end{pmatrix} \boldsymbol{\Sigma}^{-1} \xi =$$

$$= \begin{bmatrix} \mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A})^{11}, & -(\mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A})^{11}\mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{S}(\mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{S})^{-1} \\ -(\mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{S})^{22}\mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{A}(\mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{A})^{-1}, & (\mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{S})^{22} \end{bmatrix} \cdot \begin{pmatrix} \mathbf{A}'\boldsymbol{\Sigma}^{-1} \\ \mathbf{S}'\boldsymbol{\Sigma}^{-1} \end{pmatrix} \xi =$$

$$\begin{bmatrix} [\mathbf{A}'\boldsymbol{\Sigma}^{-1}(\mathbf{I} - \mathbf{P}_{\mathbf{S}}^{\boldsymbol{\Sigma}^{-1}})\mathbf{A}]^{-1}\mathbf{A}'\boldsymbol{\Sigma}^{-1} - [\mathbf{A}'\boldsymbol{\Sigma}^{-1}(\mathbf{I} - \mathbf{P}_{\mathbf{S}}^{\boldsymbol{\Sigma}^{-1}})\mathbf{A}]^{-1}\mathbf{A}'\boldsymbol{\Sigma}^{-1}\mathbf{P}_{\mathbf{S}}^{\boldsymbol{\Sigma}^{-1}} \\ [\mathbf{S}'\boldsymbol{\Sigma}^{-1}(\mathbf{I} - \mathbf{P}_{\mathbf{A}}^{\boldsymbol{\Sigma}^{-1}})\mathbf{S}]^{-1}\mathbf{S}'\boldsymbol{\Sigma}^{-1} - [\mathbf{S}'\boldsymbol{\Sigma}^{-1}(\mathbf{I} - \mathbf{P}_{\mathbf{A}}^{\boldsymbol{\Sigma}^{-1}})\mathbf{S}]^{-1}\mathbf{S}'\boldsymbol{\Sigma}^{-1}\mathbf{P}_{\mathbf{A}}^{\boldsymbol{\Sigma}^{-1}} \end{bmatrix} \xi$$

$$= \begin{bmatrix} (\mathbf{A}'\mathbf{L}\mathbf{A})^{-1}\mathbf{A}'\mathbf{L} \\ (\mathbf{S}'\mathbf{K}\mathbf{S})^{-1}\mathbf{S}'\mathbf{K} \end{bmatrix} \xi \Rightarrow \mathbf{A}\hat{\boldsymbol{\Theta}} = \mathbf{P}_{\mathbf{A}}\xi, \ \mathbf{S}\hat{\boldsymbol{\vartheta}} = \mathbf{P}_{\mathbf{S}}\xi.$$

Furthermore it is obviously valid that $\mathcal{M}(\mathbf{P}_{\mathbf{A}}) = \mathcal{M}(\mathbf{A}) = \mathcal{M}_1$, $\mathbf{P}_{\mathbf{A}}^2 = \mathbf{P}_{\mathbf{A}}$, $\mathcal{M}(\mathbf{P}_{\mathbf{S}}) = \mathcal{M}(\mathbf{S}) = \mathcal{M}_2$, $\mathbf{P}_{\mathbf{S}}^2 = \mathbf{P}_{\mathbf{S}}$, $\mathbf{P}_{\mathbf{A}}\mathbf{P}_{\mathbf{S}} = \mathbf{P}_{\mathbf{S}}\mathbf{P}_{\mathbf{A}} = \mathbf{0}$, $\mathbf{P}_{(\mathbf{A}, \mathbf{S})}^{\boldsymbol{\Sigma}^{-1}} = \mathbf{P}_{\mathbf{A}} + \mathbf{P}_{\mathbf{S}}$ and $(\mathbf{P}_{(\mathbf{A}, \mathbf{S})}^{\boldsymbol{\Sigma}^{-1}})'\boldsymbol{\Sigma}^{-1}(\mathbf{I} - \mathbf{P}_{\mathbf{A}} - \mathbf{P}_{\mathbf{S}}) = \mathbf{0}$.

REFERENCES

[1] KUBÁČEK, L.: Locally best quadratic estimators. Math. Slovaca 35, 1985, 393—408.

[2] RAO, C. R., MITRA, S. K.: Generalized Inverse of Matrices and Its Applications. J. Wiley, N. York 1971.

[3] RAO, C. R.: Unified theory of linear estimation. Sankhya, ser. A, 33, 1971, 371—394.

[4] RAO, C. R.: Linear Statistical Inference and Its Applications. J. Wiley, N. York 1965.

[5] RAO, C. R., YANAI, H.: General definition and decomposition of projectors and some applications to statistical problems. Journal of Statist. Planning and Inference 3, 1979,

[6] RAO, C. R., YANAI, H.: Generalized inverse of linear transformations — A geometric approach. Technical Report No. 83—15, Center for Multivariate Analysis, University of Pitsburgh, 1983.

Received June 18, 1984

Matematický ústav SAV
Obrancov mieru 49
814 73 Bratislava

ИСКЛЮЧЕНИЕ МЕШАЮЩИХ ПАРАМЕТРОВ В РЕГРЕССИОННОЙ МОДЕЛИ

Lubomír Kubáček

Резюме

В регрессионной модели $(\xi, (\mathbf{A}, \mathbf{S})(\boldsymbol{\Theta}', \boldsymbol{\vartheta}')', \boldsymbol{\Sigma})$ с мешающим параметром $\boldsymbol{\vartheta}$ исследуется класс всех исключающих трансформаций $\mathcal{T} = \{\mathbf{T}\colon \mathbf{TA} = \mathbf{A}, \mathbf{TS} = \mathbf{0}\}$. Исключающая трансформация является оптимальной, если трансформированная модель $(\mathbf{T}\xi, \mathbf{A}\boldsymbol{\Theta}, \mathbf{T}\boldsymbol{\Sigma}\mathbf{T}')$ позволяет построить такую линейную несмещенную оценку вектора $\mathbf{A}\boldsymbol{\Theta}$, у которой ковариационная матрица такая же самая, как у наилучшей линейной оценки в первоначальной модели. Найден класс оптимальных исключающих трансформаций.